Cosmic censorship beyond General Relativity:
Collapsing charged thin shells in low energy effective string theory

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Resumo

O colapso gravitacional de camadas finas em Einstein-Maxwell-dilaton é estudado recorrendo a soluções que representam buracos negros (ou singularidades despidas) quadridimensionais e esfericamente simétricos em teoria de cordas efectiva a baixas energias, com o objectivo de testar a conjectura da censura cósmica fraca.

Através do formalismo de Darmois-Israel, obtemos condições de junção que descrevem a união de dois espaços-tempo nesta teoria por camadas finas do género tempo. As condições de junção, juntamente com as condições de energia fraca e dominante, são usadas para estudar as posições permitidas de camadas finas estáticas cujo conteúdo de matéria é um fluído perfeito. Além disso, estudamos o colapso gravitacional de camadas finas feitas de pó, cujos espaços-tempo interior e exterior são ambos descritos por soluções de Einstein-Maxwell-dilaton estáticas com igual carga dilatónica, que é necessário para que dinâmica exista. Para este colapso, mostra-se que a condição de energia fraca impõe que a censura cósmica seja sempre satisfeita e que a camada ou ressalta ou colapsa para um buraco negro.

Finalmente, investigamos o colapso de camadas finas que unem um exterior de Einstein-Maxwel-dilaton que varia com o tempo, e radia sob a forma de um fluído nulo, com um interior estático. Neste caso, se as condições de energia para o fluído nulo são satisfeitas, a camada fina ou colapsa para um horizonte de eventos futuro, ou ressalta para infinito ou então a condição de energia fraca da camada fina é violada antes de colapsar para uma singularidade despida, preservando assim a conjectura da censura cósmica.

Palavras Chave: Censura Cósmica Fraca; Teoria Einstein-Maxwell-dilaton; Camada fina; Colapso gravitacional; Buracos negros.

Abstract

The thin shell collapse in Einstein-Maxwell-dilaton is studied by resorting to solutions which represent four-dimensional, spherically symmetric black holes (or naked singularities) in low energy effective string theory, with the aim of testing the weak cosmic censorship conjecture.

Through the Darmois-Israel formalism, we obtain the junction conditions that describe the matching of two spacetimes in this theory through timelike thin shells. The junction conditions, together with the weak and dominant energy conditions, are used to study the allowed positions of static thin shells whose matter content is a perfect fluid. Moreover, we study the collapse of thin shells made of dust, whose interior and exterior spacetimes are both described by static Einstein-Maxwell-dilaton solutions with equal dilaton charge, which is required for motion to be allowed. For this collapse, it is shown that the weak energy condition imposes that cosmic censorship is always satisfied and the shell either bounces or collapses into a black hole.

Finally, we investigate the collapse of thin shells joining a time-dependent Einstein-Maxwell-dilaton exterior, that is radiating in the form of a null fluid, with a static interior. In this case, it is shown that if the energy conditions for the null fluid stress-energy tensor are satisfied, then the thin shell either collapses into a future event horizon, bounces back to infinity or otherwise the weak energy condition of the thin shell is violated before collapsing into a naked singularity, once more upholding the cosmic censorship conjecture.

Keywords: Weak cosmic censorship; Einstein-Maxwell-dilaton theory; Thin shell; Gravitational collapse; Black holes.

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1 Introduction

1.1 Motivation

Cosmic censorship was conjectured by Penrose [1] almost half a century ago but it is still a subject of current debate and is considered one of the most important open problems in general relativity (GR). It posits, in its weak version, that any curvature singularities that might form during the evolution of generic regular initial data, with physically reasonable matter, always appear hidden behind event horizons, and thus are inside black holes. The cosmic censorship conjecture (CCC) is a cornerstone of classical general relativity. GR is expected to breakdown near curvature singularities (where quantum physics kicks in). In the absence of horizons - which essentially act as one way membranes, allowing signals to propagate inside but not outside - enclosing such regions, a distant observer would be able to see the quantum nature of the high curvature region. This would configure a breakdown of predictability within GR. Hence, CCC provides a safeguard to GR, guaranteeing its self-consistency as a classical theory.

Since its genesis, CCC has been continually tested. Some of such tests consist of very simple models such as the gravitational collapse of fluids and destruction of horizons through the use of test particles. One interesting setting to use these tests is the spherical gravitational collapse of electrically charged thin shells, that while not as realistic as rotating models which are known to be non-spherical, it provides relatively simple exact solutions of the field equations. If the collapse were to proceed until formation of a curvature singularity and the charge of the shell were sufficiently large (compared to its mass), the end state would feature a naked singularity, that is, a singularity that isn't covered by an event horizon. This would constitute a violation of cosmic censorship, but it is well known in the literature (e.g. Refs [2–4]) that such collapses never generate naked singularities when one considers the Einstein-Maxwell theory which describes electromagnetism coupled to gravity. In that theory, shells with sufficiently large charges to produce an overcharged configuration either violate basic (physically motivated) energy conditions or simply do not collapse. Instead, they bounce back at some finite radius. Similar studies for modified theories of gravity are mostly unexplored, although some examples do exist, for example Refs. [5–11].

In this work we will be interested in testing the possibility of formation of naked singularities in a four-dimensional, low energy string theory for which static spherically symmetric black hole solutions are known (e.g. [12–14]). The theory is based on the addition of a scalar field (the dilaton), that appears frequently in string theory, to the metric and Maxwell field. This field may reduce the critical value for the electric charge necessary for the appearance of a naked singularity, due to the disappearance of the event horizon of the black hole, when compared to the value obtained for Maxwell-Einstein theory. The comparison between these two theories will allow us to understand what are the aspects that low-energy string theory adds to the already known description of charged black holes. Moreover, the weak cosmic censorship conjecture and, by implication, the viability of such string-inspired models, will then be investigated in this Einstein-Maxwell-dilaton theory by studying the solutions describing the spherical collapse of charged thin shells. While this is a very simple model, its usefulness lies in the fact that it allows us to obtain exact analytical solutions. This provides us the means to infer, without the need of a perturbative approach, what are the dynamics and end results of this type of collapse.

1.2 Outline

Our investigation will focus on the study of the gravitational collapse of thin shells in this Einstein-Maxwell-dilaton theory in 4 dimensions and will be based on the solution found by Gibbons and Maeda [13, 14] and also independently found by Garfinkle, Horowitz and Strominger [12]. Our method will revolve on determining the conditions that are necessary to match two such solutions through the use of a thin shell of matter separating them. With these conditions determined, we wish to solve analytically the dynamics of the collapse of these thin shells to obtain the regions where they may exist. Moreover, we also wish to study if it is possible to form naked singularities with matter shells obeying standard energy conditions. I.e. if in this type of collapse cosmic censorship holds. Our interest is intrinsically connected to the several particularities that this solution offers, such as a lower extremization value (which should enhance the possibility of appearance of naked singularities), the behavior around the extremal horizon and of course as a way to infer the viability of string theory in these applications. Finally it will also serve as a way of comparing what will be obtained with the known results of Einstein-Maxwell present for example in [2–4].

This thesis is organised as follows. In chapter 2 we start by introducing the work that has been done so far in the main subjects that we aim to tackle in this thesis. The chapter will therefore focus on describing some of the most well known results concerning gravitational collapse, black hole solutions, CCC, tests to the CCC through the use of thin matter shells and charged particles, black holes in string theory and radiating black hole solutions. We then move on to chapter 3 where the solution obtained by Gibbons and Maeda [13, 14] and also independently by Garfinkle, Horowitz and Strominger [12] shall be reviewed. To proceed with the study of thin shell collapse, in chapter 4 we review the Darmois-Israel junction conditions and also determine new junction conditions that are related to the Maxwell and dilaton field in the theory we aim to study. These will constitute novel conditions that represent original work and consequently cannot be found in the literature. All of the junction conditions obtained will allow us to determine, in the following chapters, the motion and behavior of thin shells. In chapter 5 we resort to the junction conditions to study static shells in this theory and determine the regions where their energy conditions are satisfied and thus where they are physically allowed. Moving on, in chapter 6 the gravitational collapse of dynamical thin shells is studied, with the purpose of assessing what are the allowed regions of collapse of the parameter space and if it is possible to overcharge such shells and thus violate the CCC. Finally, in chapter 7 we obtain a novel time-dependent radiating solution. With it, we apply the previously determined junction conditions to investigate the gravitational collapse of thin shells with these exterior solutions to understand if with these solutions it is possible to violate the CCC. We end this thesis by describing in chapter 8 the main conclusions that were found in the work done and also point out some relevant work that should be pursued to have a more global overview of the CCC in low energy, four-dimensional effective string theory.

State of the Art 2

2.1 Gravitational Collapse

Studies of gravitational collapse consist in determining what is the final fate of a system constituted by matter[1], which due to the force of gravity tends to be drawn towards its own center. Several forces may act to counterbalance this collapse, such as internal pressure, electrostatic forces, centrifugal barriers, etc. Considering our physical reality, the gravitational collapse of stars near the end of their life cycle is an important open problem in astrophysics and cosmology [15].

For our work we are interested in reviewing some of the general results that have been found in the investigation of the dynamics of systems going through gravitational collapse to better understand what kind of solutions may appear and through them the validity of the theory and model being used. A seminal study was carried out by Oppenheimer and Snyder in 1939 [16]. By assuming an initial sphere of dust of uniform density they were able to prove that as the collapse progresses an event horizon starts to develop and after the radius of the sphere becomes smaller than this horizon the sphere will collapse to a spacetime singularity. It is important to note that the formation of the horizon earlier than the singularity is a key component to the formation of a black hole. Even though this mechanism represents a highly simplified case it became the basis of the general mechanism that is used to describe the formation of black holes. Nonetheless, if one takes into consideration inhomogeneities in the initial density profile of the sphere, the behavior of the horizon can vary drastically and the final outcome of the gravitational collapse might in some cases become a naked singularity [15]. The appearance of naked singularities from gravitational collapse was also numerically studied by Choptuik [17], where he demonstrated that for the gravitational collapse of a massless scalar field, in some fine-tuned situations, such singularities arise.

While singularities inside black holes by themselves are not problematic as they are kept away from outside observers due to the event horizon, the same cannot be said about naked singularities. The existence of this type of singularities together with the incompleteness of geodesics due to it, makes it impossible to predict entire evolutions of spacetime. Furthermore, near those singularities it would be expected for the physical behavior to be described by a quantum theory. The cosmic censorship conjecture was thus created to address these issues and exclude conditions that may lead to the appearance of naked singularites that arise from physically acceptable gravitational collapse models.

[1] Vacuum gravitational collapse is also possible, though not in spherical symmetry.

2.2 Black Holes

In 1916, two months after Einstein first published the Einstein field equations [18], the first non-trivial exact solution was discovered for the vacuum situation, the Schwarzschild solution [19]. This solution describes a spherically symmetric gravitational field in vacuum generated by a point mass at its origin, and Birkhoff's theorem guarantees the uniqueness of this solution. Moreover, the spacetime (outside the event horizon) must be static and as an added condition is asymptotically flat. Among other things this implies that although the spherically symmetric matter sourcing the Schwarzschild spacetime may pulsate it doesn't propagate any disturbances and as such this type of object cannot emit gravitational waves. At a certain radius there exists a null hypersurface called the event horizon beyond which the events occurring in its interior cannot influence an exterior observer. Thus, these objects were dubbed black holes for not even light can escape from inside of the horizon. In its interior lies a spacelike singularity meaning that it is inescapable for any observer that crosses the event horizon. Furthermore, due to the ever increasing tidal forces acting on the observer while nearing the singularity, the observer would be teared apart due to an ever increasing stretching in the radial direction and compression in the other directions.

A similar solution, also describing a spherically symmetric gravitational field can be obtained for a point mass with charge, and is similarly guaranteed by Birkhoff's theorem to be static and unique. This solution can be obtained through the Einstein-Maxwell field equations and is called the Reissner-Nordström solution [20, 21]. It is also asymptotically flat, has an intrinsic singularity at the origin and for the limit of vanishing charge becomes the Schwarzschild solution, but it has two null hypersurfaces meaning that it presents two horizons. More importantly, these two horizons merge into one for the case where the charge of the black hole equals its mass. In this situation we have what is called an extremal black hole. If the charge were to be bigger than the mass, then both of the horizons would disappear and a naked singularity would form. Several tests which will be analyzed in the following sections have been conducted and have shown that for this particular theory it is not possible for a naked singularity to form as the result of gravitational collapse with thin shells and non-exotic matter (i.e. matter satisfying standard energy conditions).

The two horizons present in this solution, which divide the spacetime into three different regions, are the event horizon, corresponding to the exterior horizon, and the Cauchy horizon. In the region exterior to the event horizon the time coordinate is timelike and the radial coordinate is spacelike; this is also the case for the innermost region (the region inside the Cauchy horizon). Nonetheless, in the region between both horizons the character of these coordinates is inverted. What this tells us is that the singularity is timelike as it is escapable. Furthermore an observer cannot remain in a static orbit between the two horizons as he always falls into the inner horizon. Inside this horizon, at a certain distance from the singularity a chargeless particle will be repelled due to a gravitational barrier imposed by the relation between the black hole's mass and charge [22], and so the particle never falls into the singularity.

Finally, another interesting aspect of this solution appears when considering its maximal analytic extension. An observer that is at the innermost region can cross the inner horizon, which is actually a distinct copy of the one he would have crossed before, and will find himself in a new region of spacelike time. As such he will fall through an outer horizon and end up in a new exterior asymptotically flat region from which he can once again reenter the black hole, giving rise to an “infinite lattice of asymptotically flat universes connected by black-hole tunnels” [22]. This can be observed in the Penrose diagrams of Fig. 3.1, the subextremal and extremal RN solution are represented in the lower row.

It is clear that none of these two solutions describes physically realistic black holes. In particular, a strong backreaction phenomenon known as mass inflation [23] effectively cuts off spacetime at the inner Cauchy horizon, thus precluding the existence of the above-mentioned peculiar many-world trajectories. Nonetheless they continue to be important as tools to study and predict possible more complicated phenomena at least as a first approximation, due to how they may resemble other solutions albeit in a simpler way.

The most general solution of the Einstein-Maxwell theory in 4 dimensions is the Kerr-Newman [24] solution which describes the spacetime created by a rotating and charged black hole. This solution is stationary, axisymmetric and asymptotically flat but is no longer spherically symmetric. For the case without angular momentum it becomes Reissner-Nordström, it has two horizons that merge into one for an extreme condition such as what happened in Reissner-Nordström and it has an intrinsic timelike singularity inside the innermost horizon which in this case corresponds to a ring singularity whose radius is given by its angular momentum. Furthermore, it has a region exterior to the event horizon, called the ergosphere: inside this region all particles must be orbiting in the spinning direction of the black hole. On the outer boundary of this region – the ergosurface – only a light may maintain a stationary orbit as viewed by an observer at infinity. We won't go into much more detail in this solution. It will suffice to say, the non-spherical symmetry makes it much more difficult to study, and considering the solution has several properties in common with Reissner-Nordström, we often use this simpler solution to study specific aspects of Kerr-Newman in a simplified, spherically symmetric context.

2.3 Weak Cosmic Censorship

The weak cosmic censorship conjecture (WCC), first proposed by Penrose in 1969 [1] posits that any curvature singularities that might form during the evolution of generic regular initial data, with physically reasonable matter, will always be hidden inside a black hole's event horizon. There is another conjecture which is closely related to this one, although they are mathematically independent — the strong cosmic censorship conjecture. It posits that, generically, timelike singularities never occcur, this means that not even an observer falling into a black hole would observe them. In the work present in this thesis we will nonetheless focus exclusively on the WCC.

As it has already been mentioned it is necessary for GR to guarantee its self-consistency as a classical theory. That being said, no general proof of the conjecture has ever been found and it remains one of the most important problems in GR. Part of the problem inherent to this conjecture is that at the moment we still have no exact definitions of what could constitute “generic” regular data nor what is “physically reasonable” matter. An example of an attempt at a formulation may be found in [25], which defines generic regular initial data as a hypersurface whose induced space metric and extrinsic curvature are asymptotically flat and the word "generic" is used to rule out examples of collapse into naked singularities that due to their specificness or fine-tunning are physically impossible to achieve, such as Choptuik's results [17]. For the "physically reasonable" matter, one is to interpret that this is referring to matter that is coupled to the Einstein field equations. The stress-energy tensor of this "physically reasonable" matter should satisfy suitable energy conditions and should include the Maxwell and Klein-Gordon fields. It should also exclude any type of situation in which singularities appear in flat spacetime evolutions without gravitational collapse, as these type of singularities are caused by the properties of the matter being studied and not due to gravity. For example, shock singularities which occur in perfect fluids. We arrive at the conclusion that while these definitions are not entirely precise, they allow us to infer whether some solutions that violate the conjecture may be excluded based on their assumptions.

For our work we are interested in studying the possible formation of naked singularities and in testing the weak cosmic censorship conjecture. Several tests of this hypothesis exist. For our investigation the most relevant ones can be encountered in 2.4. Due to also corresponding to tests involving the gravitational collapse of thin shells, they will be used to draw a parallelism between the results we will obtain and those already described in the literature. In section 2.5 we will also present some studies that have been made through the use of test particles to perturb extremal and near extremal black holes. To conclude, it is worth stating that due to its implications on our knowledge of the universe, cosmic censorship is a remarkable open problem in general relativity that is far from being settled.

2.4 Studies of Collapsing Thin Shells

To properly study the gravitational collapse of thin shells, the formalism of hpersurfaces needs to be introduced. For our work we will be interested in the specific case of timelike hypersurfaces and as such we will start by explaining the steps necessary to properly approach these physical objects, mostly by following [22].

An hypersurface Σ may be defined, in a 4-dimensional spacetime manifold, as a 3-dimensional submanifold that is obtained either by restricting the coordinates of the spacetime manifold using a constraint equation, or, equivalently, by parameterizing these same coordinates in terms of induced coordinates on the hypersurface. From these two basic definitions we can obtain the induced metric in Σ. Starting from the general coordinates, we are interested in defining vectors that are tangent to curves that are contained in Σ. These vectors are obtained from the derivative of the spacetime coordinates in terms of the hypersurface ones and can be used to compute the tangent components of tensors along a hypersurface. By computing the displacement along a hypersurface, it is simple to obtain the induced metric as it is the projection of the spacetime metric on Σ. The normal to the hypersurface can also be easily defined by computing the gradient of the constraint equation above mentioned, properly normalized for a timelike hypersurface.

To properly study the relations between the spacetime and an hypersurface we also need to define its extrinsic curvature which is a symmetric tensor that describes how the hypersurface is embedded in the spacetime enveloping it. The trace of the extrinsic curvature is relevant in variational studies in spacetimes with boundaries. In those approaches it corresponds to the Gibbons-Hawking-York boundary term of the action [26, 27]. Considering a congruence of geodesics that are orthogonally intersecting the hypersurface, its type of curvature depends on the sign of this trace: a positive trace corresponds to a diverging congruence which results in a convex hypersurface while a negative trace will result in a concave hypersurface.

We have defined the tools necessary to study the intrinsic and extrinsic aspects of a hypersurface embedded in spacetime. Equipped with these devices we will focus on the study of situations where a hypersurface divides spacetime into two regions $\mathcal{V}^+$ and $\mathcal{V}^-$, each with its own set of metric and coordinates. We are interested in knowing what are the conditions that allow both of the regions to be joined smoothly at Σ. As we will see this leads to obtaining finite difference equations that will be dependent only on the induced metric and extrinsic curvature. For what we are studying, our interest is in the case where the hypersurface corresponds to a thin shell. In this particular case the equations become the junction conditions which were obtained from work done by Darmois [28] and which was further developed by Israel [29]. They are two conditions, the first one tells us that there should be no jump in the induced metric between the two different regions and arises in the requisite of a well defined induced geometry on the hypersurface. The second junction condition, also known as Lanczos equation, shows that if there is a jump in the extrinsic curvature between the two regions this corresponds to the existence on the hypersurface of a singularity of spacetime due to the presence of a delta-function part of the stress-energy tensor along the hypersurface. This delta-function is just the surface matter distribution on Σ. Finally, it can be shown that this jump referent to the second junction condition implies that there is a discontinuity of the first derivative of the spacetime metric by the hypersurface as shown in [22].

Having stated the Darmois-Israel junction conditions we are now prepared to review the subject of collapsing thin shells. Our interest will be focused on several recent results that have been studied in general relativity and some alternative theories which have analyzed the possibility of formation of naked singularities arising from these collapses. Therefore, they are of interest due to their relation to the cosmic censorship conjecture.

When studying the Einstein-Maxwell equations, one finds as a static solution the Reissner-Nordström metric which shows that in the case of overextremal black holes, i.e. black holes whose charge is larger than its mass, a naked singularity arises. This leads to the question of whether the gravitational collapse of a thin shell enveloped in this metric would allow the formation of such a black hole. This problem was studied by Boulware in 1973 [2] and revisited by Hubeny in 1998 [3]. More specifically, the Darmois-Israel formalism was used to obtain the time development of a thin shell whose exterior was defined by the Reissner-Nordström metric and the interior by the Minkowski metric, i.e. an empty interior. With this initial setup in mind — and to determine the outcome of the thought experiement — Boulware used the fact that the normal on the space outside of the shell has to be positive for such a shell because it must be a spacelike vector and can only change signs inside an event horizon. From this he showed that the only way for the normal to remain positive and the shell to collapse to a naked singularity is if the shell's proper mass is negative meaning that it would have to have negative energy density on its surface. Some other studies that have been done in physically similar conditions, also pertaining to charged fluid shells in 4-dimensional spacetimes, can be found in [29–31]. They show the behavior of such shells in different situations related to their motion and stability but nonetheless don't tackle the subject of cosmic censorship.

More recently, Gao and Lemos [4] generalized the results obtained by Boulware [2] to a spacetime in d dimensions, where they studied the dynamics of such a collapse in the d-dimensional equivalent of the Reissner-Nordström black hole. The interest of such a study lies particularly on the corroboration of CCC in higher dimensions and also in the use of the results as a first approximation to scenarios of collapse in models that have extra large dimensions beyond the usual 4. By once more analyzing the relations between the normals to the shell and its proper mass, it was deduced that for a positive proper mass it is not possible for the collapse of the thin shell to either create by itself or overcharge a preexistent black hole. From these results, it may be concluded that spherical gravitational collapse in d dimensions doesn't seem to present qualitative differences from the usual 4 dimensions.

The possibility of formation of naked singularities is not restricted to charged black-holes, another example of solutions that contain this particular behavior are the ones involving rotating black holes. As such, when breaching the subject of testing the cosmic censorship conjecture through the gravitational collapse of thin shells one cannot ignore spacetimes containing rotation. Moreover, while it isn't expectable for charged black holes to represent physical scenarios in our universe, the same cannot be said about rotating black holes. Note that the usual reliance on charged black holes instead of rotating ones is mostly due to the fact that rotation adds technical complications. As a consequence, most of the studies for this type of spacetimes have been done in $2+1$ dimensions but some investigations in higher dimensions also exist. We now briefly review them.

Starting off by the work of Peleg and Steif in 1995 [32], they showed that when considering gluing two static spacetimes along a timelike hypersurface in $d = 3$ with or without a cosmological constant it is possible under certain conditions for naked singularities to form. For the case with rotation in $2+1$ dimensions, we have the work done by Crisostomo and Olea [33] where they studied the thin shells with a Hamiltonian treatment instead of the junction conditions,+ and also the work done by Mann, Oh and Park [34]. In [33], through an Hamiltonian treatment they were able to prove that it would be impossible to form naked singularities through the collapse of a thin shell due to violation of energy conservation and the fact that the shell would bounce beforehand. Also, in the case of a shell collapsing with an empty interior it cannot collapse into a naked singularity due to requirements related to the absence of an horizon and due to the explicit form of the metric that prevent the shell from reaching the origin. In [34] it is proven that for several situations involving $2+1$ rotating thin shells, the formation of naked singularities is possible as long as the shells have pressure, and a centrifugal barrier may in some cases be responsible for preventing such an occurrence.For rotating black holes in higher dimensions, exact solutions have been obtained quite recently for the case of $d = 5$ and with a cosmological constant in [35]. Using the Myers-Perry solution [36] to describe the spacetime both inside and outside a thin shell described as an imperfect fluid, assuming that the two (in general, independent) angular momenta are equal for each of the two regions, they were able to obtain exact solutions of rotating thin shells from rest at infinity collapsing onto rotating black holes, which had previously only been done for 3 spacetimes dimensions. None of these solutions obtained violates cosmic censorship for in this case, due to the first junction condition, subextremal interior solutions force the global solution, after the gravitational collapse of the shell, to also be subextremal.

To conclude this section, it's important to retain that the gravitational collapse of thin shells allows us to study cosmic censorship through exact solutions without the need of any sort of perturbative analysis. The tests that have been conducted in Einstein-Maxwell theory show in a clear manner that, disregarding negative energy density solutions, cosmic censorship is upheld in the gravitational collapse of thin shells.

2.5 Overextremizing Black Holes with Test Particles

Although there has been a lot of work developed concerning the gravitational collapse of thin shells which analyzed the possible formation of naked singularities either by overextremizing a preexisting black hole or by generating one due to collapse, this is not the only method that has been used to study this subject. One can also discuss such results when testing the possibility of obtaining superextremal black holes by perturbing them. This is done through the use of test particles falling into either extremal or subextremal black holes and seeing the effect they produce on them. In this section we intend to give a brief overview of some of the results that have been obtained using this method.

The earliest test to the possibility of overextremizing black holes was conducted by Wald in 1974 [37]. There, he attempted, in one case to overextremize a Kerr-Newman black hole either by capturing a particle of high charge-to-mass ratio or high angular momentum, and in the other case drop a high spin to mass ratio test body into a chargeless extremized black hole along its symmetry axis. For the first case, it was concluded that the energy conditions would always be violated for a naked singularity to appear. Such test particles wouldn't be captured by the black hole and instead repelled either due to the electromagnetic and centrifugal repulsions. In the second case, due to the spin-spin interactions between the two bodies, a repulsion appears that once more avoids the formation of a naked singularity.

While the result for extremal black holes seems to be more or less settled, the same cannot be said for near extremal black holes. As an example, Hubeny [3] showed that for Reissner-Nordström black holes, there are several configurations in which charged test particles can enter this near extremal black hole and overcharge it. However, this effect was only obtained while neglecting backreaction due to the particle. Even so, a proof was indeed given that showed that these effects may be arbitrarily small depending on the configurations, but may still not suffice for it to be ignored. A similar approach was investigated by Jacobson and Sotiriou [38], in this case trying to overspin a near extremal Kerr black hole with a high angular momentum test body. There, they were able to deduce that for those conditions, if neglecting backreaction and radiative effects, it would be possible to overspin the black hole and obtain a naked singularity. This possible violation of the cosmic censorship conjecture is further investigated in [39], where it was shown that although some of the orbits studied in [38] indeed had negligible dissipative radiation effects, all of those orbits had a conservative self-force that affects the orbit of the test particles in a non-negligible order. Even though that force cannot be calculated yet, it appears to present the necessary sign so that it prevents the appearance of naked singularities in these conditions.

For black holes in higher dimensions, we also have the studies conducted in [40], which focused on generalizing the thought experiment approached by Wald to the case of d-dimensional Myers-Perry black holes [36] and neutral [41] and dipole [42] black rings in asymptotically flat spacetimes. The focus of this paper lies on the possibility of test particles with angular momentum falling along the equatorial plane being able to overspin such black holes. They prove it to not be possible, at least if starting with extremal black holes. For Myers-Perry black holes in asymptotically AdS spacetimes a numerical study was done by Rocha and Santarelli in [43], that consisted in the use of massless particles moving in null geodesics preserving the equal angular momenta character of the background. They once more concluded that although the black hole would increase its spin it would still at most be extremal due to also becoming more massive.

Many other studies may also be found in this area related to several different circumstances such as quantum tunneling, presence of a cosmological constant and using different black hole solutions, that specify situations in which the cosmic censorship conjecture is violated. Nonetheless, it is important to notice that backreaction and radiation effects, which are usually ignored in those studies, are expected to be what maintains the validity of the conjecture.

2.6 Black Holes in String Theory

General relativity in higher dimensions than the usual four has been a subject of increasing interest in recent times. It sets the ground for the existence of new physical entities and effects such as [44] the appearance of extended black objects like for example black strings and black branes, added independent rotation planes to solutions, weakening of gravity due to it being diluted through more dimensions, etc. Associated to these added possibilities is the study of alternative theories to general relativity that may appear due to, for example, the fact that GR is non-renormalizable (string theory) and due to the failure of the standard cosmology model to convincingly explain several observations (e.g., the current accelerated expansion of the universe). This presents a need to depart from classical GR and several alternative theories have been proposed. Examples are theories with additional fields (scalar, vector and tensor) coupled to gravity, theories with higher order terms of the Riemann tensor and higher order derivatives of the field equations, etc. It is clear that all of these possibilities create a vast field of research. In our case we will be interested in studying the topic of black holes in string theory. More specifically, we are interested in studying a string theory inspired model that is called the Einstein-Maxwell-dilaton theory and is a generalization of Einstein-Maxwell for the case where there exists a coupling between the electromagnetic tensor and a scalar field (the dilaton). In general, these type of models of gravity coupled with scalar fields and Maxwell terms are among the simplest extensions of GR and arise naturally in supergravity theories that represent low energy effective descriptions of string theories [45–48].

It is known [49, 50] that low-energy string theory and general relativity have agreeable results in their description of static uncharged black holes as long as the mass parameter is large when compared to the Planck mass and we are in regions far from the singularity. In contrast, near that region Planck-scale correction terms arises in string theory [12] and as such Einstein's equation stops being a good approximation for the string theory equation. When one introduces charge into the problem the same cannot be said [12], due to the presence of the dilaton in string theory and the fact that it is coupled with the electromagnetic tensor. Consequently, the existence of charge stipulates that for a non-vanishing electromagnetic tensor solution, there must exist a nonconstant dilaton. This obviously produces a severance between solutions in this theory and the Reissner-Nordström solution. Furthermore, as shown in [14], this dilaton coupling may also change the causal structure of the solutions.

For the coupling constant that we will be considering for the most part of the thesis, this theory represents a four-dimensional, low energy effective description of heterotic [$E_8 \times E_8$ or $SO(32)$] string theory. One such solution that may be obtained in this theory, and is of interest to us, is described in [12]. It describes spherically symmetric, static, asymptotically flat charged black hole solutions. This solution can be completely described by the mass and charge of the black hole and by the dilaton charge which depends on the previous two parameters and also on the asymptotically constant value of the field. Moreover, the theory has a duality symmetry which relates electrically charged solutions to magnetically charged solutions and inverts the sign of the dilaton field. This solution only has one (event) horizon at the Schwarzschild radius instead of the two present in Reissner-Nordström. It also has a decreased surface area for spheres of constant radius and time, which at a certain radius becomes singular. Moreover, in the case of an extremal black hole it defines a singularity not covered by an event horizon [12]. Discussions on the effects of the dilaton charge can be found in [51–53]. What is known is that, for this solution, the dilaton field influences the extremal condition, as in some cases it affects the critical value of the charge by reducing it. The conclusion is that for the Einstein-Maxwell-dilaton theory, we have the possibility of studying gravitational collapse originating final geometries whose extremal condition can significantly depart from the one found in Einstein-Maxwell theory.

2.7 Radiating Solutions

So far we have been focusing on describing stationary spacetimes, not delving into time-dependent solutions of Einstein's equations. The presence of time-dependence increases the complexity of the problem and consequently the difficulty of the study of these spacetimes also rises. Nonetheless, simple solutions have been found to exist. One such solution, determined by Vaidya in 1951 [54, 55] presents itself as an extension of the Schwarzschild solution expressed in terms of a null coordinate and which has the particularity of possessing a time-dependent mass parameter (unlike the constant mass present in the Schwarzschild solution). It may represent a radiating or absorbing spacetime, depending on the null coordinate being used and of course on the defined direction of the timelike coordinate. While the Schwarzschild solution represents a gravitational field in vacuum, the Vaidya solution requires the existence of a stream of radial null dust that is either outgoing or ingoing, depending on whether the solution is raditating or absorbing. Moreover, on account of it being a solution of the Einstein equations, this dust that appears in the form of a stress-energy tensor, is dictated by energy-momentum conservation and satisfies standard energy conditions. Therefore, it represents a physically reasonable solution.

The extension of the Vaidya solution (and also of Reissner-Norsdtröm) to Einstein-Maxwell was determined in 1970 [56], the so-called Vaidya-Bonnor solution. It includes varying electrical charges in addition to the varying mass parameter, thus describing the emission (or absortion) of charged null dust. One feature of this solution is the presence of a charged electric current. In this case, for the stress-energy tensor to satisfy the standard energy conditions it is necessary that, in the case of a radiating solution, either the mass of the central body decreases with time or otherwise the body has to ionize and radiate charge of one sign. This could by itself pose a violation of CC but as shown in [57], it is not possible to create an extremal black hole without violating standard energy conditions. Finally, we note that this solution presumes the existence of charged null dust and currently no obvious candidates in nature appear to fill this role.

These spherically symmetric radiating solutions we have mentioned, and others more recently obtained through the same procedure, give rise to the possibility of studying gravitational collapse in models that account for radiating stars. This can be done either by resorting to the use of thin shells as for example it is done in Refs. [58–60], or even without considering thin shells, just by generalizing the Oppenheimer-Snyder study to radiating spacetimes, e.g. [61–64]. Moreover, these solutions can also be used to test CC in different settings of GR and alternative theories, as done for example in Refs. [8, 9, 65, 66]. The possibility of testing the CCC with radiating solutions is of special interest to us as we have also derived a radiating solution in Einstein-Maxwell dilaton [67], and are interested in studying whether the CCC is upheld in thin shell collapses when the exterior spacetime is described by it.

3 Static spherically symmetric black holes in Einstein-Maxwell-dilaton

In String Theory the existence of the dilaton, which couples to the electromagnetic field, implies that when we consider String Theory models then Einstein-Maxwell solutions are not good approximations for the description of charged black holes. A necessity therefore arises to study new solutions that take into account this scalar field. One such solution, that is of high interest due to representing the String Theory analog of the Reissner-Nordström solution when the scalar field is considered, was first studied by Gibbons [13] in 1982 and Gibbons and Maeda (GM) [14] in 1987. Later, Garfinkle, Horowitz and Strominger (GHS) in 1990 [12] independently obtained a new solution that was the same as the one of Gibbons and Maeda only with a rescaling of the metric by a conformal factor. This last solution is the one that our study will mostly focus on. Overall, these solutions enabled us to understand how static, charged black holes behave in four-dimensional low energy string theory.

This chapter will start with a summary of the solution obtained by GMGHS, followed by a short analysis of it with the goal of highlighting its differences with respect to the well known Schwarzschild and Reissner-Nordström solutions in pure GR and Einstein-Maxwell theories, respectively.

As we are mainly interested in studying the electric version of the GMGHS solutions we will not put emphasis on describing their magnetic counterpart, which was the one mainly considered in [12].

3.1 Field equations

The field content of the low energy effective Lagrangian obtained from string theory includes the metric $g_{\mu\nu}$, a Maxwell field A_μ with field strength $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, and a massless scalar field ϕ (the dilation), which is coupled to the field strength as $e^{-2\phi} F_{\mu\nu} F^{\mu\nu}$.

The four-dimensional low-energy Lagrangian (Ref. [12]) that describes this theory is

$$S = \int d^4x \sqrt{-g} \left[-R + 2\nabla_\mu \phi \nabla^\mu \phi + e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right] \,, \tag{3.1}$$

where R corresponds to the Ricci scalar and g is the determinant of the metric. In [12] they also obtained solutions for a more general theory with an arbitrary coupling constant between the dilaton and the Maxwell field. Note however, that for now, we are mainly interested in tackling the particular case in which this coupling constant is unitary. By varying this Lagrangian with respect to A_μ, ϕ and $g_{\mu\nu}$, setting

$G = c = 1$, we obtain the following equations:

$$\nabla_\mu \left(e^{-2\phi} F^{\mu\nu}\right) = 0\,, \tag{3.2}$$

$$\nabla^2\phi + \frac{e^{-2\phi}}{2} F_{\mu\nu}F^{\mu\nu} = 0\,, \tag{3.3}$$

$$R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2\phi}F_{\mu\alpha}F_\nu{}^\alpha - \frac{g_{\mu\nu}}{2}e^{-2\phi}F_{\alpha\beta}F^{\alpha\beta}\,. \tag{3.4}$$

Note that Eq. (3.4) was further modified to be in the same form as presented in Ref. [12]. Nonetheless, it is more convenient to write the last equation as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi\left(T^{(\mathrm{dil})}_{\mu\nu} + T^{(\mathrm{EM})}_{\mu\nu}\right)\,, \tag{3.5}$$

where

$$8\pi T^{(\mathrm{dil})}_{\mu\nu} \equiv 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}(\nabla\phi)^2\,, \qquad 8\pi T^{(\mathrm{EM})}_{\mu\nu} \equiv e^{-2\phi}\left(2F_{\mu\alpha}F^\alpha_\nu - \frac{1}{2}g_{\mu\nu}F^2\right)\,. \tag{3.6}$$

We may draw some conclusions by inspection of these equations individually. Eq. (3.2) provides an already expected result as it corresponds to the natural equivalent of Maxwell's vacuum equations ($\nabla_\mu F^{\mu\nu} = 0$) when we have the added scalar field coupled to the electromagnetic tensor. It is clear as well that Eq. (3.4) also has a direct correspondence with the Einstein-Maxwell case: if we set the scalar field to zero we obtain the usual equation which is $R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi T_{\mu\nu}$ where we define $T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F^\alpha_\nu - \frac{g_{\mu\nu}}{4}F_{\alpha\beta}F^{\alpha\beta}\right)$. Note however that from Eq. (3.3) we conclude that if $\phi = 0$ then $F^{\alpha\beta} = 0$, which is due to the coupling between the dilaton field and the electromagnetic tensor as we will see in chapter 7. Finally, Eq. (3.3) describes how the dilaton field varies due to the presence of an electric charge. Among other aspects, it is noticeable that when the electric charge is null we retrieve Laplace's equation as the coupling of the scalar field with the electromagnetic tensor is irrelevant in this case.

3.2 The GMGHS solution

Solving Eqs. (3.2)-(3.4) by assuming the solution is spherically symmetric and static produces the following metric, given in the form derived by GHS,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + r\,(r - 2D)\,d\Omega^2\,, \tag{3.7}$$

$$F = -\frac{Q}{r^2}dt \wedge dr\,, \tag{3.8}$$

$$e^{2\phi} = e^{2\phi_0}\left(1 - \frac{Q^2e^{-2\phi_0}}{Mr}\right) = e^{2\phi_0}\left(1 - \frac{2D}{r}\right)\,. \tag{3.9}$$

where M is the mass of the spacetime and D is the scalar charge, which is determined by the electric charge Q and the mass M, in addition to the asymptotic value of the scalar field, ϕ_0,

$$D = \frac{Q^2e^{-2\phi_0}}{2M}\,. \tag{3.10}$$

The scalar charge D corresponds to the Noether charge associated with the invariance of the Langragian under shifts in the scalar field, simultaneously with rescalings of the Maxwell field. It is obtained by integrating the gradient of the scalar field over a two sphere at infinity, i.e.,

$$D = \frac{1}{4\pi}\int d^2\Sigma^\mu\nabla_\mu\phi, \tag{3.11}$$

in which we defined

$$d^2\Sigma_\mu = \epsilon_{\mu\nu\lambda}\frac{\partial x^\nu}{\partial x^2}\frac{\partial x^\lambda}{\partial x^3}d^2\Omega, \tag{3.12}$$

where we considered x^2 and x^3 as the angular coordinates and $\epsilon_{\mu\nu\lambda} = \sqrt{-h}\,[\mu\nu\lambda]$ corresponds to the fully antisymmetric Levi-Civita tensor, with h being the determinant of the angular part of the metric.

Taking into consideration Eq. (3.2) we can obtain magnetically charged solutions via Electric-Magnetic duality,

$$\tilde{F}_{\mu\nu} = \frac{1}{2}e^{-2\phi}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}\,. \tag{3.13}$$

Furthermore, noting that Eqs. (3.3) and (3.4) are invariant under $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$ and $\phi \to -\phi$ we conclude that, although magnetically charged solutions require a rescaling of F and ϕ, the metric remains unchanged.

Similar to what happens in the Reissner-Nordström solution, this metric also presents a condition on the parameters so that the solution describes a naked singularity. In this case, one finds that the event horizon disappears when

$$D > M \qquad \Leftrightarrow \qquad Q^2 \geq 2M^2e^{2\phi_0}\,. \tag{3.14}$$

The above condition comes from the fact that for the GMGHS solutions, the surface $r = 2D$, is actually a singular point (where space ends). Nonetheless, the horizon continues to be at $r = 2M$. This implies that the extremality condition occurs when the horizon is located at the same surface as the singularity. Moreover, for the black hole to be overcharged it is necessary for $D > M$, so that the singular surface is outside of the black hole horizon.

In terms of the horizons of the two metrics, for Reissner-Nordström we have two horizons: a Cauchy horizon at $r = M - \sqrt{M^2 - Q^2}$, which is known to be unstable, e.g. [23], and the usual event horizon at $r = M + \sqrt{M^2 - Q^2}$. As both of these expressions evidence, in the extremal case the two horizons become degenerate at $r = M$, and they disappear completely in the overcharged case. For the GMGHS solution only one event horizon appears, at $r = 2M$ in the coordinate system used above[1].

Compared to Reissner-Nordström, in which the overextremal condition is $Q \geq M$, we can observe through Eq. (3.14) that by controlling the asymptotic value of ϕ, i.e. ϕ_0, we may have situations where the extremal case might happen for values of $Q < M$. This is of interest as it might potentially lead to a violation of the CCC.

When compared to the Schwarzschild case, we start by noticing that the metric obtained by GMGHS only differs to the Schwarzschild solution in the $d\Omega^2$ component, which is r^2 instead of $r\,(r - 2D)$. This tells us that the causal structure of the spacetime described by the GMGHS solution is quite similar to the Schwarzschild one. Note that this is only valid as long as we don't consider an extremal or overcharged black hole which has no Schwarzschild analog. In the non-extremal case the singularity is spacelike in contrast to Reissner-Norsdtröm's timelike singularity. However, for an extremal or overcharged black hole in the GMGHS framework we retrieve this timelike singularity, and the spacetime becomes similar to that of an overcharged Reissner-Norsdtröm black hole.

[1]In Ref. [12] it was also shown that as long as a coupling exists between the dilaton and the electromagnetic field the inner horizon will always be singular.

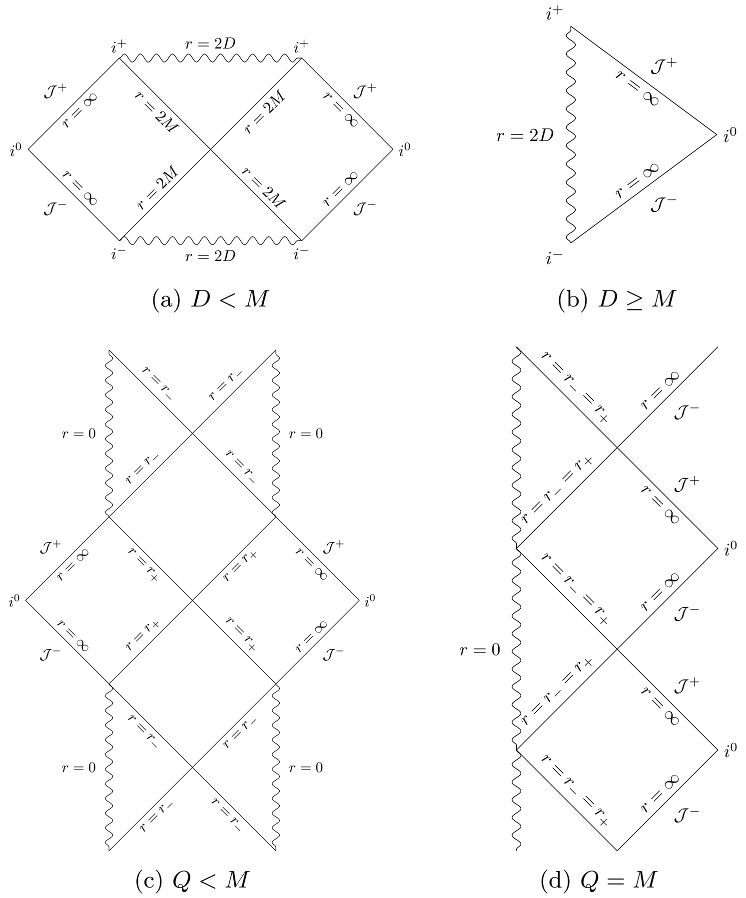


Figure 3.1: The subextremal GMGHS solution is represented at the top left. In that diagram, by making D go to zero we obtain the Penrose diagram of the Schwarzschild solution. To its right we have the diagram for an overcharged GMGHS solution, which, when one sets D to zero is the same as the overcharged Reissner-Nordström solution. At the bottom row we have on the left the subextremal Reissner-Nordström solution while to its right we have the extremal case. In all the diagrams we have i^o representing spacelike infinity, i^- and i^+ represent respectively past and future timelike infinity and $\mathcal{J}^-$ and $\mathcal{J}^+$ denote respectively past and future null infinity.

A visual representation of the differences between the causal structures of the GMGHS, Schwarzschild and Reissner-Nordström solutions can be observed in Fig. 3.1 which is an adapation of a figure from Ref. [14]. It represents the Penrose diagrams of the enumerated solutions for subextremal, extremal and overextremal values of their charges.

4 Junction conditions

Having already reviewed the metric that will be considered we are now interested in gluing two such spacetimes along a hypersurface by applying the junction conditions obtained by Darmois [28] and further developed by Israel [29]. Through this procedure we intend to determine the potential that governs the radial dynamics of a thin shell located at a timelike hypersurface separating the two different spacetimes.

We start this chapter by reviewing, in section 4.1 the procedure described in Ref. [22] to obtain the equations of the Darmois-Israel formalism. Also following the procedure done in Ref. [22] we set out to obtain, in section 4.2, the equation that determines the conservation of the surface stress-energy tensor of a shell, which is also presented for example in Refs. [22, 68, 69].

Afterwards, in section 4.3 we follow a similar procedure to the one we review to obtain the junction conditions that define the behavior of the fields ϕ and A_μ in order to determine how their transition between the two different spacetimes delimited by the thin shell occurs. Note that these conditions are in general theory-dependent and in this context the results we obtain are, to the best of our knowledge, novel, i.e. they are not described in the literature.

After all the junction conditions are determined, our focus relies on applying them to the solution we are studying, firstly by considering the solution in the metric derived by GMGHS and investigating the usefulness of using the system of coordinates used by the authors; this is done in section 4.4. We note afterwards that it is more convenient to introduce a different coordinate system in section 4.5.

Finally, in section 4.6 we determine the analytical expression of the energy density of the shell by solving the equations that we previously obtained through the Darmois-Israel formalism.

4.1 Thin Shell formalism

We start by considering a d dimensional spacetime which is partioned in two regions $\mathcal{V}^+$ and $\mathcal{V}^-$ by a hypersurface which we name Σ. For simplicity, we will assume that Σ is either spacelike or timelike. Each region, is defined by their coordinate patch $x^\alpha_\pm$ and metric $g^\pm_{\alpha\beta}$ where the suffix $+$ corresponds to the region $\mathcal{V}^+$ whereas $-$ corresponds to $\mathcal{V}^-$. With this setup defined we now wish to determine the conditions that allow for a smooth junction of both regions at Σ.

We shall start by assuming that the hypersurface is parametrized by the coordinate system y^a on both sides of the hypersurface[1]. Moreover, we will also consider a system of local coordinates x^α on the neighborhood of Σ which evidently overlaps with both $x^\alpha_\pm$ on an open region of $\mathcal{V}^\pm$. The usefulness of this coordinate system relies on the fact that it can be used to describe both sides of Σ, thus, it simplifies our computations. Taking this information into account, the hypersurface may be parametrized by

$$x^\alpha = x^\alpha(y^a)\,. \tag{4.1}$$

We may then differentiate this equation to obtain the tangent vectors on Σ

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}\,, \tag{4.2}$$

which will enable us to project tensorial quantities on the hypersurface. While this last equation provides us $d-1$ vectors that are tangent to curves of constant y^a, we still require a description of geodesics leaving Σ. To do so, we consider a congruence of geodesics orthogonally intersecting Σ. This congruence is parametrized by a parameter l which for timelike shells corresponds to the proper distance along the geodesics while for a spacelike shell it would instead be interpreted as a proper time. By convention we shall consider l to be positive in $\mathcal{V}^+$, negative in $\mathcal{V}^-$ and zero on Σ. This construction defines a normal vector n^α, perpendicular to Eq. (4.2). This vector field determines the displacement away from the hypersurface along a geodesic as $dx^\alpha = n^\alpha dl$ and also

$$n_\alpha = \epsilon \frac{\partial l}{\partial x^\alpha}\,, \tag{4.3}$$

where $n^\alpha n_\alpha = \epsilon = \pm 1$ depending on whether the shell is timelike ($\epsilon = 1$) or spacelike ($\epsilon = -1$).

Most of what will be studied in this thesis involves comparing the jump of different quantities across Σ. Thus, it is useful to use the following compact notation to represent this jump

$$[A] \equiv A(\mathcal{V}^+)|_\Sigma - A(\mathcal{V}^-)|_\Sigma\,, \tag{4.4}$$

where A is the quantity we want to study. Since by definition x^α, y^a and l are continuous across Σ we have

$$[n^\alpha] = [e^\alpha_a] = 0\,. \tag{4.5}$$

Finally, we will also be interested in determining the metric intrinsic to the hypersurface, which ideally should not depend on the metric of the rest of the spacetime. Its definition arises naturally from the concept of displacement inside the hypersurface as

$$ds^2_\Sigma = g_{\alpha\beta} x^\alpha x^\beta = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b} dy^b\right) = h_{ab} dy^a dy^b\,, \tag{4.6}$$

where h_{ab} is the induced metric, given explicitly by

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b\,. \tag{4.7}$$

It is easy to see that this metric is invariant under a $x^\alpha \to x^{\alpha'}$ transformation and it behaves like a tensor for $y^a \to y^{a'}$ transformations, which is due to the fact that it is a $(d-1)$-tensor. This means that it allows a description of the hypersurface which is coordinate independent. Moreover, it will be quite useful for raising and lowering indices of $(d-1)$-tensors on the hypersurface.

[1]From here on we shall use greek indices when considering the d dimensional spacetime whereas latin indices will refer to the $d-1$ dimensional hypersurface.

4.1.1 First Junction Condition

We will use a distributional approach to aid us in determining quantities across the hypersurface. To do so, we start by recalling the Heaviside distribution $\Theta(l)$ which equals $+1$ for positive l, 0 for negative l and is indeterminate if $l=0$. Note also that

$$\Theta(l)^2 = \Theta(l)\,, \qquad \Theta(l)\Theta(-l) = 0\,, \qquad \frac{d}{dl}\Theta(l) = \delta(l)\,, \tag{4.8}$$

where $\delta(l)$ represents the Dirac distribution. We also note that $\Theta(l)\delta(l)$ is not defined as a distribution. In this setting, the spacetime metric $g_{\alpha\beta}$ is expressed by

$$g_{\alpha\beta} = \Theta(l)g^{+}_{\alpha\beta} + \Theta(-l)g^{-}_{\alpha\beta}\,. \tag{4.9}$$

The question we now have is whether Eq. (4.9) is a valid distributional solution to the Einstein field equations

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}\,, \tag{4.10}$$

where $R_{\alpha\beta}$ is the Ricci curvature tensor and $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar curvature. Recall that the Ricci curvature tensor is obtained from the Riemann tensor through $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$, and the explicit form of the Riemann tensor is

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\delta\beta,\gamma} - \Gamma^{\alpha}{}_{\gamma\beta,\delta} + \Gamma^{\alpha}{}_{\gamma\lambda}\Gamma^{\lambda}{}_{\delta\beta} - \Gamma^{\alpha}{}_{\delta\lambda}\Gamma^{\lambda}{}_{\gamma\beta}\,, \tag{4.11}$$

where we have used the notation $f_{,\alpha} = \frac{\partial f}{\partial x^{\alpha}}$ and Γ corresponds to the Christoffel symbol given by

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}\left(g_{\beta\lambda,\gamma} + g_{\gamma\lambda,\beta} - g_{\beta\gamma,\lambda}\right)\,. \tag{4.12}$$

For Eq. (4.10) to be well-behaved it is necessary for its geometrical quantities to be properly defined as distributions. To assure this we need to test how the derivative of Eq. (4.9) behaves as it influences directly Eq. (4.12). A simple calculation yields

$$g_{\alpha\beta,\gamma} = \Theta(l)g^{+}_{\alpha\beta,\gamma} + \Theta(-l)g^{-}_{\alpha\beta,\gamma} + \epsilon\delta(l)\left[g_{\alpha\beta}\right]n_{\gamma}\,. \tag{4.13}$$

While the first two terms are well-behaved, the last one is problematic as it will give rise to terms of the form $\Theta(l)\delta(l)$ when computed in the Christoffel symbols. With the purpose of eliminating this problematic term we impose that the metric is continuous across Σ, i.e. $[g_{\alpha\beta}] = 0$. As it stands, this condition only holds for the coordinate system x^{α} that we previously defined, nonetheless, recalling that $[e^{\alpha}_{a}] = 0$ and Eq. (4.7), we can obtain a coordinate invariant condition

$$\left[g_{\alpha\beta}\right]e^{\alpha}_{a}e^{\beta}_{b} = \left[g_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}\right] = \left[h_{ab}\right] = 0\,. \tag{4.14}$$

This condition that we have obtained is called the first junction condition and it states that the induced metric must be the same on both sides of Σ for the hypersurface to have a well-behaved geometry. Note that Eq. (4.14), due to the fact that the induced metric is a (symmetric) $(d-1)$-tensor, only imposes $\Sigma_{i=1}^{d-1}i$ conditions while $[g_{\alpha\beta}] = 0$ produces $\Sigma_{i=1}^{d}i$ conditions. This happens because while the induced metric condition is coordinate independent, $[g_{\alpha\beta}] = 0$ is not and d of its conditions correspond to imposing the coordinate system to be continuous across Σ, i.e. $[x^{\alpha}] = 0$.

4.1.2 Second Junction Condition

The results obtained so far determine that the Christoffel symbols do not present problems and are well defined. From Eqs. (4.12), (4.13) and (4.14) we have

$$\Gamma^{\alpha}_{\ \beta\gamma} = \Theta(l)\Gamma^{+\alpha}_{\ \ \beta\gamma} + \Theta(-l)\Gamma^{-\alpha}_{\ \ \beta\gamma} \,. \tag{4.15}$$

As we are interested in evaluating whether the Einstein equations are well defined we will need to determine the Riemann tensor. To do so, we require the derivatives of this last expression which are easily shown to be

$$\Gamma^{\alpha}_{\ \beta\gamma,\delta} = \Theta(l)\Gamma^{+\alpha}_{\ \ \beta\gamma,\delta} + \Theta(-l)\Gamma^{-\alpha}_{\ \ \beta\gamma,\delta} + \epsilon\delta(l)\left[\Gamma^{\alpha}_{\ \beta\gamma}\right] n_{\delta} \,. \tag{4.16}$$

Recalling Eq. (4.11), we observe that the $\delta(l)$ term of this last equation is not problematic as terms of the type $\Theta(l)\delta(l)$ never appear. Taking this into account, the Riemann tensor is

$$R^{\alpha}_{\ \beta\gamma\delta} = \Theta(l)R^{+\alpha}_{\ \ \beta\gamma\delta} + \Theta(-l)R^{-\alpha}_{\ \ \beta\gamma\delta} + \delta(l)A^{\alpha}_{\ \beta\gamma\delta} \,, \tag{4.17}$$

with

$$A^{\alpha}_{\ \beta\gamma\delta} = \epsilon\left(\left[\Gamma^{\alpha}_{\ \beta\delta}\right] n_{\gamma} - \left[\Gamma^{\alpha}_{\beta\gamma}\right] n_{\delta}\right) \,. \tag{4.18}$$

Note that while Γ is not a tensor, the quantity A, which corresponds to the difference between two Christoffel symbols, is (as would be expected otherwise the Riemann tensor would be ill-defined). Moreover, we observe that while this $\delta(l)$ term imposes the existence of a curvature singularity at the hypersurface, the second junction condition will be used to either eliminate this term or to provide it with a sound physical interpretation.

To better understand this singular term we start by recalling that $g_{\alpha\beta}$ is continuous across Σ, therefore its tangential derivatives must also be continuous. Consequently $g_{\alpha\beta,\gamma}$ may only be discontinuous along the normal vector n^{α}. Explicitely this means there must exist a tensor $\kappa_{\alpha\beta}$ such that

$$\left[g_{\alpha\beta,\gamma}\right] = \kappa_{\alpha\beta} n_{\gamma} \Leftrightarrow \kappa_{\alpha\beta} = \epsilon\left[g_{\alpha\beta,\gamma}\right] n^{\gamma} \,. \tag{4.19}$$

Inserting Eq. (4.19) into Eq. (4.12) yields

$$\left[\Gamma^{\alpha}_{\beta\gamma}\right] = \frac{1}{2}\left(\kappa^{\alpha}_{\ \beta} n_{\gamma} + \kappa^{\alpha}_{\ \gamma} n_{\beta} - \kappa_{\beta\gamma} n^{\alpha}\right) \,. \tag{4.20}$$

Using this result, we are able to determine the singular part of the Riemann tensor

$$A^{\alpha}_{\ \beta\gamma\delta} = \frac{\epsilon}{2}\left(\kappa^{\alpha}_{\ \delta} n_{\beta} n_{\gamma} - \kappa^{\alpha}_{\ \gamma} n_{\beta} n_{\delta} - \kappa_{\beta\delta} n^{\alpha} n_{\gamma} + \kappa_{\beta\gamma} n^{\alpha} n_{\delta}\right) \,. \tag{4.21}$$

Moreover, by contracting the first and third indices of this last expression we may determine the singular part of the Ricci tensor

$$A_{\alpha\beta} \equiv A^{\gamma}_{\ \alpha\gamma\beta} = \frac{\epsilon}{2}\left(\kappa_{\gamma\alpha} n^{\gamma} n_{\beta} + \kappa_{\gamma\beta} n^{\gamma} n_{\alpha} - \kappa n_{\alpha} n_{\beta} - \epsilon\kappa_{\alpha\beta}\right) \,, \tag{4.22}$$

where we used $\kappa \equiv \kappa^{\alpha}_{\ \alpha}$. Finally, the singular part of the Ricci scalar is

$$A \equiv A^{\alpha}_{\ \alpha} = \epsilon\left(\kappa_{\alpha\beta} n^{\alpha} n^{\beta} - \epsilon\kappa\right) \,. \tag{4.23}$$

With this we obtain the $\delta(l)$ part of the Einstein tensor $G_{\alpha\beta}$. Meanwhile, the stress-energy tensor of Eq. (4.10) may also be decomposed in the form

$$T_{\alpha\beta} = \Theta(l)T^{+}_{\alpha\beta} + \Theta(-l)T^{-}_{\alpha\beta} + \delta(l)S_{\alpha\beta}\,. \tag{4.24}$$

The first two terms of the right-hand side are associated to the regions $\mathcal{V}^{\pm}$ whereas the singular term must then be associated to the hypersurface, i.e. it is the surface stress-energy tensor of Σ. Furthermore, there is an association between $S_{\alpha\beta}$ and the singular component of the Einstein tensor which reads

$$A_{\alpha\beta} - Ag_{\alpha\beta} \equiv 8\pi S_{\alpha\beta}\,. \tag{4.25}$$

The conclusion is that if we have a non-null $S_{\alpha\beta}$ there must exist a distribution of matter on the hypersurface. To this type of hypersurfaces we give the name of thin matter shells. By combining Eqs. (4.22), (4.23) and (4.25) we may determine $S_{\alpha\beta}$ explicitely, yielding

$$16\pi\epsilon S_{\alpha\beta} = \kappa_{\gamma\alpha}n^{\gamma}n_{\beta} + \kappa_{\gamma\beta}n^{\gamma}n_{\alpha} - \kappa n_{\alpha}n_{\beta} - \epsilon\kappa_{\alpha\beta} - \left(\kappa_{\gamma\delta}n^{\gamma}n^{\delta} - \epsilon\kappa\right)g_{\alpha\beta}\,. \tag{4.26}$$

Contracting this with n^{β} yields $S_{\alpha\beta}n^{\beta} = 0$, which means that $S_{\alpha\beta}$ is tangent to the hypersurface. We can then decompose $S_{\alpha\beta}$ by projecting it on the hypersurface with the tangent vectors determined in Eq. (4.2), obtaining this way the $(d-1)$-tensor $S_{ab} = S_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}$ which reads

$$16\pi S_{ab} = -\kappa_{ab} + h^{cd}\kappa_{cd}h_{ab}\,, \tag{4.27}$$

where we used $\kappa_{ab} = \kappa_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}$. At this point, although we already obtained the second condition necessary for the Einstein equations to be well defined in this setup we are still relying on the tensor κ which is not a usual geometrical quantity. There is however a relation between the tensor κ and the extrinsic curvature tensor K which allows us to express this condition in terms of the extrinsic curvature, as we shall see.

We start by recalling that the extrinsic curvature K_{ab} is defined as

$$K_{ab} = \nabla_{\beta}n_{\alpha}e^{\alpha}_{a}e^{\beta}_{b}\,, \tag{4.28}$$

where ∇ denotes the covariant derivative. Meanwhile, from Eq. (4.5) we have

$$[\nabla_{\beta}n_{\alpha}] = -\left[\Gamma^{\gamma}_{\alpha\beta}\right]n_{\gamma} = \frac{1}{2}\left(\epsilon\kappa_{\alpha\beta} - \kappa_{\gamma\alpha}n_{\beta}n^{\gamma} - \kappa_{\gamma\beta}n_{\alpha}n^{\gamma}\right)\,. \tag{4.29}$$

By inserting Eq. (4.29) in Eq. (4.28) we obtain

$$[K_{ab}] = [\nabla_{\beta}n_{\alpha}]\,e^{\alpha}_{a}e^{\beta}_{b} = \frac{\epsilon}{2}\kappa_{ab}\,, \tag{4.30}$$

which associates the jump in the extrinsic curvature with the surface stress-energy tensor. Defining $K \equiv h^{ab}K_{ab}$, we obtain

$$-8\pi\epsilon S_{ab} = [K_{ab}] - [K]\,h_{ab} \Leftrightarrow -8\pi\epsilon\left(S_{ab} - \frac{S}{d-2}h_{ab}\right) = [K_{ab}]\,. \tag{4.31}$$

From Eq. (4.31) we conclude that a smooth transition across the hypersurface requires the extrinsic curvature to be equal on both sides of it. $[K_{ab}] = 0$ is then our second junction condition and, like the first junction condition, it too is coordinate independent. We also note that when this condition is violated then there exists a curvature singularity at the hypersurface, nonetheless this singularity has a physical meaning as it corresponds to a surface stress-energy tensor present at the hypersurface.

4.2 Conservation of the stress-energy tensor in the presence of thin matter shells

As we have seen from the second junction condition, it is possible to describe well defined hypersurfaces with curvature singularities by assigning them an also singular surface stress-energy tensor. The question we now want to answer is how does the conservation equation behave for this type of stress-energy tensor?

To answer this question we will need to first introduce some new definitions. We start by noting that the induced metric from Eq. (4.7) allows us to define a compatible connection $\hat{\Gamma}$ which is defined as

$$\hat{\Gamma}_{cab} = \frac{1}{2}\left(h_{ca,b} + h_{cb,a} - h_{ab,c}\right) , \tag{4.32}$$

which is easily seen as the induced metric equivalent to Eq. (4.12). Moreover it can also be shown that $\hat{\Gamma}_{cab} = e^{\gamma}_{c} e^{\beta}_{b} \nabla_{\beta} e_{a\gamma}$. We can then use a new intrinsic covariant derivative ${}^{d-1}\nabla$ of $(d-1)$-tensors defined by the usual means in terms of the connection $\hat{\Gamma}$. For a given vector A_{α} tangent to Σ, i.e. orthogonal to n^{α}, this covariant derivative is used to project its covariant derivative $(A_{\alpha;\beta})$ onto the hypersurface. Note that while we use vector fields, the generalization to higher orders is trivial.

With the intention of using a more compact notation, for the rest of this section we will always use ; to represent the covariant derivative in terms of Γ and $|$ to represent the intrinsic covariant derivative in terms of $\hat{\Gamma}$.

From these previous definitions we conclude that $A_{a|b} \equiv A_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b}$ corresponds to the tangent components of the vector $A^{\alpha}{}_{;\beta} e^{\beta}_{b}$. However, we still miss information about the normal components of this vector. To obtain them we start by noticing that we can decompose the metric of the spacetime in two terms, one normal to the hypersurface and another one tangent to it, in the following manner

$$g^{\alpha\beta} = \epsilon n^{\alpha} n^{\beta} + h^{ab} e^{\alpha}_{a} e^{\beta}_{b} , \tag{4.33}$$

where h^{ab} is the inverse of the induced metric. Inserting Eq. (4.33) into $A^{\alpha}{}_{;\beta} e^{\beta}_{b}$, taking into account that $A \cdot n = 0$, yields after some computations

$$A^{\alpha}{}_{;\beta} e^{\beta}_{b} = A^{\alpha}{}_{|b} - \epsilon A^{a} n^{\alpha} K_{ab} , \tag{4.34}$$

which means that the normal part of the vector field $A^{\alpha}{}_{;\beta} e^{\beta}_{b}$ vanishes only when the extrinsic curvature is null. One important property of the extrinsic curvature that has not been mentioned before is that it is symmetric. To show this we start by noticing that

$$e^{\alpha}_{a;\beta} e^{\beta}_{b} = e^{\alpha}_{a,b} + e^{\gamma}_{a} \Gamma^{\alpha}{}_{\gamma\beta} e^{\beta}_{b} = e^{\alpha}_{b,a} + e^{\beta}_{b} \Gamma^{\alpha}{}_{\beta\gamma} e^{\gamma}_{a} = e^{\alpha}_{b;\beta} e^{\beta}_{a} , \tag{4.35}$$

where in the second equality we used the symmetry of second derivatives. This result comes directly from the fact that the tangent vectors are Lie transported along one another. Eq. (4.35) together with the fact that the tangent vectors are orthogonal to n allows us to write

$$K_{ab} = n_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b} = -n_{\alpha} e^{\alpha}_{a;\beta} e^{\beta}_{b} = -n_{\alpha} e^{\alpha}_{b;\beta} e^{\beta}_{a} = n_{\alpha;\beta} e^{\alpha}_{b} e^{\beta}_{a} = K_{ba} , \tag{4.36}$$

which proves that the extrinsic curvature is a symmetric tensor.

4.2.1 Gauss-Codazzi equations

The definition of an intrinsic covariant derivative allows us to define a purely intrinsic Riemann tensor $R^c{}_{dab}$ which can be written explicitely as

$$R^c{}_{dab} = \hat{\Gamma}^c{}_{db,a} - \hat{\Gamma}^c{}_{da,b} + \hat{\Gamma}^c{}_{ma}\hat{\Gamma}^m{}_{db} - \hat{\Gamma}^c{}_{mb}\hat{\Gamma}^m{}_{da} \,. \tag{4.37}$$

We now wish to understand what is the relation between the intrinsic Riemann tensor and the general Riemann tensor. To do so, we must evaluate the latter one on the hypersurface. After some computations we are able to obtain

$$R^\lambda{}_{\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c = R^m{}_{abc} e^\lambda_m + \epsilon \left(K_{ab|c} - K_{ac|b}\right) n^\lambda + \epsilon \left(K_{ab} n^\lambda{}_{;\gamma} e^\gamma_c - K_{ac} n^\lambda{}_{;\beta} e^\beta_b\right) \,. \tag{4.38}$$

Contracting this last equation with $g_{\lambda\eta} e^\eta_d$ yields

$$R_{\eta\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c e^\eta_d = R_{dabc} + \epsilon \left(K_{ab}K_{cd} - K_{ac}K_{bd}\right) \,, \tag{4.39}$$

which shows the relation between the intrinsic and the general Rieman curvature tensors. If we instead contract Eq. (4.38) with n_λ, we obtain

$$R_{\perp\alpha\beta\gamma} e^\alpha_a e^\beta_b e^\gamma_c = K_{ab|c} - K_{ac|b} \,, \tag{4.40}$$

where we used the notation $A_\alpha n^\alpha \equiv A_\perp$ and also of the fact $n^\alpha{}_{;\beta} n_\alpha = \left(n^\alpha n_\alpha\right)_{;\beta}/2 = 0$.

Eqs. (4.38) and (4.40) are named Gauss-Codazzi equations and they show how some of the components of the Riemann tensor may be written using only terms of the intrinsic and extrinsic curvatures of a hypersurface. Note that for the other missing components of the Riemann tensor we cannot express them using only the extrinsic and intrinsic curvature.

To better grasp the physical implications of these equations it is useful to contract them so they are in the form present in the Eistein field equations. Recalling Eq. (4.33) the Ricci tensor reads

$$R_{\alpha\beta} = \epsilon R_{\gamma\alpha\lambda\beta} n^\gamma n^\lambda + h^{cd} R_{\gamma\alpha\lambda\beta} e^\gamma_c e^\lambda_d \,, \tag{4.41}$$

and the Ricci scalar yields

$$R = 2\epsilon h^{ab} R_{\gamma\alpha\lambda\beta} e^\alpha_a e^\beta_b n^\gamma n^\lambda + h^{ab} h^{cd} R_{\gamma\alpha\lambda\beta} e^\alpha_a e^\beta_b e^\gamma_c e^\lambda_d \,. \tag{4.42}$$

Eqs. (4.41) and (4.42) may then be used to obtain the Einstein tensor

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = \epsilon R_{\gamma\alpha\lambda\beta} n^\gamma n^\lambda + h^{cd} R_{\gamma\alpha\lambda\beta} e^\gamma_c e^\lambda_d - \left(\epsilon h^{ab} n^\gamma n^\lambda + \frac{h^{ab}h^{cd}}{2} e^\gamma_c e^\lambda_d\right) R_{\gamma\mu\lambda\nu} e^\mu_a e^\nu_b g_{\alpha\beta} \,. \tag{4.43}$$

Contracting Eq. (4.43) with $n^\alpha n^\beta$ and inserting Eq. (4.39) into it yields

$$G_{\perp\perp} \equiv -\frac{\epsilon}{2} R_{\alpha\beta\gamma\lambda} e^\alpha_a e^\beta_b e^\gamma_c e^\lambda_d h^{ac} h^{bd} = -\frac{\epsilon}{2}\left[{}^{d-1}R + \epsilon\left(K_{ab}K^{ab} - K^2\right)\right] \,, \tag{4.44}$$

where ${}^{d-1}R$ is the intrinsic Ricci scalar. If instead we contract Eq. (4.43) with $e^\alpha_a n^\beta$ we obtain

$$G_{a\perp} \equiv h^{cd} R_{\gamma\alpha\lambda\perp} e^\alpha_a e^\gamma_c e^\lambda_d = K^c{}_{a|c} - K_{,a} \,. \tag{4.45}$$

Both Eqs. (4.44) and (4.45) are extremely useful when considering initial-value problems in General Relativity as they provide the constraints that the variables of the problem have to satisfy. Moreover, due to their explicit dependence on the induced metric and extrinsic curvature we conclude that for such a problem to be well posed these tensor fields are required to satisfy both equations.

For our work we won't be interested in studying the initial-value problem in this context, nonetheless Eq.(4.45) is of use. Recalling Eq. (4.10) we have

$$G_{a\perp} \equiv K^{c}{}_{a|c} - K_{,a} = 8\pi T_{\alpha\perp} e^{\alpha}_{a} \equiv 8\pi j_a \,, \tag{4.46}$$

where j_a is the stress-energy density current of the shell. Finally, recalling Eq. (4.31), we may compute the jump of Eq. (4.46) across the shell and obtain

$$S^{b}{}_{a|b} + \epsilon \, [T_{\alpha\perp} e^{\alpha}_{a}] = S^{b}{}_{a|b} + \epsilon \, [j_a] = 0 \,. \tag{4.47}$$

Eq. (4.47) shows that even if there is energy conservation in the d dimensional spacetime, there is no certainty that the the hypersurface will have a conserved surface stress-energy tensor. Moreover, it shows that the energy leaving from the hypersurface corresponds to the difference in the energy flow across the shell as one would expect.

4.3 Junction conditions of the dilaton and electomagnetic field

We will now analyse which additional constraints, other than the Darmois-Israel junction conditions, should be imposed so that the behavior of the electric and dilaton fields close to the shell are physically reasonable when considering GMGHS solutions. To this end we will continue to follow the method used in [22]. Our interest is in determining the junction conditions for the electromagnetic tensor $F^{\alpha\beta}$ and the dilaton field ϕ.

We will start by investigating the constraints for the scalar field ϕ. We may define ϕ in a region close to the hypersurface as

$$\phi = \phi^{+}\Theta(l) + \phi^{-}\Theta(-l) \,. \tag{4.48}$$

From Eq. (3.3) we know that for it to be well-behaved it is required that the laplacian of the dilaton field is properly defined as a distribution. To do so, we first need to differentiate Eq. (4.48) which reads

$$\phi_{;\beta} = \phi^{-}_{;\beta}\Theta(-l) + \phi^{+}_{;\beta}\Theta(l) + \epsilon \, [\phi] \, n_{\beta}\delta(l) \,. \tag{4.49}$$

Once more the term that causes problems is the term that is singular, $[\phi] \, n_{\beta}\delta(l)$. This term causes problems because it originates terms of the type $\Theta(l)\delta(l)$ in Eq. (3.1), which as we have mentioned before are ill-defined. Thus, for the scalar field to be well-behaved we necessarily need

$$[\phi] = 0 \,. \tag{4.50}$$

We then proceed to determine the divergence of Eq. (4.49), which reads

$$\nabla^2\phi = \nabla^2\phi^{-}\Theta(-l) + \nabla^2\phi^{+}\Theta(l) + \epsilon \, [\phi_{;\beta}] \, n^{\beta}\delta(l) \,. \tag{4.51}$$

Although we once more have a singular term, it does not present any problems to our equations. Moreover, if this term is not null then it will have a reasonable physical meaning. Since it represents a discontinuity of the variation of ϕ across the shell, it implies the existence of a scalar charge density on the shell, which we may define as $-4\pi\rho\delta(l)$. This source term would be represented in Eq. (3.1) as $16\pi\rho\phi\delta(l)$ and would consequently modify Eq. (3.3) to also have on its right-hand side the $-4\pi\rho\delta(l)$ term. We then conclude the following junction condition

$$\epsilon[\phi_{,\perp}] = -4\pi\rho\,. \tag{4.52}$$

Although we have obtained the junction conditions that arise from the field ϕ we are still missing the ones coming from the electromagnetic tensor. To determine the latter ones we need to first consider the fundamental field related to it, i.e. the Maxwell field A_α. Decomposing this field in the usual form we have

$$A_\alpha = A_\alpha^+\Theta(l) + A_\alpha^-\Theta(-l)\,. \tag{4.53}$$

Recalling that $F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha}$ and using Eq. (4.53) we obtain

$$F_{\alpha\beta} = F_{\alpha\beta}^+\Theta(l) + F_{\alpha\beta}^-\Theta(-l) + \epsilon\left[A_\alpha n_\beta - A_\beta n_\alpha\right]\delta(l)\,. \tag{4.54}$$

Because in Eq. (3.3) we have terms of the type F^2 we notice right away that the $\delta(l)$-term of Eq. (4.54) is problematic and therefore must be zero. We then have

$$[A_\alpha n_\beta] = [A_\beta n_\alpha] \Leftrightarrow [A_\alpha] = \epsilon\,[A_\perp]\,n_\alpha\,. \tag{4.55}$$

This last equation shows that for the electromagnetic tensor to be well-behaved it is necessary for the jump of the Maxwell field across Σ to only have components normal to the hypersurface. From this condition we can determine how the jump of the tangential components of $F_{\alpha\beta}$ behave. Recalling Eq. (4.35) we have

$$\left[A_{\alpha;\beta}e^\alpha_a e^\beta_b\right] = -\left[A_\alpha e^\alpha_{a;\beta} e^\beta_b\right] = -\left[A_\alpha e^\alpha_{b;\beta} e^\beta_a\right] = \left[A_{\alpha;\beta}e^\alpha_b e^\beta_a\right]\,, \tag{4.56}$$

inserting this into F_{ab} yields

$$[F_{ab}] = [F_{\alpha\beta}]\,e^\alpha_a e^\beta_b = [A_{\alpha;\beta} - A_{\beta;\alpha}]\,e^\alpha_a e^\beta_b = \left[A_{\alpha;\beta}e^\alpha_b e^\beta_a - A_{\beta;\alpha}e^\alpha_a e^\beta_b\right] = 0\,. \tag{4.57}$$

We conclude that the tangential components of the electromagnetic tensor do not vary across Σ. This result is the same as the one obtained in Ref. [68] where instead of taking our approach, the author relied on the equivalence principle and the fact that this result occurs in Relativistic Electrodynamics.

Although we have determined the behavior of the tangential components of the electromagnetic tensor across the hypersurface, we still don't know how the normal components vary. To study these components we start by recalling that in Eq. (4.54) we required that the singular term vanished. Using this information, we want to obtain Eq. (3.2) in a neighborhood of the hypersurface to see which terms might be problematic. By performing this computation we obtain

$$\left(e^{-2\phi}F^{\alpha\beta}\right)_{;\beta} = e^{-2\phi}\left\{\left(F^{\alpha\beta}_{+;\beta} - 2\phi^+_{,\beta}F^{\alpha\beta}_+\right)\Theta(l) + \left(F^{\alpha\beta}_{-;\beta} - 2\phi^-_{,\beta}F^{\alpha\beta}_-\right)\Theta(-l) + \epsilon\left[F^{\alpha\beta}\right]n_\beta\delta(l)\right\}\,. \tag{4.58}$$

We once again obtain a singular term, this time related to the normal components of the electromagnetic tensor. The only non-trivial component of the electromagnetic tensor that may originate through this previous equation a non-null singular term on the hypersurface is then the component $F_{\alpha\beta}e^{\alpha}_{a}n^{\beta}$. To understand the physical meaning of this component not being null we recall the Lagrangian of Eq. (3.1) and note that it describes our theory in the absence of charged sources. To correct this, we could add a current J^{μ} by inserting in the Lagrangian a term with a minimal coupling between the Maxwell field and J^{μ}. This would in turn modify the right-hand side of Eq. (3.2) yielding instead $-4\pi J^{\nu}$.

From this reasoning, we reach the conclusion that a non-null singular term in Eq. (4.58) may appear due to the existence of a source of charge in the hypersurface, which can be explicitely represented by

$$e^{-2\phi}\epsilon\,[F_{a\perp}] = -4\pi s_a\,, \tag{4.59}$$

where we have defined the surface charge current as $s^{\alpha} = \sigma_e u^{\alpha}|_{\Sigma}$, σ_e is the surface density of electric charge on the shell and u^{α} is the 4-velocity of the current. Note also that Eq. (4.59) reduces to the classical result obtained in Ref. [68] when $\phi = 0$.

4.4 Thin matter shells in GMGHS spacetimes

Our ultimate goal is to study, through the use of the thin shell formalism, whether it is possible for an overcharged shell (i.e. a shell whose mass and charge parameters lead to Eq. (3.14) arising in the spacetime exterior to the shell) to fully collapse and consequently give rise to a naked singularity.

For us to properly use the formalism we are first interested in finding the induced metric on a timelike hypersurface Σ. To this end let us consider $\Sigma = \{x^{\mu} : t = \mathcal{T}(\tau),\, r = R(\tau)\}$ which is parametrized by the coordinates $y^a = \{\tau, \theta, \varphi\}$, in which τ corresponds to the proper time of the shell.

Letting $f(r) = \left(1 - \frac{2M}{r}\right)$, we determine, from Eq. (4.7)

$$ds^2 = h_{ab}dy^a dy^b = -\left[-f\left(\frac{d\mathcal{T}}{d\tau}\right)^2 + \frac{1}{f}\left(\frac{dR}{d\tau}\right)^2\right]d\tau^2 + R(R-2D)d\Omega^2\,. \tag{4.60}$$

Let us now consider matter on the shell corresponding to a perfect fluid, so that its energy momentum tensor is, following Ref. [69],

$$S_{ab} = \sigma u_a u_b + p\left(h_{ab} + \epsilon u_a u_b\right), \tag{4.61}$$

where $u = \frac{\partial}{\partial\tau}$ is the fluid's 3-velocity, $\epsilon = \pm 1$ determines whether the shell is spacelike or timelike, p corresponds to a transverse pressure on the shell and $h_{ab} + \epsilon u_a u_b$ is the projector onto the two space orthogonal to u^a (in the rest frame of the shell).[2] As we are only interested in the study of timelike shells we then impose $\epsilon = 1$.

Using an overdot to represent a derivative with respect to τ, Eq. (4.47) yields

$$-\frac{2\left(R-D\right)\left(\sigma+p\right)\dot{R} + R\left(R-2D\right)\dot{\sigma}}{R\left(R-2D\right)} = -\left[T_{\alpha\perp}e^{\alpha}_{\tau}\right]. \tag{4.62}$$

We are interested in determining the normal to the shell, n. To do so, we start by defining the 4-velocity of an observer as $u^{\alpha} = \dot{\mathcal{T}}\partial_t + \dot{R}\partial_r$. Using the normalization condition $u^2 = -1$, we obtain

$$\dot{\mathcal{T}} = \pm\sqrt{\frac{f+\dot{R}^2}{f^2}}\,. \tag{4.63}$$

[2] While for a timelike shell σ has the clear interpretation of being the shell's surface energy density, for a spacelike shell this is not the case. In the latter situation σ has to correspond instead to a (longitudinal) pressure of the shell.

With it, we are ready to define the vector field normal to the shell $n^\alpha = n^t\partial_t + n^r\partial_r$. Knowing that it is perpendicular to u we have $n^\alpha u_\alpha = 0$. Moreover it is normalized so that $n^2 = 1$. This enables us to obtain the following relations:

$$n^t = \pm\sqrt{\frac{(n^r)^2 - f}{f^2}}\,, \tag{4.64}$$

$$n^r = \pm\sqrt{f + \dot{R}^2}\,. \tag{4.65}$$

With n defined, we may determine the extrinsic curvature of the shell, K_{ab}. Evaluating Eq. (4.28) we obtain

$$K_{\theta\theta} = (R - D)\, n^r = (R - D)\sqrt{\left(1 - \frac{2M}{R}\right) + \dot{R}^2}\,, \tag{4.66}$$

$$K_{\tau\tau} = -\frac{\frac{M}{R^2} + \ddot{R}}{\sqrt{\left(1 - \frac{2M}{R}\right) + \dot{R}^2}} = -\frac{1}{\dot{R}}\frac{d}{d\tau}\left(\frac{K_{\theta\theta}}{R - D}\right). \tag{4.67}$$

In these coordinates, using Eq. (4.31), we have the following equations to be satisfied

$$[K_{\theta\theta}] = -4\pi\sigma R\,(R - 2D)\;, \tag{4.68}$$

$$[K_{\tau\tau}] = -4\pi\,(\sigma + 2p)\;. \tag{4.69}$$

In principle, we could now use Eq. (4.47) and Eq. (4.67) to check whether these relations are consistent with each other. However, when we consider two different GMGHS solutions for the interior and exterior metric, the radial coordinate is discontinuous across the shell (due to the discontinuity of the parameter D). In fact, continuity of the induced metric imposes

$$R_o(R_o - 2D_o) = R_i(R_i - 2D_i)\,, \tag{4.70}$$

so that $R_o \neq R_i$ if $D_o \neq D_i$. Here we have used the subscript o to identify the dilaton charge D and radial coordinate r of the spacetime exterior to the shell, while the subscript i is the equivalent for the spacetime inside the shell.

We arrive at the conclusion that this coordinate system is not well adapted for performing the matching calculations because the coordinate r is, in general, not continuous across the shell. In the next section we simplify the problem by changing to a better suited coordinate system.

Still on the topic of the coordinate system, it is interesting to note that Eqs. (4.50), (4.52), (4.57) and (4.59) may provide some further constraints on the study of a collapsing thin shell in this theory. If we compute Eq. (4.59) in the current coordinate system we obtain

$$e^{-2\phi}\,[F_{\tau\perp}] = -e^{-2\phi}\left[\frac{Q}{R^2}\right] = -4\pi s_\tau\,. \tag{4.71}$$

Apart from the factor $e^{-2\phi}$, this is exactly the same result that was obtained in Refs. [68, 70].

Although we have stated previously that this radial coordinate is not a convenient choice as it is not invariant across the shell, we will now show that for the case of a dynamic shell this is irrelevant because the dilaton junction conditions impose much stronger constraints. To prove this statement we will now consider Eq. (4.50), which reads

$$e^{2\phi_0}\left(1 - \frac{2D}{R_o}\right) = e^{2\phi_i}|_\Sigma \Leftrightarrow R_o = \frac{2De^{2\phi_0}}{e^{2\phi_0} - e^{2\phi_i}|_\Sigma}\,, \tag{4.72}$$

where we used ϕ_0 to represent the asymptotic value of the scalar field for the exterior spacetime and we considered a general interior dilaton scalar field ϕ_i which is being evaluated at the hypersurface. Eq. (4.72) shows that for a general field ϕ_i we necessarily need to have the radius of the shell fixed and in this situation only static solutions may be found. This continues to be true in general when we consider an interior GMGHS solution, however, in the particular case that both GMGHS spacetimes have the same value for D and ϕ_0 we are able to have Eq. (4.72) be trivially satisfied, therefore allowing dynamics of the thin shell in this particular case.

4.5 Changing radial coordinate to areal radius

Let us now convert the metric to a new coordinate system such that $r(r-2D)=\varrho^2$. In this new coordinate system the radial location of the shell is determined by $\varrho=\mathcal{R}(\tau)$ and the metric becomes

$$ds^2 = -A\,dt^2 + (AC)^{-1}d\varrho^2 + \varrho^2 d\Omega^2\,. \tag{4.73}$$

Here, we defined the metric functions

$$A(\varrho) \equiv 1 - \frac{2M}{D+\sqrt{D^2+\varrho^2}}\,, \qquad C(\varrho) \equiv 1 + \left(\frac{D}{\varrho}\right)^2\,. \tag{4.74}$$

We conclude that the coordinate ϱ is precisely the areal radius. Note that in this coordinate system, the event horizon is located at $\varrho = 2\sqrt{M(M-D)}$.

Considering $u=\partial_\tau=\dot{\mathcal{T}}\partial_t+\dot{\mathcal{R}}\partial_\varrho$ once again as the velocity of the comoving observer, we obtain, using $u^2=-1$, the normalisation condition

$$A\dot{\mathcal{T}}^2 - \frac{\dot{\mathcal{R}}^2}{AC} = 1\,. \tag{4.75}$$

Inserting this into Eq. (4.73) yields the induced metric on the shell,

$${}^3ds^2 = -d\tau^2 + \mathcal{R}^2 d\Omega^2\,. \tag{4.76}$$

With the induced metric known we may once more determine the restrictions imposed on the density of the shell. Conservation of the energy-momentum tensor yields, from Eq. (4.47)

$$S^b_{\ \tau|b} = -\frac{2\,(\sigma+p)\,\dot{\mathcal{R}} + \mathcal{R}\dot{\sigma}}{\mathcal{R}} = -\left[T_{\tau\perp}\right]\,. \tag{4.77}$$

We are again interested in determining the normal to the shell. By applying the same relations that were used in the previous section we obtain

$$\begin{aligned} \dot{\mathcal{T}} &= \pm\sqrt{\frac{AC+\dot{\mathcal{R}}^2}{A^2C}}\,, && (4.78)\\ n^t &= \pm\sqrt{\frac{(n^\varrho)^2-AC}{A^2C}}\,, && (4.79)\\ n^\varrho &= \pm\sqrt{AC+\dot{\mathcal{R}}^2}\,. && (4.80) \end{aligned}$$

With this information and resorting to Eq. (3.5) we obtain for the GMGHS solution

$$T_{\alpha\perp}e^\alpha_\tau = -\dot{\mathcal{R}}\frac{\partial_{\mathcal{R}}C}{8\pi C}\frac{n^\varrho}{\mathcal{R}}\,. \tag{4.81}$$

Here and in the remainder of this section the metric functions A and C are to be considered as being evaluated at $\varrho = \mathcal{R}$. Using the normal of the shell determined we may compute the extrinsic curvature K_{ab}, whose only non vanishing components are

$$K_{\theta\theta} = \mathcal{R} n^{\varrho} = \frac{K_{\varphi\varphi}}{\sin^2\theta}\,, \tag{4.82}$$

$$K_{\tau\tau} = \frac{-C^2 \partial_{\mathcal{R}}(A) + \partial_{\mathcal{R}}(C)\dot{\mathcal{R}}^2 - 2C\ddot{\mathcal{R}}}{2Cn^{\varrho}} = -\frac{\sqrt{C}}{\dot{\mathcal{R}}}\frac{d}{d\tau}\left(\frac{K_{\theta\theta}}{\mathcal{R}\sqrt{C}}\right)\,. \tag{4.83}$$

Finally, applying Eq. (4.31) in this coordinate system yields

$$[K_{\theta\theta}] = -4\pi\mathcal{R}^2\sigma\,, \tag{4.84}$$

$$[K_{\tau\tau}] = -4\pi\left(\sigma + 2p\right)\,. \tag{4.85}$$

Note that the above equations are consistent with the (non-)conservation of the shell's stress-energy tensor. Indeed, using successively Eqs. (4.83), (4.84), (4.81) and (4.77), we recover Eq. (4.85):

$$[K_{\tau\tau}] = -\left[\frac{\sqrt{C}}{\dot{\mathcal{R}}}\frac{d}{d\tau}\left(\frac{K_{\theta\theta}}{\mathcal{R}\sqrt{C}}\right)\right] = 4\pi\left(\sigma + \frac{\mathcal{R}}{\dot{\mathcal{R}}}\dot{\sigma}\right) + \left[\frac{\partial_{\mathcal{R}}C}{2C}n^{\varrho}\right] = -4\pi\left(\sigma + 2p\right)\,. \tag{4.86}$$

As a consistency check all the equations above reduce to the expressions we obtained in section 4.4 if we set $D = 0$, which implies $C = 1$ and $\varrho = r$.

It is important to notice that if we were to join smoothly a GMGHS solution with another spacetime with a different non-trivial stress-energy tensor we would expect that the first two equalities of Eq. (4.86) to be posed in a more general form. The reason why this would happen is due to the term $[T_{\alpha\perp}e^{\alpha}_{\tau}]$ not being what was determined in Eq. (4.81) on both sides of the shell.

Nonetheless, the results of Eqs. (4.84) and (4.85), together with Eq. (4.47) represent the minimum general conditions that must be satisfied so that two different spacetimes may be joined smoothly through a hypersurface.

4.6 Energy density and radial potential for a thin shell of dust

To properly determine the surface energy density in terms of $\mathcal{R}$ we will consider a shell of dust so as to simplify Eq. (4.47) by considering $p = 0$. Moreover, we will consider only the case where the interior is another GMGHS solution. Within this assumption, we will determine the most general case, i.e. $D_i \neq D_o$, but keeping in mind that the dilaton junction conditions impose $D_i = D_o$ in order to allow motion of the shell, i.e. $\dot{\mathcal{R}} \neq 0$.

We start by noticing that Eq. (4.84), with some manipulation, reads

$$\sqrt{A_o C_o + \dot{\mathcal{R}}^2} = \frac{A_i C_i - A_o C_o}{8\pi\mathcal{R}\sigma} - 2\pi\mathcal{R}\sigma\,, \tag{4.87}$$

and similarly

$$\sqrt{A_i C_i + \dot{\mathcal{R}}^2} = \frac{A_i C_i - A_o C_o}{8\pi\mathcal{R}\sigma} + 2\pi\mathcal{R}\sigma\,. \tag{4.88}$$

Moreover, from Eq. (4.77) we can determine a differential equation for σ

$$\frac{d\sigma}{d\mathcal{R}} + 2\frac{\sigma}{\mathcal{R}} = \frac{\dot{\sigma}}{\dot{\mathcal{R}}} + 2\frac{\sigma}{\mathcal{R}} = \frac{1}{8\pi\mathcal{R}}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i}\sqrt{A_i C_i + \dot{\mathcal{R}}^2} - \frac{\partial_{\mathcal{R}}C_o}{C_o}\sqrt{A_o C_o + \dot{\mathcal{R}}^2}\right)\,. \tag{4.89}$$

Upon inserting Eqs. (4.87) and (4.88) in Eq. (4.89) we obtain a non-linear differential equation for σ. For the uncharged case ($D_o = Q_o = D_i = Q_i = 0$) the right hand side vanishes and we just recover the usual $\sigma = const.\mathcal{R}^{-2}$ as a solution. For charged spacetimes the solution differs.

In fact, it is possible to convert the above equation into a linear ordinary differential equation. Defining $\mathcal{M} \equiv 4\pi\mathcal{R}^2\sigma$ we get

$$\frac{d\mathcal{M}}{d\mathcal{R}} = \frac{\mathcal{R}}{2}\left[\mathcal{R}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i} - \frac{\partial_{\mathcal{R}}C_o}{C_o}\right)\frac{A_iC_i - A_oC_o}{2\mathcal{M}} + \frac{\mathcal{M}}{2\mathcal{R}}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i} + \frac{\partial_{\mathcal{R}}C_o}{C_o}\right)\right]. \tag{4.90}$$

Note that in the uncharged case ($D_i = D_o = 0$) we obtain simply $d\mathcal{M}/d\mathcal{R} = 0$ and so $\mathcal{M}$ is just a constant. Multiplying by $2\mathcal{M}$ we obtain

$$\frac{d\left(\mathcal{M}^2\right)}{d\mathcal{R}} - \frac{\mathcal{M}^2}{2}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i} + \frac{\partial_{\mathcal{R}}C_o}{C_o}\right) = \frac{\mathcal{R}^2}{2}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i} - \frac{\partial_{\mathcal{R}}C_o}{C_o}\right)(A_iC_i - A_oC_o) . \tag{4.91}$$

This has the generic form

$$F'(\mathcal{R}) + P(\mathcal{R})F(\mathcal{R}) = Q(\mathcal{R}), \tag{4.92}$$

with

$$F(r) = \mathcal{M}^2, \quad P(\mathcal{R}) = -\frac{1}{2}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i} + \frac{\partial_{\mathcal{R}}C_o}{C_o}\right), \quad Q(\mathcal{R}) = \frac{\mathcal{R}^2}{2}\left(\frac{\partial_{\mathcal{R}}C_i}{C_i} - \frac{\partial_{\mathcal{R}}C_o}{C_o}\right)(A_iC_i - A_oC_o), \tag{4.93}$$

and a simple solution can be given in closed form, yielding

$$\begin{aligned}\mathcal{M}^2 =& \frac{\gamma\sqrt{D_o^2+\mathcal{R}^2}\sqrt{D_i^2+\mathcal{R}^2} - (D_o^2 - D_i^2)\left(2M_o\sqrt{D_o^2+\mathcal{R}^2} - 2M_i\sqrt{D_i^2+\mathcal{R}^2}\right)}{\mathcal{R}^2}\\ &+ \frac{2(D_o^3M_o + D_i^3M_i) + 2\mathcal{R}^2(D_oM_o + D_iM_i) - \mathcal{R}^2(D_o^2 + D_i^2) - 2D_o^2D_i^2}{\mathcal{R}^2},\end{aligned} \tag{4.94}$$

where we have used γ as an integration constant.

We wish to determine γ and to do so we start by noticing that when $D_i = D_o = 0$ we should just get $\mathcal{M}^2 = m^2$. This m^2 value that we are imposing to be the constant of the general solution of these differential equations, comes directly from Israel's study in which he considered a Schwarzschild exterior and an empty interior (Ref. [29]). There it is proven that $M \geq m \geq 0$ for the shell to collapse from infinity. Nonetheless, in the case we are studying, the dilaton charge in general won't be null, thus we are interested in determining whether γ also has terms that depend on this charge. To do so, we start by expanding Eq. (4.94) when $\mathcal{R}$ goes to infinity and recall that for it to agree with the studies in Refs. [2, 4, 29] we should obtain once more m^2. This assumption yields

$$\gamma = m^2 - 2D_iM_i + D_i^2 - 2D_oM_o + D_o^2, \tag{4.95}$$

When we consider $D_i = D_o = D$, Eq. (4.94) may be rewritten in a clearer manner as

$$\mathcal{M} = m\sqrt{1 + \frac{D^2}{\mathcal{R}^2}} = m\sqrt{C}. \tag{4.96}$$

This last expression shows that $\mathcal{M}$ is always non-negative as long as $m \geq 0$.

After obtaining the solution for $\mathcal{M}$ we can compute the potential that describes the motion of the shell. This potential is defined by $\dot{\mathcal{R}}^2$ and can be determined from Eq. (4.87) or (4.88). When we compute this quantity, we obtain a one dimensional effective potential which allows classical motion in the regions where $\dot{\mathcal{R}}^2$ is positive. In contrast, regions where the potential becomes negative are classically forbidden, while points in which the potential vanishes represent turning points (or static configurations if $\ddot{\mathcal{R}} = 0$).

From Eqs. (4.87) and (4.88) we obtain

$$V(\mathcal{R}) = -A_o C_o + \frac{\mathcal{R}^2}{4\mathcal{M}^2}\left(A_i C_i - A_o C_o - \frac{\mathcal{M}^2}{\mathcal{R}^2}\right)^2 = -A_i C_i + \frac{\mathcal{R}^2}{4\mathcal{M}^2}\left(A_i C_i - A_o C_o + \frac{\mathcal{M}^2}{\mathcal{R}^2}\right)^2 , \quad (4.97)$$

which is the potential we intend to analyse in the next chapters.

4.7 Conclusions

In this chapter we have determined the junction conditions necessary to match two static solutions of the Einstein-Maxwell-dilaton theory and study dynamic thin shells. When applied to the particular theory we intend to study we have shown that all the junction conditions that are necessary only allow dynamics for a particular subset of solutions for both of the spacetimes separated by a thin shell.

This tells us that when considering the GMGHS solution in the static case we may use it to join two spacetimes that have the same scalar field on them. Furthermore, from Eq. (4.47) we have shown that although the shell's stress-energy tensor S_{ab} is not locally conserved, this fact has a physical explanation that arises from the junction conditions. Note that, for the Schwarzschild and Reissner-Nordström solutions, local conservation is guaranteed because the second term of Eq. (4.47) vanishes. In our case this does not happen, which allows us to infer that the cause is the presence of the scalar field which is a source of radiation in this theory.

To conclude, we now have the tools to study the collapse of a thin shell with the characteristics above specified and also to determine what are the the static thin shell solutions that the formalism allows. Both of these topics will be tackled in the next two chapters.

5 Self-gravitating Static Shells in Einstein-Maxwell-dilaton theory

In this chapter we will analyze the static shell solutions that arise from the application of the thin shell formalism. We start by determining in section 5.1 what are the conditions that influence the possibility of existence of static shells. In particular, we will determine for what combinations of conserved charges can one obtain static shell configurations. For this we will always be considering a GMGHS solution to describe the exterior spacetime while the interior spacetime will be Minkowski.

Afterwards, in section 5.2 we study if the previously obtained static shells do actually obey both the Weak and Dominant Energy Conditions, a characteristic that should be at least fulfilled for them to be physically reasonable solutions.

5.1 Static shell solutions

In this chapter, due to constantly resorting to different radial coordinates we will rely on the same definitions as in the previous chapter: R corresponds to the radial location of the shell as defined in Eq. (3.7) and $\mathcal{R}$ corresponds to the areal radial location of the shell. Recall that $R(R-2D) = \mathcal{R}^2$.

We have already proved, in section 4.4 that when we join smoothly a GMGHS exterior solution with a Minkowski interior we have our radius R fixed as defined in Eq. (4.72), which yields

$$\mathcal{R} = \frac{D}{\sinh(\phi_0)}\,. \tag{5.1}$$

As we are considering static shells we are interested in imposing $\dot{\mathcal{R}} = 0$ and $\ddot{\mathcal{R}} = 0$. When we apply this to Eqs. (4.82) - (4.85) we get

$$\begin{aligned} K_{\theta\theta} &= \mathcal{R}\sqrt{AC}\,, \qquad [K_{\theta\theta}] = -4\pi\mathcal{R}^2\sigma\,, & (5.2)\\ K_{\tau\tau} &= -\frac{\partial_{\mathcal{R}}A}{2}\sqrt{\frac{C}{A}}\quad , \qquad [K_{\tau\tau}] = -4\pi\left(\sigma+2p\right)\,. & (5.3)\end{aligned}$$

Moreover, Eq. (4.47) in this context reads

$$2(\sigma+p) + \mathcal{R}\frac{d\sigma}{d\mathcal{R}} + \frac{\partial_{\mathcal{R}}C}{8\pi}\sqrt{\frac{A}{C}} = 0\,, \tag{5.4}$$

which is shown, from Eqs. (5.2) and (5.3) to be trivially satisfied. Thus, no further requirements arise from Eq. (4.47).

We may now invert Eqs. (5.2) and (5.3) to determine the values of σ and p. This way we obtain

$$\begin{aligned} \sigma &= \frac{1}{4\pi\mathcal{R}}\left(1-\sqrt{\left(1-\frac{2M}{D+\sqrt{\mathcal{R}^2+D^2}}\right)\left(1+\frac{D^2}{\mathcal{R}^2}\right)}\right), & (5.5)\\ p &= -\frac{\sigma}{2}+\frac{M}{8\pi\left(D+\sqrt{\mathcal{R}^2+D^2}\right)^{\frac{3}{2}}\sqrt{\left(D-2M+\sqrt{\mathcal{R}^2+D^2}\right)}}. & (5.6)\end{aligned}$$

One curious fact in this regime is that, if we were to consider a shell made of dust, i.e. $p=0$, then Eqs. (5.5) and (5.6) would impose $D=M$. This means that a static shell made of dust that joins a Minkowski interior with a GMGHS exterior would necessarily have to be extremal. Moreover, it is important to state that no requirements on $\mathcal{R}$ appear. This is physically sensible because for extremal black holes the gravitational and scalar attractive forces are exactly cancelled by the electric repulsion, for example enabling the existence of static configurations of multiple black holes [12]. These static solutions are an extension of studies such as the one present in Ref. [71] applied to the Majumdar-Papatetrou solutions [72, 73] when one adds the dilaton scalar field.

Until now we have only considered thin shells with an empty interior, nonetheless it is also possible to generalize these static solutions so that the interior of the shell is also a GMGHS solution. We start by reviewing the junction condition presented in Eq. (4.50) to obtain the new values of $\mathcal{R}$ that are allowed. In this case we consider two solutions described by their dilaton charge D, energy M and asymptotic value of the dilaton field ϕ_0 distinguished by the suffixes i and o for the inner and outer solution. Other than the trivial value of $\mathcal{R}=0$, the junction condition for the dilaton field yields

$$\mathcal{R}=\begin{cases}\frac{2e^{\left(\phi_{0i}+\phi_{0o}\right)}\sqrt{2D_iD_o\cosh(\phi_{0i}-\phi_{0o})+D_i^2+D_o^2}}{\left|e^{2\phi_{0i}}-e^{2\phi_{0o}}\right|}, & \\ \frac{2e^{\left(\phi_{0i}+\phi_{0o}\right)}\sqrt{D_i^2+D_o^2-2D_iD_o\cosh(\phi_{0i}-\phi_{0o})}}{\left|e^{2\phi_{0i}}-e^{2\phi_{0o}}\right|}, & \text{if } D_i^2+D_o^2-2D_iD_o\cosh\left(\phi_{0i}-\phi_{0o}\right)\geq 0\end{cases}. \qquad (5.7)$$

The generalization of Eqs. (5.5) and (5.6) reads

$$\begin{aligned} \sigma &= \frac{1}{4\pi\mathcal{R}}\left(\sqrt{\left(1-\frac{2M_i}{D_i+\sqrt{\mathcal{R}^2+D_i^2}}\right)\left(1+\frac{D_i^2}{\mathcal{R}^2}\right)}-\sqrt{\left(1-\frac{2M_o}{D_o+\sqrt{\mathcal{R}^2+D_o^2}}\right)\left(1+\frac{D_o^2}{r^2}\right)}\right) & (5.8)\\ p &= -\frac{\sigma}{2}+\left(\frac{\partial_{\mathcal{R}}A_o}{2}\sqrt{\frac{C_o}{A_o}}-\frac{\partial_{\mathcal{R}}A_i}{2}\sqrt{\frac{C_i}{A_i}}\right). & (5.9)\end{aligned}$$

Finally, we note that it is not possible to have a static timelike shell inside the horizon. The reason is as follows.

We start by assuming that we have $\varrho_{H_i}=2\sqrt{M_i(M_i-D_i)}\leq\varrho_{H_o}=2\sqrt{M_o(M_o-D_o)}$, where we have defined ϱ_H as the apparent event horizon of the interior and exterior solutions respectively. We are interested in determining the values of $\mathcal{R}\leq\varrho_{H_i}$ in which we may find static timelike thin shells. For such an $\mathcal{R}$ the spacetime metric will be of the form

$$ds^2=-\frac{d\varrho^2}{|A|C}+|A|dt^2+\varrho^2d\Omega^2\,. \qquad (5.10)$$

Following the same procedure as previously done in section 4.5 to retrieve the junction conditions for this case, we obtain $-\frac{\dot{\mathcal{R}}^2}{|A|C}+|A|\dot{\mathcal{T}}^2=-1$, so that the shell may be timelike. From this equation we immediately conclude that if we set $\dot{\mathcal{R}}=0$, it gives rise to a contradition as the left hand side becomes positive while the right hand side is clearly negative. This means that we are not allowed to have a timelike shell at a constant $\mathcal{R}$ lower than the apparent event horizons, as initially predicted. The reason why this happens is clear as inside that region the coordinates ϱ and t invert their character, becoming timelike and spacelike respectively, thus, for a shell to exist in this region at a constant $\mathcal{R}$ it necessarily needs to be spacelike.

5.2 Energy Conditions

In this section we are interested in analyzing whether these static shells are concordant to both the Weak and Dominant Energy Conditions. We start by describing the physical meaning of both these conditons. Following this we determine the requirements that timelike shells with an empty interior need to fulfill for these energy conditions to be upheld. Finally, we briefly discuss the more generic case of shells with a GMGHS interior and exterior. We will consider the general case of shells with $p \neq 0$.

A brief description of the Weak and Dominant Energy Conditions, adopting the notation of Ref. [74], follows. The Weak Enegy Condition (WEC) states that for a given stress-energy tensor any observer will measure a non-negative energy density. Written explicitely this yields $T_{\mu\nu} X^\mu X^\nu \geq 0$, for all timelike vectors X^μ. The Dominant Energy Condition (DEC), on the other hand, demands that all observers should measure the flow of energy-momentum to be causal, i.e. not faster than the speed of light. Explicitely, this means that for any given causal, future-pointing, vector Y^μ we have that $-T^{\mu\nu} Y_\nu$ is also causal and future-pointing.

When considering a perfect fluid in an orthonormal frame, i.e. $T_{\mu\nu} = diag(\sigma, p, p, p)$, the energy conditions are as follows. The WEC produces two conditions, $\sigma \geq 0$ and $\sigma + p \geq 0$. In the case of the DEC, the condition required is $\sigma \geq |p|$.

Starting with timelike shells with an empty interior, we note that when we consider $p = 0$, by resorting to Eq. (5.5) we obtain

$$\sigma = \frac{e^{-\phi_0} \sinh^2(\phi_0)}{4\pi D} \geq 0 \,, \tag{5.11}$$

where we considered $D \geq 0$ and we replaced the value of $\mathcal{R}$ with the one we found in Eq. (5.1). This goes to show that, in the extremal case, we are allowed to have the shell at any value of $\mathcal{R}$.

For the empty interior case with $p \neq 0$, the WEC produces two conditions. The first condition is

$$\sigma \geq 0 \Leftrightarrow \begin{cases} \mathcal{R} \geq 2\sqrt{M\,(M-D)}, & \text{if } M \geq D \geq 0 \\ \mathcal{R} \geq \dfrac{\sqrt{\sqrt{\frac{D^7 (D+8M)}{M^4}} + \frac{D^2 (D^2 + 4DM - 8M^2)}{M^2}}}{2\sqrt{2}}, & \text{if } D > M \end{cases} \,, \tag{5.12}$$

The second condition that is imposed by the WEC is

$$\sigma + p \geq 0 \Leftrightarrow \begin{cases} \mathcal{R} \geq 2\sqrt{M\,(M-D)}, & \text{if } M \geq D \geq 0 \\ \mathcal{R} \geq \varrho_-(D, M), & \text{if } D > M \end{cases} \,. \tag{5.13}$$

The analytical expression of ϱ_- corresponds to a cubic root which has a complicated expression. This expression can nonetheless be interpreted clearly if we instead resort to the radial coordinate r (that defines the position of the shell as R). In this coordinate system, by taking into consideration that $\sigma + p$

is continuous for all values of R we may retrieve that the behavior of the shell is as follows. Expanding $\sigma + p$ both when $R \to \infty$ and when $R \to 2D$ it can be shown that in general we have $\sigma + p < 0$ when $R \to 2D$ and $\sigma + p > 0$ when $R \to \infty$. Thus, when we have $D > M$, by continuity there must exist a ϱ_- such that $\sigma + p = 0$ when $\mathcal{R} = \varrho_-$ and if $\mathcal{R} < \varrho_-$ then $\sigma + p < 0$, violating the WEC. Note that in the extremal case we have that $\varrho_- = 0$.

The union of both conditions produced by the WEC is represented graphically in Fig. 5.1 by the union of the blue and orange regions.

From the figure we can observe that, for a subextremal shell, we may never have a timelike static shell inside of the BH horizon, which is in accordance with the fact that inside of this horizon the coordinates t and ϱ change their character and the shell would become spacelike. Moreover, we notice that the value of $\varrho = 0$ is only allowed for an extremal shell as it had been previously mentioned. Finally, for an overcharged shell, a minimum ϱ in which it can be placed, always exists.

We will now discuss the Dominant Energy Condition. For it to be satisfied we need to have $\sigma \geq |p|$, which means that adding on the WEC we also need to impose $\sigma - p \geq 0$. This leads us to the following result

$$\sigma - p \geq 0 \Leftrightarrow \mathcal{R} \geq \varrho_-^*(M, D)\,, \tag{5.14}$$

where ϱ_-^* is once more the solution of a cubic equation on M and D and whose requirements can be seen in Fig. 5.1. We have represented in this figure, in blue, the result of Eq. (5.14) and the union of the blue region with the orange region represent the region that satisfies the Weak Energy Condition.

One interesting aspect that is noticeable from Fig. 5.1 is that when $D \to 0$, for the DEC to be fulfilled we have a lower value of ϱ that is higher than $2\sqrt{M(M-D)}$. While this may appear strange, it is in agreement with previous works on this subject concerning the Schwarzschild case such as Refs. [75, 76] in which they show that the DEC is only satisfied if $1 - 2M/\varrho \geq 1/5$. This last one being the result we also obtain for $D = 0$.

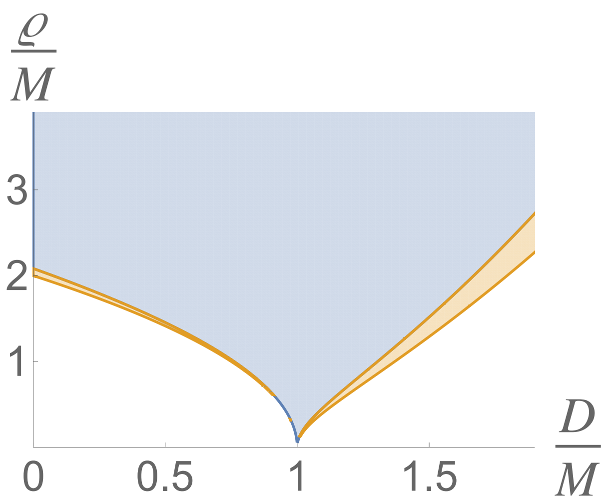


Figure 5.1: The values of $\mathcal{R}$ for which the DEC is satisfied correspond to the blue region and the ones where the WEC is satisfied correspond to the union between the blue and orange region. The white region below the curve $\mathcal{R} = 2\sqrt{M(M-D)}$ corresponds to the interior of the horizon. The orange region above it corresponds to another region, this time outside the horizon, which is forbidden by the Dominant Energy Condition.

It is possible to perform an analysis similar to the previous one but with a GMGHS interior instead of Minkowski. However, the large dimensionality of the parameter space (assuming we normalize everything in terms M_o we still have D_i, D_o and M_i) does not allow to present interesting results as clearly as in the empty interior case.

5.3 Conclusions

In this chapter we have shown that for the Energy Conditions to be fulfilled we can never have overextremal shells at $\varrho = 0$. We have also shown that the existence of a shell at $\varrho = 0$ may only occur for an extremal shell. This however, is a consequence of focusing our current study on static shells because as we have shown before, when we consider timelike shells we cannot have them at a constant radius inside the event horizon. It is then of interest to understand whether it is possible to have a thin shell collapse that might originate the usual GMGHS black hole solution. Note however that for this collapse to happen we would need to have a GMGHS interior as shown in chapter 4.

Another issue that is unavoidable with this analysis is that we are not able to study the stability of these static solutions. Such studies rely on the dynamic properties of these shells, specifically on the sign of the second derivative of the potential that governs their motion. In the approach we followed so far, the junction conditions forbid any kind of motion whatsoever. Note that we have restricted our study to $\dot{\mathcal{R}} = 0$ from the beginning and so we did not extract the radial potential. Therefore, we do not have information about (the signs of) its derivatives and are unable to retrieve through them a physical description of the stability of these static shells.

Finally, we focused on static solutions with flat interiors. It is possible to consider GMGHS interiors but as it is a problem that depends on five different variables (M_o, M_i, D_o, D_i, $\mathcal{R}$) we were not able to find general analytical solutions that described how both the WEC and the DEC depended on them. Nonetheless, in chapter 6 we will analyse these type of shells by considering that they are dust shells. Moreover, they will be studied in a dynamical framework, i.e. we will impose $D_o = D_i$ in order to allow motion.

6 Dynamical Shells and cosmic censorship

In this chapter we will be interested in examining the behavior of the potential of a shell whose interior and exterior are both described by GMGHS solutions. For that purpose we will focus on shells that are formed at $\mathcal{R} \to \infty$ and analyse their dynamics to assess whether or not they may collapse into a black hole or even a naked singularity.

Due to the requirement set by Eq. (4.50), our study will be focused on the situations in which the interior of the shell is either described by a black hole, i.e. $M_i > D$, or when that interior is extremal, i.e. $M_i = D$. As both of these situations are completely described by the same equations we will analyze them jointly.

The analysis of the potential will be used to determine the behavior of the shells in both cases. It will be centered on determining which are the regions where the potential is greater or equal to zero and through it infer what are the radii at which movement of the shell is allowed. The potential that governs the dynamics of a shell is given by

$$\dot{\mathcal{R}}^2 = V(\mathcal{R}). \tag{6.1}$$

This last equation shows that if $V(\mathcal{R})$ were to be smaller than zero then it's kinetic energy would be negative. As this is not physically realistic we conclude that the regions where the potential is negative portray boundaries that represent turning points of a moving shell on which it will bounce back.

After determining the regions where the shell may move, we will focus on reviewing whether or not the energy density that we defined in section 4.6 obeys the Weak Energy Condition and see if this further restricts the dynamics of these shells.

6.1 Shell collapsing from infinity with a Black Hole interior

We are interested in studying the potential as defined in Eq. (4.97). By inserting the expression obtained for $\mathcal{M}^2$ defined in Eq. (4.96) where we consider $D_i = D_o = D$ as imposed by Eq. (4.50), we obtain the following expression for the potential of the shell

$$V(\mathcal{R}) = \frac{\left(\mathcal{R}^2 + D^2\right)\left(\frac{\left(m^2+2\left(\sqrt{\mathcal{R}^2+D^2}-D\right)(M_o-M_i)\right)^2}{m^2} - 4\left(\mathcal{R}^2 - 2\left(\sqrt{R^2+D^2}-D\right)M_i\right)\right)}{4\mathcal{R}^4}. \tag{6.2}$$

Our purpose is to identify the regions in which motion is allowed and where the shell suffers a bounce. To do so, we are interested in determining all the places where $V(\mathcal{R}) \geq 0$. We start by determining the condition that is required for the shell to be initially at infinity, i.e. $V(\mathcal{R}\to\infty)\geq 0$, which yields

$$V(\mathcal{R}) \xrightarrow{\mathcal{R}\to+\infty} -1+\frac{(M_o-M_i)^2}{m^2}+\mathcal{O}\left(\frac{1}{\mathcal{R}}\right) . \tag{6.3}$$

Therefore, for the shell to exist at infinity we require that $M_o - M_i \geq m$, with $M_o - M_i = m$ in the case where the shell is initially at rest at infinity. Note that this result was expected since the exterior spacetime has energy M_o, which is the total energy of the spacetime, and the interior spacetime has energy M_i. What this means is that $M_o - M_i$ should represent the energy contained in the shell. In principle, Eq. (6.3) would also allow for $M_i - M_o \geq m$, however as will be shown later, this would violate the energy conditions of the shell.

From Eq. (6.2), it is clear that for $V(\mathcal{R}) \leq 0$, the second term of the numerator must also be non-positive. When multiplied by m^2, this term, which we shall henceforth refer to as $\bar{V}$, reads

$$\bar{V} = \left[m^2 + 2(R-2D)\left(M_o - M_i\right)\right]^2 - 4m^2(R-2D)\left(R-2M_i\right) , \tag{6.4}$$

where we have used the coordinate change $\mathcal{R}^2 = R(R-2D)$ to simplify the expressions that we will be analysing. Exclusively for the analysis of $\bar{V}$ we shall continue using the coordinate R.

When we consider that the shell is initially at rest at infinity, Eq. (6.4) can be further simplified yielding

$$\bar{V}_{\mathrm{rest}} = m^4 + 4m^2(R-2D)(M_o+M_i-2D) . \tag{6.5}$$

For the subextremal case, i.e. $M_o > M_i > D$, Eq. (6.5) shows that the potential will always be positive and, consequently, the shell will always collapse into a black hole. Moreover, also for the subextremal case, since $\bar{V} \geq \bar{V}_{\mathrm{rest}}$ we conclude that even if the shell is not initially at rest at infinity it will also collapse into a black hole.

We also note that for shells initially at rest at infinity, it is possible for them to collapse into a black hole even if the interior spacetime is overcharged as long as $M_o + M_i \geq 2D$. As before, this is also true even for the cases in which the shell is not initially at rest.

Until now we have focused on the cases in which the exterior spacetime was subextremal, nonetheless, to study the CCC we need to investigate what happens to Eq. (6.4) when we consider extremal and overcharged exteriors, i.e. $D \geq M_o$. From Eq. (6.5) we already know that if $D \geq M_o$ then we have

$$\begin{cases} \bar{V}_{\mathrm{rest}}(R) > 0, & \text{if } R < 2D + \frac{m^2}{8D-8M_i-4m} \\ \bar{V}_{\mathrm{rest}}(R) \leq 0, & \text{if } R \geq 2D + \frac{m^2}{8D-8M_i-4m} \end{cases} . \tag{6.6}$$

This leads to the conclusion that even if the exterior spacetime is not overcharged and we simply have $2D \geq M_o + M_i$, then the shell cannot be formed at infinity and it has a maximum radius at which it bounces.

The only case that we are now missing is what happens when $M_o - M_i > m$ and the shell is overcharged. To analyse this situation it is useful to expand Eq. (6.4), which reads

$$\bar{V} = m^4 - 4m^2\left(2D - M_o - M_i\right)\left(R-2D\right) + 4\left[\left(M_o-M_i\right)^2 - m^2\right]\left(R-2D\right)^2 . \tag{6.7}$$

The shell bounces if $\bar{V} \leq 0$ for values of $R > 2D$. To determine if such values exist we obtain the zeroes of Eq. (6.7) which are

$$R_- = 2D + \frac{m^2}{2\left(2D - M_i - M_o - \sqrt{4\left(D - M_o\right)\left(D - M_i\right) + m^2}\right)}, \tag{6.8}$$

$$R_+ = 2D + \frac{m^2}{2\left(2D - M_i - M_o + \sqrt{4\left(D - M_o\right)\left(D - M_i\right) + m^2}\right)}. \tag{6.9}$$

Several conclusions may be drawn from Eqs. (6.8) and (6.9). First, we note that as long as $\left(M_o - M_i\right)^2 \geq m^2$, Eq. (6.7) always becomes negative for $R_- < R < R_+$. Additionally, $R_+ > 2D$ because we are considering overcharged spacetimes. Thus, $\bar{V}$ will always become negative before reaching the singularity and a shell coming from infinity will bounce at $R = R_+$.

One interesting possibility that needs to be investigated is whether it is possible to have $R_- = R_+$. In that situation, $\bar{V}$ would always be non-negative and the shell could collapse into a naked singularity. Nonetheless, in the case of overcharged spacetimes this is not possible. To show it, we start by noting that for $R_- = R_+$ we would require

$$4\left(D - M_o\right)\left(D - M_i\right) + m^2 = 0\,. \tag{6.10}$$

However, as we are considering $D \geq M_o$ and $D \geq M_i$, the right-hand side of Eq. (6.10) will always be positive. Therefore, Eq. (6.10) can never be satisfied and it is impossible to have the shell collapsing into a naked singularity when both the interior and exterior spacetimes are overcharged.

We finish this section by presenting the plots of the potential we are studying for different values of m, D and M_i, where all these values have been normalized in terms of M_o. The plots can be seen in Fig. 6.1. Note that, here we are once more using the coordinate $\mathcal{R}$ instead of R.

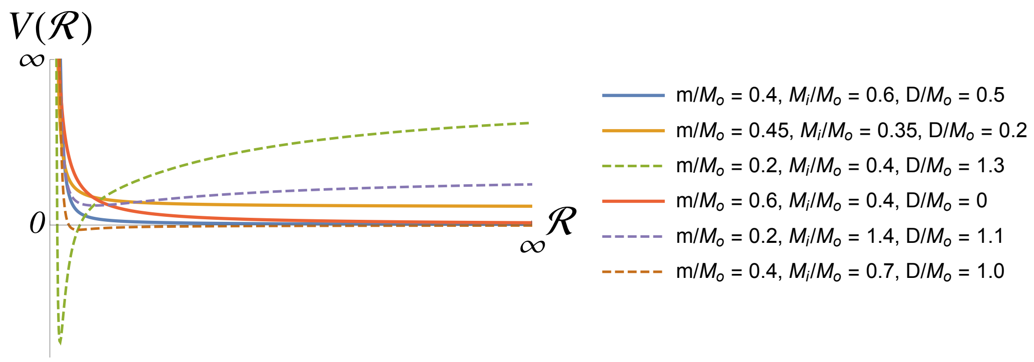


Figure 6.1: In blue we have an example of the potential when $M_o > D > M_i$ and $M_o - M_i = m$, in this case the shell is at rest at infinity and collapses into a black hole. The orange curve represents a situation in which collapse occurs from infinity, although the shell initially is not at rest and both spacetimes are subextremal. The potential represented by the green curve corresponds to a collapse with an overcharged interior and exterior, note that the shell bounces before it reaches the singularity. The red curve represents collapse with an interior and exterior Schwarzschild solution and with the shell initially at rest at infinity. The purple curve represents complete collapse into the singularity when $M_i > D > M_o$, i.e. the interior spacetime is undercharged and the exterior is overcharged. Finally, the brown curve represents the collapse of a shell with an overcharged interior and an extremal exterior.

Supposedly, the purple curve of Fig. 6.1 violates CC, however recalling Eq. (4.84) it is easy to see that the conditions imposed would mean that the surface energy density σ would have to be negative for some values of $\mathcal{R}$ which, as we'll see in the next section, violates the energy conditions.

6.2 Energy conditions

To determine completely what are the relations between the four variables of this study (D, M_o, M_i and m) we will verify what is necessary for the shell to obey the Weak Energy Condition. Considering that for these dynamic shells we are assuming they are made of dust, therefore setting $p = 0$, the condition is reduced to

$$\sigma \geq 0\,. \tag{6.11}$$

Recall that for dust the Weak and Dominant Energy Conditions are the same. The expression for $\mathcal{M} = 4\pi\mathcal{R}^2\sigma$ was obtained previously in Eq. (4.96). It is easy to note that for Eq. (6.11) to be verified we need to impose $\mathcal{M} > 0$ so that σ might have a positive, real value. Considering this, it is straightforward to see that the only requisite we get from the energy conditions is $m \geq 0$.

We conclude that the energy conditions do not seem to impose any restrictions on the motion of the shell as long as we have $m \geq 0$. Taking this into consideration, we note that for Eq. (4.84) to satisfy the WEC for all values of $\mathcal{R}$ we need to have $M_o \geq M_i$. This shows us that we cannot overcharge these solutions by considering a subextremal interior and an overcharged exterior in which we have $M_i > M_o$, without violating the energy conditions. Therefore, the CCC is also upheld in this context.

6.3 Conclusions

One of the main issues that was borne out of our study is that it is not possible to tackle the interesting case of a subextremal solution that is being overcharged simply by matching two GMGHS spacetimes. This would imply in this context $D_o > M_o \geq M_i > D_i$. However, this last expression is not possible in a dynamical framework due to the requirements brought by the junction conditions of the scalar field. Therefore, we are here restricted to studying collapses from shells that have at most an extremal interior if we are to obey the energy conditions.

Nonetheless, one can test CC in this framework. Since $D_o = D_i$ one can get an overextremal exterior ($M_o < D_o$) from an underextremal interior only if $M_o < M_i$. However, this would imply that the shell had negative energy and consequently violate the energy conditions. Therefore, CC is satisfied in this context.

A possible alternative to violating CC comes from generalizing the GMGHS solutions so that they might be time dependent. In this manner we may end up having the energy of the spacetimes M_o or M_i or otherwise the dilaton charge D varying with time. By using time-dependent solutions we could start with both an interior and exterior that were subextremal and that, as time advances, become overcharged due to either the dilaton charge increasing or otherwise by having the energies that describe the spacetimes decreasing. The generalization of the GMGHS solutions and the study of collapsing thin shells in this new time dependent context will be tackled in the next chapter.

7 Time dependent solutions and radiating shells

In this chapter we intend to study the gravitational collapse of thin shells that have an exterior radiating solution and a GMGHS interior, to test CC. The first part of the chapter will contain a revision of Ref. [67] where the exterior solution that we intend to use, which is a radiating solution with time independent scalar charge, was obtained. After this review, we apply the junction conditions obtained in chapter 4.5 (properly adapted to this new solution). The junction conditions will allow us to determine the potential that describes the dynamics of these thin shells and with it test CC. The structure of this chapter is as follows.

The chapter starts with a generalization of the field equations of Refs. [12–14] to radiating spacetimes in section 7.1. We then briefly summarize the static solutions of Ref. [12] for a more general coupling than the one that was considered in chapter 3, this in section 7.2. Afterwards, in section 7.3 we follow the procedure of Refs. [54–56, 77] to obtain time dependent radiating solutions in this theory. The energy conditions of these solutions will be reviewed based on Refs. [74, 77–79] in 7.4. Following this, we will focus on describing the radiating solution with time independent scalar charge in section 7.5.

Finally, in section 7.6 we determine the junction conditions of a shell with a GMGHS interior and a radiating solution with time independent scalar charge exterior by following similar works done in for example [58, 61–63]. Once we determine the junction conditions we will move on to study CC by analysing the gravitational collapse of these shells under certain energy restrictions using the same method as the one used for example in [58–60].

7.1 Field Equations

To extend the Vaidya [54, 55] and Bonner-Vaidya [56] solutions to their Einstein-Maxwell-dilaton counterparts we first redefine the Lagrangian present in Eq. (3.1). The need for this redefinition is due to the fact that we are now considering a radiating solution which describes charged radiation, thus we expect a charged current J^μ to appear. This can be represented in the Lagrangian by inserting a minimal coupling between the Maxwell field and the current. Moreover, due to the presence of charged radiation we shall also require an extra matter term $\mathcal{L}_m$ in the Lagrangian to account for the null fluid that appears. Other than these two additions, we also determine the solution for a general coupling between the dilaton and the electromagnetic tensor, by introducing the parameter a. Although this coupling constant is not required to obtain the solutions we set out to find in this chapter it is still useful to interpret how they behave. Note that in Eq. (3.1) we simply imposed $a = 1$. The new Lagragian we obtain is

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi}\left[R - 2\nabla_\mu\phi\nabla^\mu\phi - e^{-2a\phi}F_{\mu\nu}F^{\mu\nu} + 16\pi A_\mu J^\mu\right] + \mathcal{L}_m\,. \tag{7.1}$$

Some of the relevant values of the coupling parameter are $a = 1$ and $a = \sqrt{3}$. The former, as we noted in Section 2.6, corresponds to a four-dimensional low-energy effective description of (heterotic $[E_8 \times E_8$ or $SO(32)]$) string theory that we've studied so far. The latter one is instead obtained by a Kaluza-Klein reduction from five dimensions [14]. We note also that other possible values may also be obtained from compactification (and truncation) of intersecting brane solutions [41, 80].

By varying Eq. (7.1) in terms of the Maxwell field, dilaton and metric and recalling Eq. (3.6) we obtain the following field equations

$$\nabla_\mu\left(e^{-2a\phi}F^{\mu\nu}\right) = -4\pi J^\nu\,, \tag{7.2}$$

$$\nabla^2\phi + \frac{a}{2}e^{-2a\phi}F_{\mu\nu}F^{\mu\nu} = 0\,, \tag{7.3}$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \equiv 8\pi\left(T^{(\mathrm{dil})}_{\mu\nu} + T^{(\mathrm{EM})}_{\mu\nu} + T^{(\mathrm{fluid})}_{\mu\nu}\right)\,, \tag{7.4}$$

where we have defined the fluid stress-energy tensor as

$$T^{(\mathrm{fluid})}_{\mu\nu} = T^{\mathrm{m}}_{\mu\nu} + g_{\mu\nu}A_\sigma J^\sigma - 2A_{(\mu}J_{\nu)}\,, \tag{7.5}$$

and where $T^{\mathrm{m}}_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}_{\mathrm{m}}}{\partial g^{\mu\nu}}$. We will be interested in choosing $T^{(fluid)}_{\mu\nu}$ of the form of a null fluid.

Interestingly, we note that by adding the coupling parameter we are able to retrieve the Einstein-Maxwell theory when we impose $\phi = a = 0$. Recall that in Chapter 3 this did not happen due to Eq. (3.3). It is now clear that even though we could impose $\phi = 0$ we would still be considering that it coupled to the electromagnetic tensor. This means that for it to be null then $Q = 0$, which would then give rise to the Einstein vaccum field equations.

7.2 Static black hole solutions

We already described the static black hole solutions of Eqs. (7.2)-(7.4) in the absence of J^μ and when $a = 1$ in chapter 3. We also note that in Ref. [12] they also obtain a solution for generic a that for the electrically charged case reads

$$ds^2 = -\lambda dt^2 + \lambda^{-1}dr^2 + r^2 B d\Omega^2\,, \tag{7.6}$$

$$F = -\frac{Q}{r^2}dt \wedge dr\,, \tag{7.7}$$

$$e^{2\phi(r)} = e^{2\phi_0}(1 - r_-/r)^{\frac{2a}{1+a^2}}\,, \tag{7.8}$$

with

$$\lambda(r) = (1 - r_+/r)(1 - r_-/r)^{\frac{1-a^2}{1+a^2}}\,, \tag{7.9}$$

$$B(r) = (1 - r_-/r)^{\frac{2a^2}{1+a^2}}\,, \tag{7.10}$$

where the physical mass M, the electric charge Q, and the dilatonic charge D are related to $r_\pm$ by

$$M = \frac{r_+}{2} + \frac{1-a^2}{1+a^2}\frac{r_-}{2}\,, \qquad Q^2 = e^{2a\phi_0}\frac{r_+ r_-}{1+a^2}\,, \qquad D = \frac{a}{1+a^2} r_-\,. \tag{7.11}$$

As we have discussed in the previous section, when a goes to 0 we obtain the Reissner-Nordström solution with a constant decoupled scalar field. In that situation r_- and r_+ have a clear physical meaning corresponding respectively to the Cauchy horizon and event horizon of the Reissner-Nordström solution. However, when considering a nonvanishing a the surface $r = r_-$ becomes singular as it has a vanishing area. Recalling the Electric-Magnetic duality for these solutions, present in Eq. (3.13), it is interesting to note that the sign of a allows to switch between electric (positive a) and magnetic (negative a) solutions. Thus, for any given a these solutions only depend on three quantities, $r_\pm$ and ϕ_0. This last one however may be trivially generated since Eqs. (7.2)-(7.4), in the absence of sources (i.e. $J^\mu = 0$), are invariant under a simultaneous shift of the dilaton field $\phi \to \phi + \phi_0$ and a rescaling of the Maxwell field $A_\mu \to e^{a\phi_0} A_\mu$. Therefore, only $r_\pm$ are independent parameters for these solutions when considering an arbitrary a. This goes in agreement with the fact that although there are three conserved charges (M, Q and D), they are not independent. Indeed they satisfy

$$a^2 e^{-2a\phi_0} Q^2 = 2aMD - (1-a^2)D^2\,, \tag{7.12}$$

which reads $D = 0$ if we set a to be zero and yields the result of Eq. (3.10) for $a = 1$.

7.3 Time Dependent solutions

There are several methods that can be used to solve Eqs. (7.2)-(7.4). In our case we will adapt the one done for example in Refs. [56, 77]. It consists of using the static solution we already have written in the retarded/advanced Eddington-Finkelstein $u := t - \varepsilon r_*$ (with $dr/dr_* = \lambda$ and $\varepsilon = \pm 1$ for the retarded and advanced coordinate respectively). Recall that the retarded coordinate is usually used to describe radiating spacetimes while the advanced coordinate describes absorbing spacetimes. Additionally, we also need to consider the mass and electric charge as functions of u, which implies $r_+ = r_+(u)$ and $r_- = r_-(u)$ in the equations determined in the previous sections. In practice, this ansatz means

$$\begin{aligned} ds^2 &= -\lambda(u,r)du^2 - 2\varepsilon du dr + r^2 B(u,r) d\Omega^2\,, && (7.13)\\ F &= -\frac{Q(u)}{r^2} du \wedge dr\,, && (7.14)\\ e^{2\phi(u,r)} &= e^{2\phi_0}\left(1 - \frac{r_-(u)}{r}\right)^{\frac{2a}{1+a^2}}\,. && (7.15) \end{aligned}$$

Solving this ansatz for Eq. (7.2) yields

$$J^\nu = -\frac{e^{-2a\phi(u,r)}}{4\pi r^2} Q'(u) \delta^\nu_r\,, \tag{7.16}$$

where the prime denotes a derivative with respect to u. Note that δ now represents the Kronecker delta and not the Dirac distribution (recall that $\delta^\nu_\mu = 0$ if $\nu \neq \mu$ and $\delta^\nu_\mu = 1$ if $\nu = \mu$). From Eq. (7.16) we conclude that as long as the electric charge Q is not constant we have a radial current, which is nevertheless divergence free, i.e. $\nabla_\mu J^\mu = 0$.

By inserting the ansatz we are considering into the dilaton field equation we obtain

$$a\left(1-a^2\right)r_-'(u)=0\,. \tag{7.17}$$

Consequently, if $a=0$ or $a=\pm1$ Eq. (7.3) imposes no restrictions on r_-, however for all other values of a this equation imposes r_- to be constant[1]. Note that $a=0$ corresponds to having Einstein-Maxwell (with a constant decoupled scalar field) and this case has already been analysed in Ref. [56].

Of the analysis of the field equations we are just missing the conditions that Eq. (7.4) imposes on our ansatz. To determine these conditions we first have to define explicitely $T_{\mu\nu}^{(\mathrm{fluid})}$. For that purpose it is useful to introduce the following (future-pointing in contravariant form) null vectors:

$$\ell_\mu=-\partial_\mu u=-\delta_\mu^u\,,\qquad w_\mu=\frac{1}{2}g_{uu}\delta_\mu^u-\varepsilon\delta_\mu^r\,, \tag{7.18}$$

which satisfy $\ell_\mu\ell^\mu=0$, $w_\mu w^\mu=0$ and $\ell_\mu w^\mu=-1$. These vectors then allow us to write $T_{\mu\nu}^{(\mathrm{fluid})}$ in the form of a null fluid, as defined in [77] and references therein. The explicit expression of this particular stress-energy tensor is

$$8\pi T_{\mu\nu}^{(\mathrm{fluid})}=\mu\ell_\mu\ell_\nu+(\rho+P)(\ell_\mu w_\nu+\ell_\nu w_\mu)+Pg_{\mu\nu}\,, \tag{7.19}$$

where P and ρ are respectively generalizations of the pressure and energy density of a perfect fluid (Eq. (4.61)) to the null fluid case and μ is an energy density that receives contributions from both the matter Lagrangian, $\mathcal{L}_{\mathrm{m}}$, and from the current terms, $A_{(\mu}J_{\nu)}$, in Eq. (7.5). For our solutions, the term $A_\sigma J^\sigma$ in Eq. (7.5) vanishes as A_μ only has a uu-component while from Eq. (7.16), J^μ only has an rr-component. Moreover, we also note that the term $A_{(\mu}J_{\nu)}$ only has a uu-component, which contributes to the energy density μ but not to P and ρ.

Our definition of $T_{\mu\nu}^{(\mathrm{fluid})}$ generalizes a null dust, due to the non-null values of P and ρ, which would describe a pressureless fluid with energy density μ moving with four-velocity ℓ^μ. From Eq. (7.18) we conclude that the stress-energy tensor we are using has instead an energy flux along the null vector w^μ – this is clear from the contraction of Eq. (7.19) with the different possible combinations of null tensors ℓ^μ and/or w^μ.

Solving Eq. (7.4) considering this null fluid we come to the conclusion that the fluid stress-energy tensor is of the form of Eq. (7.19) with $P=0$ and its components μ and ρ are

$$\mu_{a=1} = \frac{\varepsilon\left[(r_+(u)r_-(u))'-2rr_+'(u)\right]+2r^2r_-''(u)}{2r^2[r-r_-(u)]}\,, \tag{7.20}$$

$$\mu_{a\neq1} = -\varepsilon\frac{\left[(1+a^2)r-r_-(u)\right]}{(1+a^2)r^{\frac{3+a^2}{1+a^2}}\left[r-r_-(u)\right]^{\frac{2a^2}{1+a^2}}}r_+'(u)\,, \tag{7.21}$$

$$\rho_{a=1} = -\varepsilon\frac{1}{r[r-r_-(u)]}r_-'(u)\,, \tag{7.22}$$

$$\rho_{a\neq1} = 0\,. \tag{7.23}$$

Although with this we have obtained all the components that characterize the total stress-energy tensor $T_{\mu\nu}$ we still need to determine the energy conditions that it should satisfy. This will be done in the next section.

[1]Henceforth we restrict to $a\geq0$. This condition can be enforced without loss of generality through the symmetry $\phi\to-\phi$, $a\to-a$ of Eq. (7.1).

7.4 Energy Conditions

In this section we will discuss the energy conditions based on the terminology and definitions of Refs. [74, 79] and based on the discussions present in Refs. [77, 78].

The total stress-energy tensor is defined as

$$T_{\mu\nu} = T^{(\mathrm{dil})}_{\mu\nu} + T^{(\mathrm{EM})}_{\mu\nu} + T^{(\mathrm{fluid})}_{\mu\nu} \,. \tag{7.24}$$

We will be interested in verifying what are situations in which it obeys the weak and strong energy conditions. To do so, we will start by analysing what these energy conditions correspond to on the fluid stress-energy tensor. The reason why we start with only this component of the total stress-energy tensor is due to the fact that it allows us to obtain some restrictions that will prove useful in the analysis of the total stress-energy tensor.

Recall that the fluid stress-energy momentum tensor has been determined to be of the form present in Eq. (7.19) with $P = 0$. It is easy to identify that this tensor possesses two different spacelike eigenvectors and only one lightlike eigenvector (ℓ^μ) whose eigenvalue ($-\rho$) has geometric multiplicity equal to 2. From this fact we can identify this tensor as a tensor of type II according to the aforementioned terminology. From [74, 79] we can conclude that the weak and strong energy conditions both impose $\mu \geq 0$ and $\rho \geq 0$.

Checking these conditions for Eqs. (7.20)-(7.23), we conclude the following:

For $a \neq 1$ the condition $\rho \geq 0$ is automatically satisfied. Moreover, recalling that $r \geq r_-(u)$ for the metric to be regular, the energy conditions for μ reads

$$\varepsilon r'_+(u) \leq 0 \,, \tag{7.25}$$

which tells us that the apparent horizon must decrease (increase) for radiating (absorbing) solutions for which $\varepsilon = +1$ ($\varepsilon = -1$) which was expected as it is losing (gaining) energy. When $a = 1$, the energy condition for ρ reads

$$\varepsilon r'_-(u) \leq 0 \,, \tag{7.26}$$

while the energy condition for μ yields

$$2r^2 r''_- + \varepsilon \left[(r_+ r_-)' - 2r r'_+ \right] \geq 0 \,. \tag{7.27}$$

To better interpret these last two equations it is favorable to express them in terms of the quantities M and D. When we make this change in Eq. (7.20) we obtain

$$\mu_{a=1} = \frac{2}{r^2\,(r - 2D)} \bar{\mu} \,, \tag{7.28}$$

where we defined

$$\bar{\mu} := r^2 D'' + \varepsilon M D' - \varepsilon (r - D) M' \,. \tag{7.29}$$

Inserting this into the energy conditions yields

$$\varepsilon D' \leq 0 \,, \qquad \bar{\mu} \geq 0 \qquad \text{for } \; a = 1 \,. \tag{7.30}$$

Note that for $a = 1$, we may obtain radiating solutions in which it is possible for the mass to increase in time as long as the second derivative of the dilaton charge is sufficiently large and positive. This curious observation is similar to the Einstein-Maxwell case, where the Schwarzschild mass can also increase if the black hole ionizes at a faster rate [56].

From Eqs. (7.29) and (7.30) we note that the case where D is constant greatly simplifies the restrictions imposed by the energy conditions, only imposing

$$\mu = -\frac{2\varepsilon(r-D)}{r^2(r-2D)}M' \geq 0 \Leftrightarrow -\varepsilon M' \geq 0\,, \tag{7.31}$$

where once more we've used the fact that $r \geq 2D$ for the metric to be regular. Due to its simplicity, we will use this solution as the exterior spacetime of the shells we will study in this chapter. In this case the energy condition is reduced to saying that for $\varepsilon = +1$ ($\varepsilon = -1$) the apparent horizon shrinks (increases) in time.

Having obtained the energy conditions for the fluid component of the stress-energy tensor, we now turn our attention towards the total stress-energy tensor. Note that we will restrict our study to the case $a = 1$ as it is the one that interests us. Moreover we will not write here $T_{\mu\nu}$ explicitely as its components are quite extensive.

To analyse this tensor through the method described in Refs. [74, 79] we need to define it in an orthonormal basis. This can easily be done by computing its eigenvectors and eigenvalues, which lead to the conclusion that this tensor is of type I — it possesses one timelike eigenvector E_0 and three spacelike eigenvectors E_1, E_2 and E_3. We define λ_0 as the eigenvalue that corresponds to E_0 while λ_i with $i = 1, 2, 3$ are the eigenvalues associated to the spacelike eigenvectors. We also note that $-\lambda_0$ corresponds to the proper energy density of the spacetime, which we shall name η. Finally, we identify λ_i with the principal stresses of the spacetime, denoted by π_i.

Recall that for $T_{\mu\nu}$ to satisfy the weak energy condition it is required that $\eta \geq 0$ and $\pi_i \geq -\eta$ for $i = 1, 2, 3$. Additionally, for it to satisfy the dominant energy condition it is also necessary that $\eta \geq \pi_i$ for $i = 1, 2, 3$. In our case, we have

$$\lambda_0 = -\eta \quad = \quad -\frac{2r_-r_+(r-r_-) - 4r^2\varepsilon(r-r_-)r'_- + \sqrt{\chi(u)}}{4r^3(r-r_-)^2}\,, \tag{7.32}$$

$$\lambda_1 = \pi_1 \quad = \quad -\frac{2r_-r_+(r-r_-) - 4r^2\varepsilon(r-r_-)r'_- - \sqrt{\chi(u)}}{4r^3(r-r_-)^2}\,, \tag{7.33}$$

$$\lambda_2 = \pi_2 \quad = \quad -\frac{r_-\left[2r(\varepsilon r r'_- - r_+) + r_-(r+r_+)\right]}{4r^3(r-r_-)^2}\,, \tag{7.34}$$

$$\lambda_3 = \pi_3 \quad = \quad \pi_2\,, \tag{7.35}$$

where we have defined

$$\chi(u) = r_-^2\left[\left(2r^2\varepsilon r'_- + r_-(r-r_+)\right)^2 + 4r^2(r-r_-)\left(2r^2r''_- - \varepsilon 2rr'_+ + \varepsilon(r_+r_-)'\right)\right]. \tag{7.36}$$

For the energy conditions to be satisfied we first need to assure that $\chi(u)$ is positive. From Eq. (7.36) it is easy to see that the first term is always positive, however the second term may be problematic. Luckily, this problematic term is exactly the same as the one obtained previously in Eq. (7.27) and which is required to be positive for the fluid stress-energy tensor to satisfy the weak and strong energy condition. Therefore, if we assure that the fluid stress-energy tensor satisfies these energy conditions we assure that we will have no problem with $\chi(u)$.

From Eqs. (7.26) and (7.32), we can easily show that $\eta = -\lambda_0 \geq 0$. Moreover, it is straighforward from Eq. (7.33) that $\pi_1 \geq -\eta$. Additionally, since $2rr_+ \geq r_-(r + r_+)$, we conclude from Eqs. (7.34) and (7.35) that $\pi_2 = \pi_3 \geq 0 \geq -\eta$. Thus, the weak energy condition is satisfied. Finally, from Eqs. (7.32) and (7.33) we have $\eta - \pi_1 \geq 0$ and it can be shown that $\eta - \pi_2 = \eta - \pi_3 \geq 0$ which means the dominant energy condition is also satisfied. We conclude that the requirements of the energy conditions on the fluid stress-energy tensor are sufficient to guarantee that the total stress-energy tensor always satisfies them too.

7.5 Radiating solution with time independent scalar charge

We have seen in the previous sections that the heterotic string case, i.e. $a = 1$, is special. Indeed from Eq. (7.11) we know that $2D(u) = r_-(u)$. Moreover, we saw that in this case if $D = \text{const}$ we have a fairly simple solution. Naturally, for this choice, the dilaton field (7.8) loses its time dependence on account of $r'_- = 0$ and the quantities that are time-dependent are M and Q.

When we consider $D = \text{const}$ we may treat both $a = 1$ and $a \neq 1$ cases simultaneously as Eq. (7.20) reduces to Eq. (7.21) and Eq. (7.22) reduces to Eq. (7.23) for $a = 1$. Moreover, we have $\rho = P = 0$ and consequently, the total stress-energy tensor is of the form of a null dust just like in the Vaidya [54, 55] and Vaidya-Bonnor [56] solutions.

For the rest of this chapter we will focus on the $a = 1$ radiating solution with time independent scalar charge. In this case we have $M(u) = r_+(u)/2$, $Q^2(u)/M(u) = 2e^{2\phi_0}D = \text{const}$ and

$$ds^2 = -\left(1 - \frac{2M(u)}{r}\right) du^2 - 2\varepsilon du dr + r^2 \left(1 - \frac{2D}{r}\right) d\Omega^2 \,, \tag{7.37}$$

$$F = -\frac{Q(u)}{r^2} du \wedge dr \,, \qquad e^{2\phi} = e^{2\phi_0}\left(1 - \frac{2D}{r}\right) \,, \tag{7.38}$$

$$\mu(u, r) = -\varepsilon \frac{2}{r^2} \frac{(r - D)}{r - 2D} M'(u) \,. \tag{7.39}$$

Finally, as we wish to test cosmic censorship and D is constant, henceforth we will always consider $\varepsilon = +1$, so that M decreases with (retarded) time and we have a chance of violating the CCC if it falls below D.

7.6 Collapse of shells with a radiating solution with time independent scalar charge exterior

Our study will focus on thin shells whose interior metric, ds_i^2 is given by Eq. (7.6) and the exterior metric by Eq. (7.37).

In the exterior spacetime, we define the shell as the hypersurface $\xi = \{x^\mu : u = \mathcal{U}(\tau),\, r = R(\tau)\}$. The first junction condition (4.14) tells us that $\theta_i = \theta_o$, $\varphi_i = \varphi_o$ and $R_i = R_o$. As in the previous chapters, we will focus on timelike shells, for which we have

$$\left(1 - \frac{2M}{R}\right) \dot{\mathcal{U}}^2 + 2\dot{\mathcal{U}}\dot{R} = 1 \,, \tag{7.40}$$

where, once again the overdot corresponds to derivatives with respect to the proper time τ. Note also that $M = M(u)$, and it is being evaluated at $u = \mathcal{U}(\tau)$.

Recall that the junction conditions for the scalar field merely impose $D_i = D_o$ and that the junction conditions for the Maxwell field determine, through Eq. (4.59), the existence of a surface density of charge at the shell which varies with R.

Solving Eq. (7.40) for $\dot{\mathcal{U}}$ yields

$$\dot{\mathcal{U}} = \frac{1}{\dot{R} + B} = \frac{B - \dot{R}}{1 - \frac{2M}{R}} \,, \tag{7.41}$$

where we have defined $B = \sqrt{A + \dot{R}^2}$ and $A = 1 - 2M/R$. With this equation we are able to determine the extrinsic curvature, which reads

$$K_{\tau\tau} = -\frac{M}{R^2 B} + \frac{\dot{M}\dot{\mathcal{U}}}{RB} - \frac{\ddot{R}}{B} \,, \tag{7.42}$$

$$K_{\theta\theta} = \frac{K_{\varphi\varphi}}{\sin^2\theta} = (R - D)B \,, \tag{7.43}$$

Once again we may obtain $K_{\tau\tau}$ through $K_{\theta\theta}$:

$$K_{\tau\tau} = -\frac{1}{\dot{R}}\frac{d}{d\tau}\left(\frac{K_{\theta\theta}}{R - D}\right) + \frac{\dot{M}\dot{\mathcal{U}}}{RB} - \frac{\dot{M}}{R\dot{R}B} \,. \tag{7.44}$$

Next, recall that the second junction condition when applied to this case yields Eqs. (4.68) and (4.69), with the difference that now $p = 0$ as we are considering a shell made of dust

$$[K_{\tau\tau}] = -4\pi\sigma \,, \tag{7.45}$$

$$[K_{\theta\theta}] = -4\pi\sigma R(R - 2D) \,. \tag{7.46}$$

Finally, we can determine how the shell radiates its energy and through it obtain a differential equation for σ by applying Eq. (4.47). This yields

$$\frac{d}{d\tau}\left[4\pi\sigma R(R - 2D)\right] = \frac{D^2\dot{R}(B_o - B_i) + (R - 2D)(R - D)\dot{\mathcal{U}}\dot{M}_o}{R(R - 2D)} \,. \tag{7.47}$$

To better understand the right-hand side of this equation we start by noticing that the second junction condition implies

$$B_o - B_i = -\frac{4\pi\sigma R(R - 2D)}{R - D} \,, \tag{7.48}$$

$$B_o + B_i = \frac{A_i - A_o}{4\pi R(R - 2D)\sigma}(R - D) \,, \tag{7.49}$$

Consequently, we have

$$B_o = \sqrt{1 - \frac{2M_o}{R} + \dot{R}^2} = \frac{A_i - A_o}{2\mathcal{M}}(R - D) - \frac{\mathcal{M}}{2(R - D)} = \frac{R - D}{R}\frac{M_o - M_i}{\mathcal{M}} - \frac{\mathcal{M}}{2(R - D)} \,, \tag{7.50}$$

where we have considered $\mathcal{M} = 4\pi\sigma R(R - 2D)$.

We are interested in following the procedure presented in for example Refs. [58–60]. These studies were done with a Schwarzschild interior and a Vaidya exterior so we expect our equations to reduce to the ones obtained therein when we take $D \to 0$. We start by defining the quantities α and $\hat{m}$ as

$$\hat{m} = M_o - M_i \,, \tag{7.51}$$

$$\alpha = \frac{\hat{m}}{\mathcal{M}} \,. \tag{7.52}$$

Applying these definitions to Eq. (7.47), yields the following expression for $\dot{\mathcal{U}}$

$$\dot{\mathcal{U}} = \frac{R\dot{\mathcal{M}}}{\dot{\hat{m}}(R-D)} + \frac{D^2\hat{m}\dot{R}}{\alpha(R-D)^2(R-2D)\dot{\hat{m}}}\,. \tag{7.53}$$

We may insert this expression into Eq. (7.41). By substituting the value of the square root by the one determined in Eq. (7.50), we obtain

$$\frac{\left[\dot{\alpha}R(R-D)(R-2D) - D^2\alpha\dot{R}\right]\left[2\alpha R(R-D)\dot{R} + 2\alpha^2(R-D)^2 - \hat{m}R\right]}{R^2(R-D)(R-2D)\left[2\alpha(R-D)\dot{R} - \hat{m}\right]} = \frac{\dot{\hat{m}}\alpha}{\hat{m}}\,. \tag{7.54}$$

Finally, we may also insert Eqs. (7.51) and (7.52) into Eq.(7.50), which then reads

$$\dot{R}^2 - \frac{M_o + M_i}{R} - \frac{\hat{m}^2}{4\alpha^2(R-D)^2} = \alpha^2\left(\frac{R-D}{R}\right)^2 - 1\,. \tag{7.55}$$

When we take D to be zero in Eqs. (7.54) and (7.55) we obtain

$$\dot{\hat{m}}\alpha(2\alpha R\dot{R} - \hat{m}) = \hat{m}\dot{\alpha}(2\alpha R\dot{R} + 2\alpha^2 R - \hat{m})\,, \tag{7.56}$$

$$\dot{R}^2 = \alpha^2 - 1 + \frac{M_o + M_i}{R} + \frac{\hat{m}^2}{4\alpha^2 R^2}\,, \tag{7.57}$$

which are exactly the same as the equations obtained in Refs. [58–60].

The problem we intend to study depends (assuming we are normalizing all quantities in regards to D) on the three unkowns $M_o(\tau)$, $\mathcal{M}(\tau)$ and $R(\tau)$. To completely determine it, we need three different equations, however we currently have only obtained two of them (Eqs. (7.54) and 7.55), which means we need to impose a third one.

In the spirit of Ref. [58], we could for example use Eq. (7.54) and assume the equation is free from singularities. Then, it should obey a relation similar to $k\left[2\alpha(R-D)\dot{R} - \hat{m}\right] = \dot{\alpha}R(R-D)(R-2D) - D^2\alpha\dot{R}$ where k is a constant. However, this option would require initial conditions not only for M_o, $\mathcal{M}$ and R but also depend on the value to be set for k which would increase the number of parameters in the study. This is of course undesirable as it would further complicate the analysis of CC in this setting.

Another option, which bypasses the k problem is to make a simpler assumption, related to Eqs. (7.47) and 7.53 and corresponds to setting $\dot{\mathcal{M}} = 0$, i.e. imposing $\mathcal{M}$ to be constant. Looking at Eq. (7.47) we reach the conclusion that this assumption corresponds to considering that all the energy that is being radiated by the exterior spacetime (leaves the shell) is counterbalanced by the energy received due to the scalar field radiation originated by the dilaton field[2]. Taking this into account, we obtain a simple expression for $\dot{\mathcal{U}}$ which can be inserted into Eq. (7.41), yielding

$$\frac{\dot{M}_o}{\dot{R}} = \frac{dM_o}{dR} = \frac{D^2\mathcal{M}}{(R-D)^2(R-2D)}(B_o + \dot{R})\,. \tag{7.58}$$

As we are studying shells collapsing from infinity and radiating, this means both $\dot{R}$ and $\dot{M}_o$ should be negative (assuming that $\dot{\mathcal{U}} \geq 0$). Note that Eq. (7.54) reduces to Eq. (7.58) when $\mathcal{M}$ is taken to be constant. As such, they represent the same restriction to the system we wish to solve.

We arrive at a system of two equations (Eqs. (7.55) and (7.58)) with two unknowns (M_o and $\dot{R}$) and, while it still requires the three initial conditions from before, it no longer requires the presence of another constant. For this reason, in our work we opted to use this simple framework to test CC.

[2]This assumption only holds for $D \neq 0$, otherwise the counterbalancing term vanishes.

Our interest lies in the attempt of overcharging an extremal or near extremal black hole. For motion to be allowed it is necessary for $\dot{R}^2$ to be positive so we will start by determining the asymptotic behavior of $\dot{R}^2$. To do so, we will change our notation and will henceforth define $M = M(R(\tau))$. As we will be interested in describing the motion of a shell collapsing from infinity we will consider $R(\tau_i) \to \infty$, where τ_i corresponds to the instant of the proper time of the shell at which it starts its collapse. Moreover, assuming the shell reaches the singularity, we have $R(\tau_f) = 2D$, where in this case τ_f is the instant of the proper time of the shell at which the shell reaches the endpoint of its collapse.

Expanding Eq. (7.55) for large values of R reads

$$\dot{R}^2 \approx \left(-1+\alpha^2\right) + \frac{M_o(\infty) + M_i - 2D\alpha^2}{R} + \mathcal{O}\left(\frac{1}{R}\right)^2 , \tag{7.59}$$

where $\hat{m} = M_o(\infty) - M_i$. Meanwhile, for small radii (near the singularity $R = 2D$), now considering $\hat{m} = M_o(2D) - M_i$, $\dot{R}^2$ reads

$$\dot{R}^2 \approx \frac{1}{4}\left(\frac{\mathcal{M}^2}{D^2} + \frac{2\left(M_o + M_i\right)}{D} + \frac{\hat{m}^2}{\mathcal{M}^2} - 4\right) + \frac{\left(D^2\alpha^2 - D\left(M_o + M_i\right) - 2\mathcal{M}^2\right)}{4D^3}(R-2D) + \mathcal{O}(R-2D)^2 . \tag{7.60}$$

From Eq. (7.59) we conclude that for the shell's collapse to be allowed then $\left(M_o(\infty) - M_i\right)^2 \geq \mathcal{M}^2$. Moreover, in the extremal case, the first term of Eq. (7.60) can be simplified to a perfect square which by definition will always be greater or equal to zero.

The fact that the shell is collapsing from infinity tells us that, at least initially[3], $\dot{R}$ corresponds to the negative root of $\dot{R}^2$ in Eq. (7.55). Therefore, the sign of Eq. (7.58) will depend only on the sign of $B_o + \dot{R}$. If we are to consider that the exterior spacetime is always radiating, i.e., $dM_o/dR \geq 0$ this imposes

$$B_o \geq -\dot{R} \Leftrightarrow 1 - \frac{2M_o}{R} \geq 0 \Leftrightarrow R \geq 2M_o . \tag{7.61}$$

Recall that in this coordinate system $2M_o$ represents the apparent horizon of the exterior spacetime. Eq. (7.61) leads to the conclusion that for CC to be tested it is necessary for $M_o(2D) \leq D$, otherwise, by continuity, there exists an $R_H > 2D$ (corresponding to the apparent horizon of the exterior spacetime) which the shell must cross. Note that for this to happen it is not needed for $\dot{R}$ to be zero, i.e., the shell does not need to bounce at (or before) R_H. Recall that the apparent horizon corresponds to a non-timelike hypersurface, that is covered, for a shell collapsing from infinity, by a future null-like event horizon. This future event horizon would then need to be crossed for the shell to reach the apparent horizon.

The issue of reaching a future event horizon is problematic since, as it can be seen from Fig. 7.1, this metric, like the Vaidya metric, is not geodesically complete and is not defined beyond this future event horizon. Several studies on the maximal extension of the Vaidya metric exist, e.g. Refs. [81, 82] where extensions into the black hole region are obtained. In those studies it is shown that while it is possible to extend the metric equipped only with the knowledge of $M(u)$ on the assymptotically flat region, its extension is not unique and a myriad of different solutions may be imposed to obtain a physically reasonable continuation of M. Moreover, the analytical extension in some cases does not offer a solution to this conundrum as it has been shown to not always obey to energy conditions [82].

[3] If there is a bounce then $\dot{R}$ will change its sign.

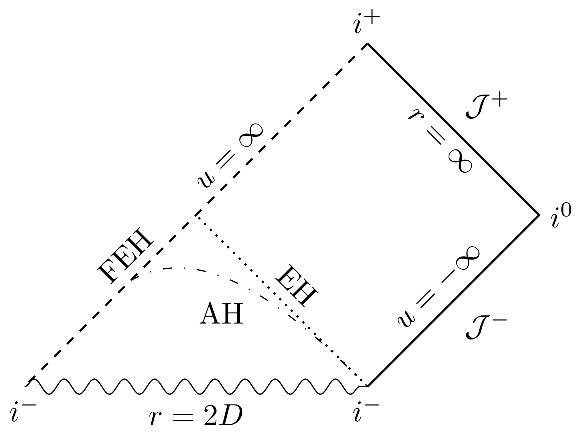


Figure 7.1: This figure represents the Penrose diagram of the Vaidya radiating spacetime. The spacelike hypersurface AH corresponds to the apparent horizon, i.e. the hypersurface $r = 2M(u)$. EH, corresponds to the event horizon which is the null-like boundary which covers the apparent horizon. FEH corresponds to the future event horizon of the spacetime, which coincides with the null hypersurface $r = 2M(u \to \infty)$.

To solve the problem of how to extend the metric, we could follow the approach of Refs. [81, 82] and define a proper continuation of M beyond the future event horizon. Nonetheless, this is outside of the scope of this thesis. Thus, we will not tackle the CCC for cases in which the shell crosses the future event horizon of the exterior spacetime.

Taking this information into account, in the following situations which will be analysed in regards of CC violation, we will always consider that the shell never crosses the future event horizon i.e. $2M_o(R) \leq R$.

We start by analysing the collapse with an extremal interior. In this case we have $M_i = D$ and from Eq. (7.48), assuming that the shell is near the singularity, we have

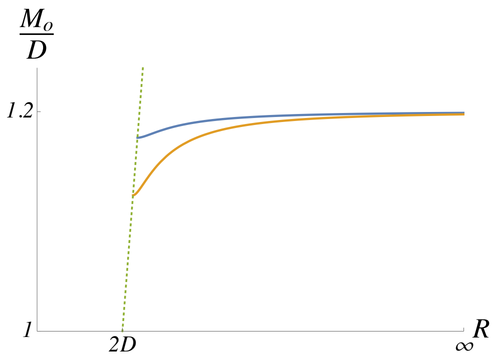
$$\sqrt{1 - \frac{M_o}{D} + \dot{R}^2} - \sqrt{\dot{R}^2} = -\frac{\mathcal{M}}{R - D}\,, \tag{7.62}$$

which, unless $\mathcal{M}$ is zero, always violates the energy conditions of the shell. Otherwise, if $\dot{R}^2$ is positive, then the right-hand side of the equation is also positive, which would mean $\mathcal{M}$ would need to be negative. The only other possibility would then be for $M_o = D$, in which case $\mathcal{M}$ could be zero, however this in turn would mean, from Eq. (7.48) that $M_o = M_i$. Consequently, the exterior spacetime would not be radiating and the interior and exterior would be exactly the same static spacetime, impossibilitating the study of collapse. We conclude that the CCC is upheld when we consider collapse with an extremal interior.

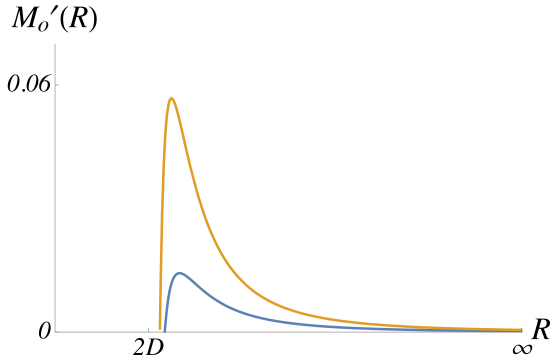
Let us consider collapse with a subextremal interior. In that case, both M_i and M_o need to be initially larger than D, moreover for CC to be violated it is necessary for $M_o(2D) \leq D$. Inserting this into Eq. (7.48) and assuming we are near the singularity yields

$$\sqrt{1 - \frac{M_o}{D} + \dot{R}^2} - \sqrt{1 - \frac{M_i}{D} + \dot{R}^2} = -\frac{\mathcal{M}}{R - D}\,. \tag{7.63}$$

On account of $1 - M_o/D \geq 0$ and $1 - M_i/D < 0$ we conclude that once again we violate the energy conditions in this framework. We observe that the left-hand side of the equation is positive which means that $\mathcal{M}$ has to be lower than zero. Thus, the CCC is also upheld in collapses with a subextremal interior



(a) M_o in terms of R



(b) M_o' in terms of R

Figure 7.2: In both subfigures we may observe in blue the behavior, when collapsing, of a shell with a subextremal exterior and an extremal interior that has $\mathcal{M} = 0.2D$ and $M_o(\infty) = 1.2D$. In orange we have the collapse of a shell with a subextremal exterior and an overcharged interior with $M_i = 0.8D$, $M_o(\infty) = 1.2D$ and $\mathcal{M} = 0.4D$. Fig. 7.2a shows us how M_o decreases from infinity in both cases until it reaches the apparent horizon $R = 2M_o$ represented by the dotted green line. At that point it satisfies $M_o'(R) = 0$ by virtue of Eq. (7.58). The behavior of the derivatives of each curve can be seen explicitly in Fig. 7.2b. Note that although we represent the quantities M_o and $M_o'(R)$ until the apparent horizon, we expect these to only be valid until the future event horizon.

in this framework. Note that this argument is quite general as it does not rely on the fact that $\mathcal{M}$ is constant.

We end this section by portraying the typical behavior of M_o when we consider initially subextremal shells collapsing from infinity with extremal and overcharged interiors, this in Fig. 7.2. We also show, in Fig. 7.3, how $V(R)$ varies for each of those shells. Only classically allowed situations are presented because otherwise Eq. (7.58) would give rise to complex values for M_o which are physically unrealistic.

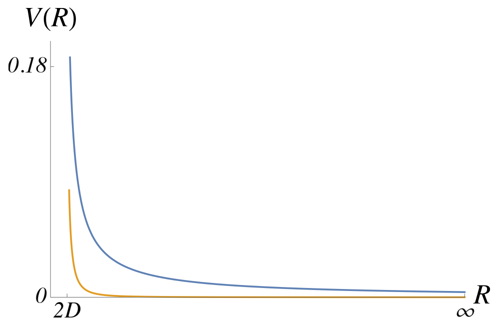


Figure 7.3: This figure presents the potentials of both of the shells described in Fig. 7.2. Both shells collapse into a future event horizon as their potential is always nonnegative and they obey the necessary energy conditions.

In both figures, we have the depicted quantities ending at $R = 2M_o(R)$. However, it should be taken into account that when the shell reaches that radius it has already crossed the future event horizon which should be the radius at which the depicted quantities end. Since we are unable to determine the future event horizon without knowing the complete behavior of M_o, we cannot precise it in the figures. Thus, we use instead the apparent horizon radius to define the boundary even if we can only guarantee that the quantities are properly defined until the future event horizon.

7.7 Conclusions

We have shown in this chapter that even when considering shells with exterior radiating solutions with time independent scalar charge and a GMGHS interior we cannot violate the CCC if we are to impose the proper mass of the shell $\mathcal{M}$ to be constant. Moreover, we note that the treatment that was used to conclude this fact can also be generalized for shells with variable proper mass as Eqs. (7.62) and (7.63) still impose the CCC to be upheld for the energy conditions related to $\mathcal{M}$ to be satisfied. We must, however, stress that we were only able to study collapse for the cases in which the shell never crossed the future event horizon of the exterior spacetime. To study these cases we would need to extend our solution beyond that horizon, which is a problem that we have not tackled in this thesis.

The junction conditions determined in 4.3 demand for the jump of the dilaton charge across the shell to vanish for motion to occur when we consider a GMGHS interior solution. Consequently, we conclude that CC may not be violated in this setting. Nonetheless, we have determined solutions that have variable D which might also be used to test CC, although in this case we may not resort to the thin shell formalism, at least with a GMGHS interior solution. Possible tests to these solutions could involve the possibility of having $D(u)$ becoming greater than $M(u)$ for initial conditions where $M(0) > D(0)$, either due to the dilaton charge increasing or due to the spacetime radiating its energy.

8 Conclusions

In this thesis we set out to investigate whether the four-dimensional low energy effective description of heterotic string theory which is represented by the GMGHS solution is consistent with the Cosmic Censorship Conjecture (CCC). To do so, we resorted to the thin shell formalism which showed us that in this context for dynamical shells, the presence of the dilaton field requires both the exterior and interior spacetime, that are joined by the thin shell, to have the same dilaton charge. Furthermore we also determined the conditions that are imposed by the formalism on the Maxwell field.

Supported by these results we determined static shell solutions, on account of the interior spacetime being flat with vanishing dilaton and Maxwell fields. We also showed that in this context, static shells with a GMGHS exterior arise as a natural extension of the known results when considering an exterior Schwarzschild solution. We were not able, however, to determine the stability of said static solutions as the jump in the dilaton charge by default does not allow the study of the potential that would govern their dynamics. Moreover, another open question that remains, is how they behave when one considers also a GMGHS interior with different dilaton charge.

Additionally, we also studied the dynamics of thin shells, when we demand all of the junction conditions to be satisfied by assuming the geometries inside and outside of the shell to be static. In this context we determined that they always collapse into black holes for the subextremal cases and that collapse into a naked singularity is not possible either due to the shells bouncing beforehand or because it would violate the weak and dominant energy conditions (which are the same on account of only having studied shells of dust). Natural extensions of these results that constitute an interesting problem are related to considering thin shells that also have pressure, i.e. their matter content is a perfect fluid instead of simply dust.

Because of the restriction imposed by the junction conditions related to the dilaton field on the possible methods of overcharging these solutions we found the need to explore a new class of time-dependent solutions that are detailed in Ref. [67]. These radiating solutions provided the means to test the CCC by considering them joined to a GMGHS interior. Equipped with these novel solutions we have also shown that, when considering a non-radiating thin shell which never crosses the future horizon of the exterior spacetime, we are not able to violate the CCC. Although we did not explore the radiating thin shell case in this context, the equations we resorted to for our proof appear to hold and suggest that, as long as the energy conditions are satisfied, cosmic censorship is upheld. We once more note that this study was only performed for shells whose matter content was dust so it is would be of interest extending these results for at least the perfect fluid case. Moreover, when tackling radiating shells, one more equation is required to fully determine the evolution. In this thesis we suggested an alternative to Refs. [58–60], as a simple way of closing the system of equations of motion in the presence of a nonvanishing dilaton charge. It is important to add that, to have a complete overview of the CCC in this setting, we would need to extend the spacetime beyond the future horizon. This would allow us to study collapse in the cases where the shell crosses the future event horizon of the exterior spacetime.

We also note that in this thesis we focused on testing the CCC resorting to the thin shell formalism due to its ability to provide exact equations to be solved. Nonetheless, the model is not the only way to perform this study. Analysis of overcharging GMGHS spacetimes with charged test particles is one current possibility, even if it doesn't include backreaction effects and neglects the finite size of the particles. Moreover, it will also prove invaluable to study the CCC for the new solutions determined in Ref. [67] by analysing how the different radiating solutions behave when subjected to the tests present in for example Refs. [8, 57, 65]. I.e. either by studying the spacetime alone and understand the end result of its evaporation or by using matching conditions to join it smoothly to another spacetime, without the presence of a thin shell inbetween, and studying its collapse. All these constitute interesting points of further work in analysing the CCC in this low energy, four-dimensional description of string theory.

On the whole, with the work that was developed in this thesis, we have shown, for the restrictions that we initially set out to study, that the CCC is upheld when considering collapsing thin shells in static, spherically symmetric solutions which constitute a description of four-dimensional, low energy effective heterotic string theory. These results hint that string theory models that give rise to these solutions seem to be consistent with the CCC being valid and therefore may be viable descriptions of string theory in this context. We note however that further exploration on this topic is necessary as our focus was only on spherically symmetric spacetimes and these represent idealized models, as we expect black holes in our universe to not be spherically symmetric due to having rotation. That said, a further analysis of GMGHS for rotating solutions is also required to solidify the conclusions that we have obtained in our work.

We end by pointing out that while this work only tackled classical aspects, one would expect, when studying gravitational collapse, that for small radii and high curvature regions of the collapse, stringy and quantum corrections should be taken into account that here were disregarded. The conclusion is, even if in our tests and in the tests suggested as future work, the CCC is shown to be valid, these corrections might end up influencing the outcome of gravitational collapse and the validity of the CCC in string theory.

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