Modeling some economical concepts in mathematical programming

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Abstract

In this work we analyze the choice and the combination of multi-objective functions in a mathematical programming problem. The study is motivated and based on the article Combining Equity and Utilitarianism in a Mathematical Programming Model of J.N.Hooker and H.P.Williams [7].

The goal is to model two combined functions in the mathematical programming problem: the utility function and the equity function. There are many definitions of equity, as well as approaches and models to this concept, and the inverse can be seen as an inequality problem. Here, we search the combination that maximizes the optimization problem.

Addressing a question raised by Hooker and Williams in [7], we analyze the choice of a new equity function for the problem of healthcare policy in allocating scarce resources to classes of patients. It is shown that the choice of a more suitable function, as the choice of an alternative MILP resolution method, allows to extend the range of possible solutions and lead to better results for this problem.

Keywords: Mathematical programming, Multi-objective programing, Optimization models

1. Introduction

This work is based on the article Combining Equity and Utilitarianism in a Mathematical Programming Model by J. N. Hooker and H. P. Williams (see reference [7]), where the authors consider the Rawlsian definition of equity, maximizing the minimum utility over individuals or classes of individuals. When the disparity of utility becomes too great, they change the objective to utilitarian.

Our goal is to change the equity function as well as change the mixed-integer linear programming (MILP) formulation of the problem to one that better suits the model. We analyze the mathematical programming model, combining an equity function with a utility function in a real problem followed by an investigation into optimization models.

We enumerate several equity functions, as possible replacements for the Rawlsian equity function, including the classic Nash bargaining solution and the egalitarian solution. The iniquity models considered here are the Atkinson index, the Gini index and the Theil index. Such functions are presented for the two-person case.

Finally, we suggest an alternative model with a more adequate equity function to solve the problem of how to allocate healthcare resources, discussed in the Hooker and Williamss article, followed by a comparative analysis of the results furnished by the two models that have been tested.

2. The initial problem

The article of J. Hooker and H. P. Williams [7] discusses the problem of combining the conflicting goals of fairness and utilitarianism in social policies in a single mathematical programming model. However, once a value for Δ has been settled upon, maximizing the social welfare function allows the same policy to be applied consistently whenever a budgeting decision is made.

$$
\begin{aligned}
&\max z \\
&\text{s.t:} \\
&z \leqslant \begin{cases} 2\min\{u_1, u_2\} + \Delta & \text{if } |u_1 - u_2| \leqslant \Delta \\ u_1 + u_2 & \text{otherwise} \end{cases} \\
&\qquad u_1, u_2 \geqslant 0
\end{aligned}
\tag{1}
$$

when the disparity of utility becomes bigger than Δ, the objective becomes progressively utilitarian. According to the authors, the level at which they set Δ is judgmental and likely to be a point of disagreement among the parties concerned. A parameter α (in this case equal to 2), transforms the linear combination in a continuous function. Hooker and

Williams make a MILP model for (1).

$$\begin{aligned}
&\max z \\
&\text{s.t:} \\
&z \leqslant (\textstyle\sum_i n_i - 1)\Delta + \sum_{i=1}^n n_i v_i \quad (a) \\
&u_i - \Delta \leqslant v_i \leqslant u_i - \Delta\delta_i, \text{ for all } i \quad (b) \\
&w \leqslant v_i \leqslant w + (M - \Delta)\delta_i, \text{ for all } i \quad (c) \\
&u_i \geqslant 0, \delta_i \in \{0,1\}, \text{ for all } i.
\end{aligned} \tag{2}$$

The model suffers from loss solutions, more specifically, the dominated solutions. This loss can lead to bad choices and bad decisions because the ideal is not always the optimal solution, the decision maker does not have a range of solutions. It is therefore important to examine other approaches with different methods of optimization.

The problem chosen and used by Hooker and Williams is the distribution of resources in health. These resources will be available for a limited number of treatments, the main issue being who should receive the treatment among groups of individuals. The candidates are individuals who can get different utilities. The treatments are pacemakers, hip replacements, and aortic valve replacements that are divided into three subgroups. Nine categories of candidates for coronary artery bypass grafts divided into three major groups. Heart transplant, kidney transplant and kidney dialysis candidates. The utility is defined as the following:

$$u_i = q_i y_i + a_i \tag{3}$$

where the utility is the average net gain measured in QALYs for a member of group i when the treatment is administered plus the gain when it is not. q_i is the average net gain when the treatment is administered and a_i the average net gain that results from medical management without the treatment in question. y_i is a binary variable that is equal to 1 when everyone in group i receives a specified treatment.

3. Mathematical Modeling of the Problem

Hooker and Williams suggest, as a follow up of their work, a new function of equity, one that already exists as an alternative to the Rawlsian definition. The models of equity and iniquity under consideration are:

- Equity models:
  - The Nash bargaining solution;
  - The egalitarian solution;

- Iniquity models:
  - The Atkinson index;
  - The Gini index;
  - The Theil index;

The Nash bargaining solution is the maximization of the Nash product [2] and for the two-person problem it takes the form:

$$arg \max_{u>u^0}(\log(u_1 - u_1^0) + \log(u_2 - u_1^0)) \tag{4}$$

After the minimum requirement is satisfied for all individuals, the remaining resources are distributed according to the conditions of each individual. Graphically, the function is $u^0 = 0$ when the value reaches the disagreement point, zero:

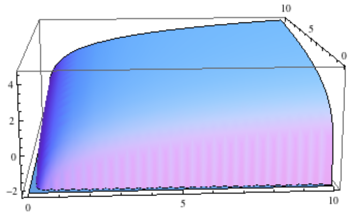

Figure 1: The Nash bargaining solution

Combining the utilitarian model with the Nash bargaining solution, we maximize z subjected to:

$$z \leqslant \begin{cases} \log(u_1 - u_1^0) + \log(u_2 - u_1^0) & \text{if } |u_1 - u_2| \leqslant \Delta \\ \quad \text{and } u > u^0 & \\ u_1 + u_2 \quad \text{if } |u_1 - u_2| > \Delta & \end{cases} \tag{5}$$

This model is expressed in terms of the individual's utility and not in "physical" amounts received. The utilities u_i for the individuals are set according to the perception of each one's utility. The difference between this model and the one used in the article is the goal of maximizing the sum of the products of the differences of each individual's utility and not the minimal use of the set of individuals. Equity is now defined in terms of the individual's well-being and not on levels of satisfaction as measured in the Rawlsian function [10].

The egalitarian solution is the point in the feasible region where the set of all individuals reach a maximum increase of equal value with respect to the disagreement point [2]:

$$\max\{u > u^0 | u_i - u_i^0 = u_j - u_j^0, \forall i, j \in N\} \tag{6}$$

For the two-person problem:

$$u_2 = u_1 - u_1^0 + u_2^0 \tag{7}$$

Once more, when combining the utilitarian model with the egalitarian solution, we maximize z subjected to:

$$z \leqslant \begin{cases} u_1 - u_1^0 + u_2^0 & \text{if } |u_1 - u_2| \leqslant \Delta \\ u_1 + u_2 & \text{if } |u_1 - u_2| > \Delta \end{cases} \tag{8}$$

The downside is the specificity of this model, making it somewhat unadaptable. As in the Nash bargaining solution, these results are measured in welfare levels.

The Atkinson index is one of the welfare measures most referenced in iniquity [3]. This index allows the variation in sensitivity of inequalities in different parts of the income distribution through a sensitivity parameter ϵ, known as "iniquity aversion parameter", which can range from 0, when the individual is indifferent to the nature of income distribution since the distribution is equitable, and infinite, when the individual is concerned with only the group income position that possess lower incomes. The Atkinson index is directly related to the class of additive functions of social welfare:

$$W = \frac{1}{N} \sum_{i=1}^{n} f(u_i) \tag{9}$$

where $f(u_i)$ is:

$$f(u_i) = \begin{cases} \frac{1}{1-\epsilon} u_i^{1-\epsilon} & \epsilon \neq 1 \\ \log u_i & \epsilon = 1 \\ u_i & \epsilon = 0, \quad \text{utilitarian situation} \end{cases} \tag{10}$$

Combining the utilitarian model with the Atkinson index, we maximize z subjected to:

$$z \leqslant \begin{cases} \log u_i & \text{if } |u_1 - u_2| \leqslant \Delta \\ & \text{and } \epsilon = 1, i = 1, 2 \\ \frac{1}{1-\epsilon} u_i^{1-\epsilon} & \text{if } |u_1 - u_2| \leqslant \Delta \\ & \text{and } \epsilon \neq 1, i = 1, 2 \\ u_i & \text{if } |u_1 - u_2| > \Delta, i = 1, 2 \end{cases} \tag{11}$$

As ϵ increases, the function assigns a greater weight to increases in low incomes in the production of welfare. This means that the function of social welfare must have $W'' < 0$, i.e. it must be concave. Graphically, when the index is lower, income distribution is more equitable. The Atkinson index is more efficient when used in a comparison between group problems. The index has the disadvantage of being unintuitive [9] .

The Gini index is a measure of statistical dispersion intended to represent the income distribution [1], developed by the Italian statistician and sociologist Corrado Gini in an article in 1912 [6] . This index measures the inequality among values of a frequency distribution, such as income levels. Analytically related to the functions of social welfare, inequality complex measure and a synthetic index [5]:

$$G = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |u_i - u_j|}{n^2 2\mu} \tag{12}$$

where $\mu = \frac{\sum_{i=1}^{n} u_i}{n}$. For the two-persons case:

$$\begin{aligned} G &= \frac{\sum_{i=1}^{2} \sum_{j=1}^{2} |u_i - u_j|}{2^2 * 2\mu} \\ &= \frac{(|u_1 - u_2| + |u_1 - u_1| + |u_2 - u_2| + |u_2 - u_1|)}{8\mu} \\ &= \frac{(|u_1 - u_2|)}{4\mu} \end{aligned} \tag{13}$$

Combining the utilitarian model with the new index:

$$z \leqslant \begin{cases} \frac{(|u_1 - u_2|)}{4\mu} & \text{if } |u_1 - u_2| \leqslant \Delta \\ u_1 + u_2 & \text{if } |u_1 - u_2| > \Delta \end{cases} \tag{14}$$

A characteristic of the Gini index is the fact that it gives information about the distribution of income and not about the characteristics of the income distribution, such as location and format. It is a good indicator for a general analysis of a problem with a large population. The graphical representation can also be compared over time, and is simple to calculate and interpret [9]. However the coefficient suffers from the disadvantage of being affected by the average value, measured by an arbitrary source [8] and does not allow comparisons between and within groups.

The Theil index is the appropriate index for data with a degree of aggregation hierarchy [1]:

$$T = \frac{1}{2} * (\frac{u_1}{\mu}) * \log(\frac{u_1}{\mu}) + \frac{1}{2} * (\frac{u_2}{\mu}) * \log(\frac{u_2}{\mu}) \tag{15}$$

where $\mu = \frac{u_1 + u_2}{2}$. Graphically:

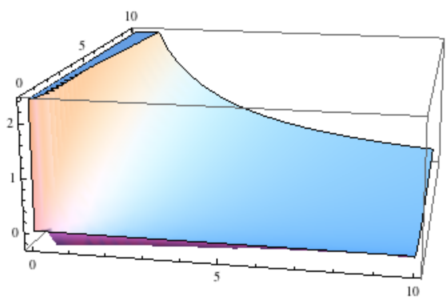

Figure 2: The Theil Index

When the function is zero, each individual has the same income, thus representing the case of perfect equality. The function on the opposite extreme is the situation of maximum iniquality. Combined with the utilitarian function:

$$z \leqslant \begin{cases} \frac{1}{2} * (\frac{u_1}{\mu_u}) * \log(\frac{u_1}{\mu}) + (\frac{1}{2}) * (\frac{u_2}{\mu}) * \log(\frac{u_2}{\mu}) \\ \quad \text{if } |u_1 - u_2| \leqslant \Delta \\ u_1 + u_2 \quad \text{if } |u_1 - u_2| > \Delta \end{cases} \tag{16}$$

The Theil index is a recommended choice as it has a very flexible structure [1]. It is also a convenient tool for data with a certain hierarchy as it can be performed for various components, dividing the analysis into groups. However, it presents a major disadvantage as its values are not always comparable across different units. It is complex to calculate and interpret and differs greatly when the distribution varies without regard to the variation in the distribution of the values [9].

After this analysis of the equity equations suggested by the authors, we conclude that the appropriate choice, having regard to the example where the new model is applied, is an equality function adapted to the problem. The proposed model suggests a change in the equity function. Keeping the utility function, which is already in the common form, the equity function is changed to one more coherent with the sample, which is representative of equitable relationships. According to the definition of equity, all interventions require at least a subgroup to receive the treatment. The previous function, the max min of utilities, is replaced by an equity function that selects at least a subset of candidates among the 9 different groups of treatments. After reviewing the existing equity functions as well the existing iniquity functions, the ideal function obtained is represented by z_2 and the goal is to maximize the minimum:

$$\begin{aligned} &\max\min z_2 = \textstyle\sum_{k=1}^{m} M_k \sum_{j=mki}^{mkn} y_j, \\ &k = 1, ..., m, j = mki, ...mkn \end{aligned} \tag{17}$$

Where M is a penalty imposed on the model, to not choose treatments at a lower cost or limited to maximize utility. $m = 9$ are the possible treatments and mkn the variation of the subgroups, which can range from 1 to 12. The restrictions are those used in the initial model (positive or zero utilities) with the addition of budget constraint and imposing at least one subgroup in each treatment:

$$\begin{aligned} &\textstyle\sum_{i}^{n} (n_i c_i y_i) \leqslant B \\ &\textstyle\sum_{j=mki}^{mkn} y_j \geqslant 1 \end{aligned} \tag{18}$$

where c_i is the added cost per patient to administer the treatment. The available budget, B, is equivalent to three million pounds.

The initial model comes from the utility function and the Rawlsian equity definition. The original utility function is the sum of all utilities while the equity function is to maximize the minimum utility between individuals:

$$\begin{cases} \max U(u) = \sum_{i=1}^{n} u_i \\ \max\min\{u_i\} & i = 1, ..., n \\ u_i \geqslant 0 \end{cases} \tag{19}$$

The proposed model:

$$\begin{cases} \max U(u) = \sum_{i=1}^{n} u_i \\ \max\min \sum_{k=1}^{m} M_k \sum_{j=mki}^{mkn} y_j \\ \ i = 1, ..., n,\ k = 1, ..., m,\ j = mki, ...mkn \\ u_i \geqslant 0 \\ M \geqslant 0 \\ y_i \in \{0, 1\} \end{cases} \tag{20}$$

M is the penalty assigned to each group k. Groups can also be divided into subgroups mkn.

The proposed method is the ϵ constraint method. This method replaced the optimization method performed. It will find solutions that will be better than those obtained by the MILP method, and will also find solutions that, while not great, could be included in the final decision.

The ϵ constraint method for the initial model, the minimum utilities and the sum of utilities, takes the form:

$$\begin{aligned} &\max \alpha \\ &\text{subject to:} \\ &\alpha \leqslant u_i, i = 1, ..., n \\ &U(u) = \textstyle\sum_{i=1}^{n} u_i \\ &U(u) \geqslant \epsilon \\ &u_i \geqslant 0, i = 1, ..., n \end{aligned} \tag{21}$$

And the suggested model after the linearization of the max min problem:

$$\begin{aligned} &\max \alpha \\ &\text{subject to:} \\ &\alpha \leqslant z_2 \\ &z_1 = \textstyle\sum_{i}^{n} (q_i y_i + a_i) \\ &z_1 \geqslant \epsilon \\ &z_2 = \textstyle\sum_{k=1}^{m} M_k \sum_{j=mki}^{mkn} y_j \\ &u_i \geqslant 0 \\ &\textstyle\sum_{i}^{n} (n_i c_i y_i) \leqslant B \\ &\textstyle\sum_{j=mki}^{mkn} y_j \geqslant 1 \end{aligned} \tag{22}$$

The ϵ constraint method allows reaching for a larger set of Pareto solutions and also picking dominated and weakly-dominated solutions. Thus, instead of choosing the parameter Δ variant to compromise between equity and utility, it also has access to more solutions and choice between the combination of the two objectives. This allows a fair analysis of the case and does not depend so much on personal judgments. The model was implemented in the IBM CPLEX program.

4. Results

The data used is the same as that in the article [7] and represented in Table 1. The explanation is the same as the one in the article [7] and is as follows:

"There are nine types of treatments, as mentioned above, and the group dimensions are an approximation based on several estimates of the relative frequency of each operation in the United States. The relative frequency of kidney dialysis patients is reduced to one-third the prevailing rate. The groups corresponding to pacemakers, hip replacements, and aortic valve replacements are divided into three subgroups, of which subgroup B represents the average cost per QALY reported by Briggs and Gray (2000). Groups A and C reflect deviations from the average and allow policy makers to consider different prognoses among patients with the same basic disease. The nine categories of candidates for coronary artery bypass grafts (CABGs) are explicitly distinguished by Briggs and Gray, and the costs per QALY reflect their estimates. The kidney dialysis candidates are categorized by expected life span while on dialysis, to reflect the fact that the cost per patient as well as the QALYs gained depend on how long the patient survives. The relative size of each category is based on survival rates for the United States reported by the National Kidney and Urologic Diseases Information Clearinghouse (2010). The annual cost per patient is derived from (a) Briggs and Grays (2000) estimate of 14,000 per QALY; (b) an average of 0.688 QALYs per year of dialysis, based on converting to a 0-1 scale the Index of Well Being for such patients reported in Evans et al. (1985), which Briggs and Gray cite as their source; and (c) an average of 0.85 additional years of life obtained for each year spent on dialysis. This results in a per- capita annual dialysis cost of (14,000)(0.688)(0.85), or about 8,200. Some categories are further subdivided by prognosis due to the high per-patient cost, because otherwise, funding a single category would consume a large fraction of the budget. The expected QALYs without intervention, given by α_i, depends entirely on such population characteristics as age, general state of health, and environment."

The QALY expected without intervention, α_i, in the new model is called a_i and depends entirely on population characteristics such as age, health status and the environment where they live. The data used does not represent any particular population, selected to represent a possible set of circumstances.

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention a_i	Subgroup size n_i
Pacemaker for atrioventricular heart block					
Subgroup A	3,500	3	1,167	13	35
Subgroup B	3,500	5	700	10	45
Subgroup C	3,500	10	350	5	35
Hip replacement					
Subgroup A	3,000	2	1,500	3	45
Subgroup B	3,000	4	750	4	45
Subgroup C	3,000	8	375	5	45
Valve replacement for aortic stenosis					
Subgroup A	4,500	3	1,500	2.5	20
Subgroup B	4,500	5	900	3	20
Subgroup C	4,500	10	450	3.5	20
CABGa for left-main disease					
Mild angina	3,000	1.25	2,400	4.75	50
Moderate angina	3,000	2.25	1,333	3.75	55
Severe angina	3,000	2.75	1,091	3.25	60
CABG for triple-vessel disease					
Mild angina	3,000	0.5	6,000	5.5	50
Moderate angina	3,000	1.25	2,400	4.75	55
Severe angina	3,000	2.25	1,333	3.75	60
CABG for double-vessel disease					
Mild angina	3,000	0.25	12,000	5.75	60
Moderate angina	3,000	0.75	4,000	5.25	65
Severe angina	3,000	1.25	2,400	4.75	70
Heart transplant	22,500	4.5	5,000	1.1	2
Kidney transplant					
Subgroup A	15,000	4	3,750	1	8
Subgroup B	15,000	6	2,500	1	8
Kidney dialysis					
Less than 1 year survival					
Subgroup A	5,000	0.1	50,000	0.3	8
12 years survival					
Subgroup B	12,000	0.4	30,000	0.6	6
25 years survival					
Subgroup C	20,000	1.2	16,667	0.5	4
Subgroup D	28,000	1.7	16,471	0.7	4
Subgroup E	36,000	2.3	15,652	0.8	4
510 years survival					
Subgroup F	46,000	3.3	13,939	0.6	3
Subgroup G	56,000	3.9	14,359	0.8	2
Subgroup H	66,000	4.7	14,043	0.9	2
Subgroup I	77,000	5.4	14,259	1.1	2
At least 10 years survival					
Subgroup J	88,000	6.5	13,538	0.9	2
Subgroup K	100,000	7.4	13,514	1	1
Subgroup L	111,000	8.2	13,537	1.2	1

Table 1: Data for Healthcare Example

The results in the article using the MILP model are a set of binary variables that determine which groups should receive particular treatments fairly, or in an utilitarian way, in terms of Δ. They concluded that combining equity and efficiency leads to interesting and unexpected results. The new model has different results, the solution being not only composed of binary variables. This will vary depending on a parameter, the variable M. Setting it optimally depends on the results obtained, to this end some simulations were performed. The M parameter is assigned with a higher value for treatments at a lower cost and with a lower value for treatments with a higher cost, thus allowing the model to make fairer decisions and not choosing the treatments that minimize the cost. Note that the options for the parameter M are multiple, and it was decided to start with a quote on a scale of 0 to 10.

Among all the simulations we chose the set:

$$M = [9, 15, 7, 15, 15, 15, 1, 2, 3]$$

with the results in figure 1.1.

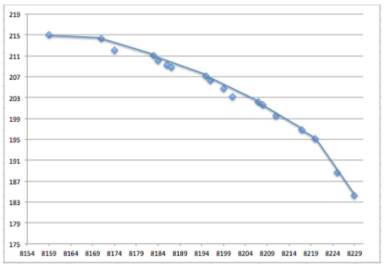

Figure 3: Results with the variation of M

We chose this set because the solutions have higher optimum values and there are also various alternatives. With the previous method, 9 of 18 solutions would not be found, corresponding to 50% of the solutions. After the tests we assumed that this set of values for M is the widest of the simulations presented.

However, it is not just around the variable M that we can draw conclusions about the new model. Another analysis is the variation in costs c_i. By varying within subgroups, namely, within each treatment, 30 simulations were performed obtaining, according to the ϵ constraint method, dominated and non-dominated variables. The aim is to carry out a statistical analysis on the non-dominated and unsupported variables. With this method we can see the dominated and weakly-dominated variables that do not appear in the other method. This way we can say that the new model is more flexible. The goal now is, from 30 simulations, to verify how much was lost by ignoring the unsupported variables. From the sensitivity analysis around the sample, we conclude:

- the amount of unsupported variables;
- the mean of unsupported variables;
- the min of unsupported variables;
- the max of unsupported variables;
- the standard deviation of unsupported variables;

It is of interest to analyze the processing time of the model by performing the same statistics around the CPU variable:

- the computing time of CPU;
- the mean of CPU;
- the min of CPU;
- the max of CPU;
- the standard deviation of CPU;

A simulation is made, for example, around the variables c_i for $i = 1, 2, 3$. It ranges up costs to c_i = [5000,3500,4000]. The graph of the solutions obtained is:

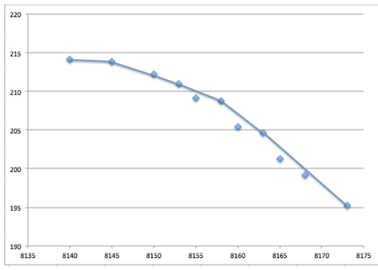

Figure 4: Results with the variation of c

Where there are 10 solutions of which only 7 are not dominated. Similarly the following simulations were recorded:

Iteration	Variable c_i	Amount of solutions	non dominated solutions	% of non dominated solutions
1	$c_i = 1, 2, 3 = [3000, 3000, 3000]$	10	7	70%
2	$c_i = 1, 2, 3 = [4000, 4000, 4000]$	10	6	60%
3	$c_i = 1, 2, 3 = [5000, 3500, 4000]$	11	7	63,64%
4	$c_i = 1, 2, 3 = [3500, 4000, 5000]$	12	7	58,33%
5	$c_i = 4, 5, 6 = [2500, 2500, 2500]$	11	6	54,55%
6	$c_i = 4, 5, 6 = [3500, 3500, 3500]$	11	8	72,73%
7	$c_i = 4, 5, 6 = [4500, 3000, 3500]$	10	6	60%
8	$c_i = 4, 5, 6 = [6000, 3000, 3500]$	8	6	75%
9	$c_i = 7, 8, 9 = [4000, 4000, 4000]$	10	8	80%
10	$c_i = 7, 8, 9 = [5000, 5000, 5000]$	11	7	63,64%
11	$c_i = 7, 8, 9 = [6000, 4500, 5000]$	11	6	54,55%
12	$c_i = 10, 11, 12 = [2500, 2500, 2500]$	7	6	85,71%
13	$c_i = 10, 11, 12 = [3500, 3500, 3500]$	10	8	80%
14	$c_i = 10, 11, 12 = [4500, 3000, 3500]$	7	7	100%
15	$c_i = 13, 14, 15 = [2500, 2500, 2500]$	8	5	62,5%
16	$c_i = 13, 14, 15 = [3500, 3500, 3500]$	10	7	70%
17	$c_i = 13, 14, 15 = [4500, 3000, 3500]$	8	8	100%
18	$c_i = 13, 14, 15 = [5000, 3500, 4500]$	8	5	62,5%
19	$c_i = 16, 17, 18 = [2500, 2500, 2500]$	11	6	54,55%
20	$c_i = 16, 17, 18 = [3500, 3500, 3500]$	11	7	63,64%
21	$c_i = 16, 17, 18 = [5000, 4000, 2000]$	8	5	62,50%
22	$c_i = 19 = [22000]$	10	7	70%
23	$c_i = 19 = [23000]$	9	6	66,67%
24	$c_i = 20, 21 = [10000, 10000]$	9	6	66,67%
25	$c_i = 20, 21 = [20000, 20000]$	8	7	87,50%
26	$c_i = 20, 21 = [17000, 15000]$	7	6	85,71%
27	$c_i = 22..33 =$ [- 500 em todas as parcelas]	10	7	70%
28	$c_i = 22..33 =$ [+ 500 em todas as parcelas]	9	5	55,56%
29	$c_i = 22..33 =$ [7000, 12000 ,22000 ,28500 ,36500 ,46000 ,58000 ,66500 ,77000 ,89000 ,110000 ,121000]	10	5	50%
30	$c_i = 22..33 =$ [5000, 14000, 20500, 30000, 36000, 46500, 56500, 68000, 78000, 88000,105000,131000]	9	8	88,89%

Table 2: Simulation results

The resulting graphics are found in section 4.1. When the variable c_i does not vary, M varies with the values previously proposed. It is therefore concluded that on average 68,66% of the solutions are non-dominated, resulting in 3,34% solutions dominated to be left aside. The standard deviation of non-dominated solutions is 0.9738, i.e. the values found tend to be close to the average. The minimum percentage of non-dominated solutions in a simulation is 50 % and the maximum rate is 100 %.

For the processing time, the results obtained are shown in the following table. The time is recorded in seconds and hundredths of seconds:

Iteration	Variable c_i	CPU (hours:min:secs:cent.secs)	accumulated CPU
1	$c_i = 1,2,3 = [3000, 3000, 3000]$	00:00:24:17	00:00:24:17
2	$c_i = 1,2,3 = [4000, 4000, 4000]$	00:00:18:56	00:00:42:73
3	$c_i = 1,2,3 = [5000, 3500, 4000]$	00:00:23:29	00:01:06:02
4	$c_i = 1,2,3 = [3500, 4000, 5000]$	00:00:15:03	00:01:21:05
5	$c_i = 4,5,6 = [2500, 2500, 2500]$	00:00:11:73	00:01:32:78
6	$c_i = 4,5,6 = [3500, 3500, 3500]$	00:00:14:12	00:01:46:90
7	$c_i = 4,5,6 = [4500, 3000, 3500]$	00:00:20:37	00:02:07:27
8	$c_i = 4,5,6 = [6000, 3000, 3500]$	00:00:23:00	00:02:30:27
9	$c_i = 7,8,9 = [4000, 4000, 4000]$	00:00:11:92	00:02:42:19
10	$c_i = 7,8,9 = [5000, 5000, 5000]$	00:00:09:45	00:02:51:64
11	$c_i = 7,8,9 = [6000, 4500, 5000]$	00:00:13:15	00:03:04:79
12	$c_i = 10,11,12 = [2500, 2500, 2500]$	00:00:12:17	00:03:16:96
13	$c_i = 10,11,12 = [3500, 3500, 3500]$	00:00:15:48	00:03:32:44
14	$c_i = 10,11,12 = [4500, 3000, 3500]$	00:00:16:71	00:03:49:15
15	$c_i = 13,14,15 = [2500, 2500, 2500]$	00:00:19:11	00:04:08:26
16	$c_i = 13,14,15 = [3500, 3500, 3500]$	00:00:15:34	00:04:23:50
17	$c_i = 13,14,15 = [4500, 3000, 3500]$	00:00:14:42	00:04:37:92
18	$c_i = 13,14,15 = [5000, 3500, 4500]$	00:00:32:75	00:05:10:67
19	$c_i = 16,17,18 = [2500, 2500, 2500]$	00:00:15:81	00:05:26:48
20	$c_i = 16,17,18 = [3500, 3500, 3500]$	00:00:18:00	00:05:44:48
21	$c_i = 16,17,18 = [5000, 4000, 2000]$	00:00:09:14	00:05:53:62
22	$c_i = 19 = [22000]$	00:00:16:71	00:06:10:33
23	$c_i = 19 = [23000]$	00:00:15:50	00:06:25:83
24	$c_i = 20,21 = [10000, 10000]$	00:00:29:01	00:06:54:84
25	$c_i = 20,21 = [20000, 20000]$	00:00:17:56	00:07:12:40
26	$c_i = 20,21 = [17000, 15000]$	00:00:21:23	00:07:33:63
27	$c_i = 22..33 =$ [- 500 em todas as parcelas]	00:00:15:85	00:07:49:48
28	$c_i = 22..33 =$ [+ 500 em todas as parcelas]	00:00:18:43	00:08:07:91
29	$c_i = 22..33 =$ [7000, 12000 ,22000 ,28500 ,36500 ,46000 ,58000 ,66500 ,77000 ,89000 ,110000 ,121000]	00:00:13:98	00:08:21:89
30	$c_i = 22..33 =$ [5000, 14000, 20500, 30000, 36000, 46500, 56500, 68000, 78000, 88000,105000,131000]	00:00:19:00	00:08:40:89

Table 3: Simulation results of CPU

The time that the program takes to solve the model is very small, a matter of seconds for each simulation. The average CPU is 00:00:17:28 seconds and hundredths of seconds, which is converted to the second value of 17,278. The standard deviation is 5.356660952, measured in seconds, the minimum time is 00:00:09:14, namely 9.14 seconds, and the maximum of 32,75 seconds.

The results of a multi-objective problem extend the range of solutions under examination. It discovers a set of solutions with different features, not only the optimums, establishing commitments between different aspects of assessment [4]. At the time of decision making, we can thus assign the treatments more fairly, taking into account the existence of more options into the analysis.

4.1. The resultants graphics

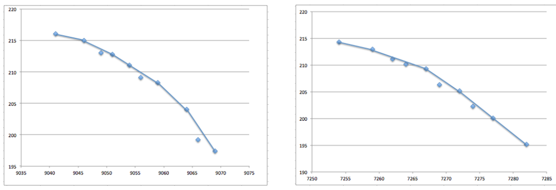

Figure 5: Simulation 1 and 2

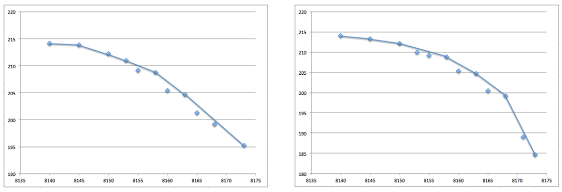

Figure 6: Simulation 3 and 4

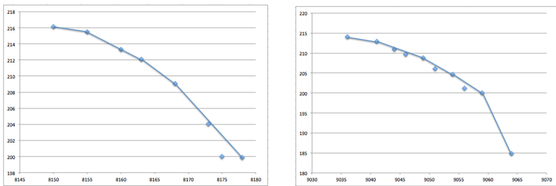

Figure 7: Simulation 5 and 6

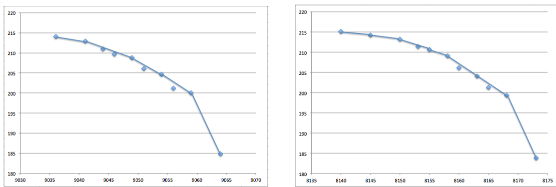

Figure 8: Simulation 7 and 8

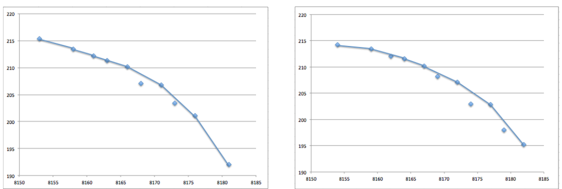

Figure 9: Simulation 9 and 10

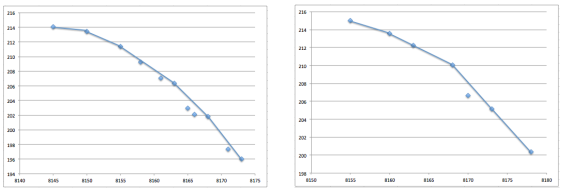

Figure 10: Simulation 11 and 12

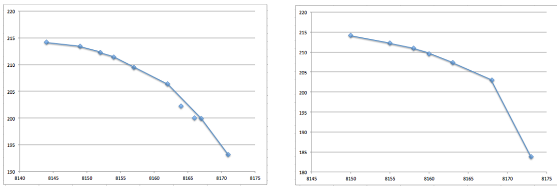

Figure 11: Simulation 13 and 14

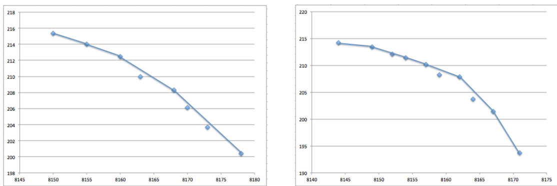

Figure 12: Simulation 15 and 16

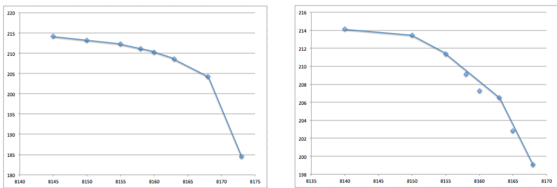

Figure 13: Simulation 17 and 18

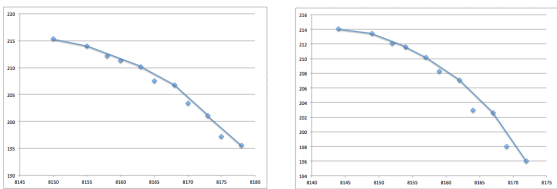

Figure 14: Simulation 19 and 20

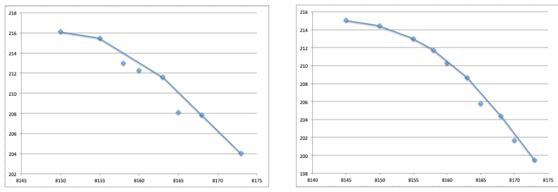

Figure 15: Simulation 21 and 22

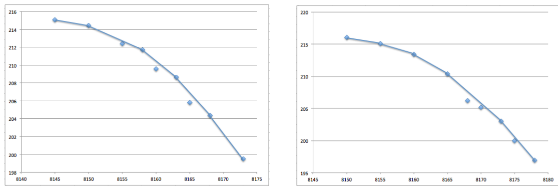

Figure 16: Simulation 23 and 24

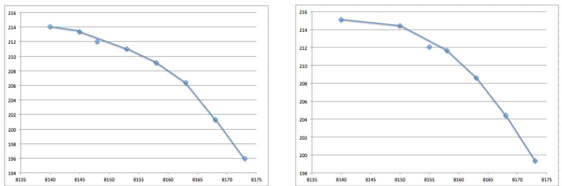

Figure 17: Simulation 25 and 26

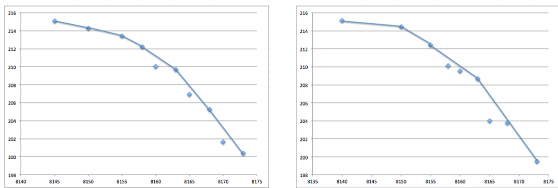

Figure 18: Simulation 27 and 28

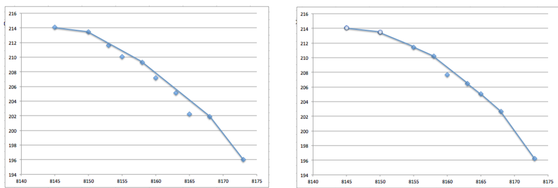

Figure 19: Simulation 29 and 30

5. Conclusions

Given the objectives presented above, i.e. the exploration of alternative proposals by Hooker and Williams in their article "Combining Equity and Utilitarianism in the Mathematical Programming

Model" [7] and the implementation of a new function and use of a new model for the optimization, we conclude that, although the results are different from the initial model, the goals of this thesis have been achieved. The establishment and operation of the combination of the different equity functions with the utility function proposed by Hooker and Williams, and the implementation of a new model for the health example in the article, achieved prominent results.

Achieving the first objective allowed us to observe the behavior of the combination of different functions with the change of the function of equity. The conclusions drawn, as well as the interpretation of each experimentation, are quite different and cannot be compared with each other since they do not arrive at the same conclusion once the results are not in the same measure. The function used by the authors is measured in levels of satisfaction, the solution of the bargaining theorem is measured in levels of well-being, and some of the iniquity measures in "physical" amounts. The main mission of this analysis is to see if these functions are able to adapt to the example of distribution of resources in health. In real problems, the definition of equity has to be adapted and chosen carefully through an exploratory analysis. Thus, it is concluded that the best function to use in the second part of the objectives, implementation of a new function of equity and implementation of a new optimization model for the problem of distribution of scarce resources in health, is an equity function built around the data. The model is a model *min max* with the implementation of the ϵ constraint method for optimization of the combining functions (equity and utility).

Achieving the second goal, lets us contribute to the article *"Combining Equity and Utilitarianism in the Mathematical Programming Model"* in a critical way, giving another hypothesis to solve the mathematical programming problem by following a different approach. The choice of the ϵ constraint method was performed after analysis and attempts to implement other methods. The results obtained are more complete in terms of the solutions. It is believed that there are still other hypotheses to solve the problem depending on the choice of other equality functions.

Based on the experience and the conclusions reached in the construction of this work, it is clear that the path to the optimal model is difficult and long. To achieve the best combination of functions and to find the optimal solutions programming problem, we need to use a trial-and-error approach.

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