Instituto Superior Técnico
Departamento de Matemática

FUNDAMENTOS DE TOPOLOGIA E ANÁLISE REAL
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1. (4.0 val.) Let $A \subset \mathbb{R}^N$ be bounded, $\mathcal{J}(\mathbb{R}^N)$ denote the class of Jordan measurable sets, c_N, $\overline{c_N}$ the Jordan and upper Jordan content. Show that:

(a) if $\overline{c_N}(A) = 0$ then $A \in \mathcal{J}(\mathbb{R}^N)$ and $int(A) = \emptyset$;

(b) if $A \in \mathcal{J}(\mathbb{R}^N)$, then $c_N(A) = 0 \Leftrightarrow int(A) = \emptyset$. Is it true in general that $\overline{c_N}(A) = 0 \Leftrightarrow int(A) = \emptyset$?

(c) let $D \subset R$ be dense in R, with R some bounded rectangle. If $D \in \mathcal{J}(\mathbb{R}^N)$, then $c_N(D) = c_N(R)$. Show that this is not necessarily true replacing $\mathcal{J}(\mathbb{R}^N)$ by the Lebesgue measurable sets $\mathcal{L}(\mathbb{R}^N)$ and c_N by the Lebesgue measure m_N.

2. (3,0 val.) Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable, $n \in \mathbb{N}$, and $E := \{x \in X : \lim f_n(x) \text{ exists in } \overline{\mathbb{R}}\}$. Show that E is measurable.

3. (3,0 val.) Let $f \in L^1([0,1])$, $f \geq 0$ and $A = \{x \in [0,1] : f(x) > 0\}$. Then A is Lebesgue measurable, $\sqrt[n]{f(x)} \in L^1([0,1])$, for all $n \in \mathbb{N}$ and

$$\lim_n \int_0^1 \sqrt[n]{f(x)}\, dx = m(A).$$

4. (3,0 val.) Prove or disprove:

(a) Let $(X, \mathcal{M}, \mu)$ be a measure space, $A_n \in \mathcal{M}$, $n \in \mathbb{N}$, such that $A_n \searrow A$, then $A \in \mathcal{M}$ e $\mu(A_n) \longrightarrow \mu(A)$.

(b) Let μ^* be an outer measure and $E \subset X$ such that for all $\epsilon > 0$ there is U μ^*-measurable with $\mu^*(U \setminus E) < \epsilon$. Then E is μ^*-measurable.

(c) If $E \in \mathcal{L}(\mathbb{R}^{N+M})$, then all its sections E^y, $y \in \mathbb{R}^M$, and E_x, $x \in \mathbb{R}^N$, are Lebesgue measurable.

5. (3,0 val.) Prove Radon-Nikodym's theorem for *σ-finite positive* measures, assuming that it holds for *finite positive* measures.

6. (4,0 val.) Let $(X, \mathcal{M}, \mu)$ be a measure space, μ a positive measure, and $\mathcal{S}_\mu$ be the space of equivalence classes of integrable simple functions μ-a.e.

(a) Show that $\mathcal{S}_\mu$ is dense in $L^p_\mu(X)$, $1 \leq p < \infty$.

Let now μ^* be an outer measure such that $\mu = \mu^*_{|\mathcal{M}}$ and $\mathcal{M}$ is the class of μ^*-measurable sets. Assume that μ^* is induced from a pre-measure on a *countable* algebra $\mathcal{A}$. Show that:

(b) For $E \in \mathcal{M}$ such that $\mu(E) < \infty$, given $\epsilon > 0$, there exists $B = \cup_{n=1}^k A_n$, $A_n \in \mathcal{A}$, such that $\mu(B \Delta E) < \epsilon$, where $B \Delta E = (B \setminus E) \cup (E \setminus B)$. (Hint: start with showing that there is finite measure set $A = \cup_{n=1}^\infty A_n$, $A_n \in \mathcal{A}$ disjoint, such that $E \subset A$ and $\mu(A \setminus E) < \epsilon$.)

(c) $L^p_\mu(X)$ is separable, i.e., contains a countable dense subset.
