Duration: **30** minutes

- Write your number and name below.
- Add your answers to this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and TWO QUESTIONS. The total of points is 4.0.

Number: **Name**:

1. An engineer admits that the times to failure (in hours) of compressors used in process plants are continuous i.i.d. r.v. with unknown constant failure rate λ ($\lambda > 0$). She simultaneously observed five process plants for 8 760 hours and three compressors were replaced during that period. **(1.5)**

Compute a 90% confidence interval for the expected time to failure of such a compressor.

- **Failure time**

  T = failure time of a compressor $\sim$ exponential(λ), $\lambda > 0$ (UNKNOWN)

  [The exponential distribution is the only continuous distribution with constant failure rate.]

- **Censored data**

  Since the test lasted for $t_0 = 8\,760$ hours and any compressor is replaced (as soon as it fails), we are dealing with a Type I/item censored testing with replacement.

- **Confidence interval for λ**

  If we take into account that

  - $CI_{(1-\alpha)\times 100\%}(\lambda) \overset{Table\ 5.16}{=} \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2\times\tilde{t}}\right] = [\lambda_L; \lambda_U]$,
  - $(1-\alpha)\times 100\% = 90\%$, $n = 5$ (five process plants were observed), $r = 3$ (replacements, i.e., failures during the test),
  - and the cumulative total time in a Type I/item censored test with replacement equals $\tilde{t} \overset{Def.\ 5.17}{=} n \times t_0 = 5 \times 8\,760 = 43\,800$,

  then we get

  $$CI_{90\%}(\lambda) = \left[\frac{F^{-1}_{\chi^2_{(2\times3)}}(0.05)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(2\times3+2)}}(0.95)}{2\times\tilde{t}}\right] \overset{tables}{=} \left[\frac{1.635}{2\times 43\,800}; \frac{15.51}{2\times 43\,800}\right]$$

- **Confidence interval for $E(T)$**

  $E(T) = 1/\lambda$ is a decreasing function of $\lambda > 0$, thus

  $$CI_{95\%}(1/\lambda) = \left[\frac{1}{\lambda_U}; \frac{1}{\lambda_L}\right] \simeq \left[\frac{2\times 43\,800}{15.51}; \frac{2\times 43\,800}{1.635}\right] \simeq [5647.969; 53577.982].$$

2. Consider an upper one-sided $\bar{X}$−chart, with control limits $LCL = -\infty$ and $UCL = \mu_0 + \gamma \times \sigma_0/\sqrt{n}$, for monitoring a normally distributed quality characteristic X. **(1.5)**

Derive, $\xi(\delta,\theta)$, the probability that this chart triggers a signal in the presence of a shift in the expected value and the standard deviation with magnitudes $\delta = \sqrt{n}(\mu-\mu_0)/\sigma_0$ and $\theta = \sigma/\sigma_0$ (respectively).

Obtain $\xi(\gamma,\theta)$ and comment on this result.

- **Quality characteristic and control statistic**

  $X \sim \text{normal}(\mu, \sigma^2)$

  $\bar{X}_N$ = mean of the N^{th} random sample of size n

  $\bar{X}_N \sim \text{normal}\left(\mu = \mu_0 + \delta\,\sigma_0/\sqrt{n}, \sigma^2/n = (\theta\,\sigma_0)^2/n\right)$, where $\delta = \sqrt{n}\,(\mu - \mu_0)/\sigma_0$ $(\delta \geq 0)$ and $\theta = \sigma/\sigma_0$ $(\theta \geq 1)$ represents a shift in the expected value and in the standard deviation σ (respectively).

- **Control limits of the upper one-sided $\bar{X}-$ chart**

  $LCL = -\infty, \quad UCL = \mu_0 + \gamma\,\sigma_0/\sqrt{n}$

- **Requested probability**

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL, UCL] \mid \delta, \theta\right) \\
&= 1 - \left[\Phi\left(\frac{UCL-\mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL-\mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\
&\overset{LCL=-\infty}{=} 1 - \Phi\left[\frac{(\mu_0 + \gamma\,\sigma_0/\sqrt{n}) - (\mu_0 + \delta\,\sigma_0/\sqrt{n})}{\frac{\theta\,\sigma_0}{\sqrt{n}}}\right] \\
&= 1 - \Phi\left(\frac{\gamma - \delta}{\theta}\right), \quad \delta \geq 0, \quad \theta \geq 1
\end{aligned}$$

- **Particular case of $\xi_\mu(\delta,\theta)$; comment**

  Since

$$\xi_\mu(\gamma,\theta) = 1 - \Phi\left(\frac{\gamma-\gamma}{\theta}\right) = 1 - \Phi(0) = 0.5, \quad \theta \geq 1,$$

  we can add that, in the presence of a (large) shift in the expected value with magnitude $\delta = \gamma$, the RL of the upper one-sided $\bar{X}-$chart is a geometric r.v. with parameter 0.5, regardless of the value of θ. Hence, in this particular case, this chart is insensitive to any changes in the standard deviation of the normally distributed quality characteristic X.

3. An upper one-sided CUSUM chart for Poisson output has been set with no head start, $LCL = 0$, $UCL = 1$, $\lambda_0 = 0.25$ and reference value $k = 1$. **(1.0)**

Calculate the probability that this chart triggers a false alarm after the collection of the first sample.

- **Upper one-sided CUSUM chart for Poisson output**

$$Z_N = \begin{cases} 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in \mathbb{N} \end{cases} \quad \text{(control statistic)}$$

  $LCL = 0, \quad UCL = 1, \quad k = 1$ (reference value), $\quad u = 0$ (initial value of Z_N, no head-start)

- **In-control run length and requested probability**

  It is represented by $RL^u(\theta = 0)$, [has a phase-type distribution,] and

$$\begin{aligned}
P\left[RL^0(0) > 1\right] &\overset{Table\ 10.3}{=} \underline{e}_0^\top \times [\mathbf{Q}(0)] \times \underline{1} \\
&= \text{sum of the entries of the 1st. line of } \mathbf{Q}(0)
\end{aligned}$$

  Since $UCL = x = k = 1$ and $\lambda_0 = 0.25$, we have: $\underline{e}_0^\top = [1 \quad 0]$; $\underline{1} = [1 \quad 1]^\top$; according to (10.8) and (10.10)

$$\mathbf{Q}(0) = \begin{bmatrix} F_{Poi(\lambda_0)}(k) & P_{Poi(\lambda_0)}(k+1) \\ F_{Poi(\lambda_0)}(k-1) & P_{Poi(\lambda_0)}(k) \end{bmatrix} = \begin{bmatrix} F_{Poisson(0.25)}(1) & P_{Poi(0.25)}(1+1) \\ F_{Poi(0.25)}(1-1) & P_{Poi(0.25)}(1) \end{bmatrix};$$

$$\begin{aligned}
P\left[RL^0(0) > 1\right] &= F_{Poisson(0.25)}(1+1) \\
&\overset{table}{=} 0.9978.
\end{aligned}$$