Lecture 1: Introduction

André Martins, Francisco Melo, Mário Figueiredo

Deep Learning Course, Winter 2022-2023

Deep Learning Course

- A new MSc-level course
- Offered jointly by DEEC and DEI
- MSc programs: MEEC, MECD, MEIC-A, MEIC-T
- 350 students enrolled this year! (192 DEEC, 158 DEI).

Course Website(s)

DEEC:

`https://fenix.tecnico.ulisboa.pt/disciplinas/AProf/2022-2023/1-semestre`

DEI:

`https://fenix.tecnico.ulisboa.pt/disciplinas/AP-Dei/2022-2023/1-semestre`

There we can find:

- Syllabus
- Lecture slides
- Literature pointers
- Practical and homework assignments
- Announcements
- ...

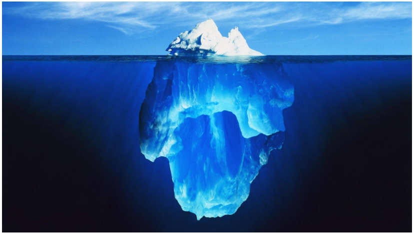
Instructors

- **Main instructors:** André Martins (DEI Alameda), Francisco Melo (DEI Tagus), Mário Figueiredo (DEEC)
- **Practical classes:** Andreas Wichert, Ben Peters, Chrysoula Zerva, Gonçalo Correia, João Fonseca, João Santinha, José Moreira, Margarida Campos, Tomás Costa
- **Location & schedule:** see course webpage in Fenix (previous slide)
- **Office hours:** see information in Fenix
- **Communication:**
  `piazza.com/tecnico.ulisboa.pt/fall2022/c88`

Please register in Piazza!!

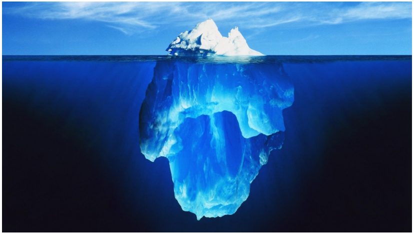
Outline

What is “‘Deep Learning”?



- Neural networks?
- Neural networks with many hidden layers?
- Anything beyond shallow (linear) models for machine learning?
- Anything that learns representations?
- A form of learning that is really intense and profound?

What is "'Deep Learning"?



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Why Did Deep Learning Become Mainstream?

Lots of recent breakthroughs:

- Object recognition
- Speech and language processing (Transformers, BERT, GPT-3)
- Machine translation
- Chatbots and dialog systems
- Self-driving cars
- Solving games (Atari, Go, StartCraft II)
- Protein design (AlphaFold)

No signs of slowing down...


smart
On my way

Microsoft's Deep Learning Project Outperforms Humans In Image Recognition

Michael Thomsen, CONTRIBUTOR

I write about tech, video games, science and culture. **FULL BIO**

Opinions expressed by Forbes Contributors are their own.

Microsoft's new breakthrough: AI that's as good as humans at listening... on the phone

Microsoft's new speech-recognition record means professional transcribers could be among the first to lose their jobs to artificial intelligence.

By Liam Tung | October 19, 2016 -- 10:10 GMT (11:10 BST) | Topic: Innovation

Who is wearing glasses?

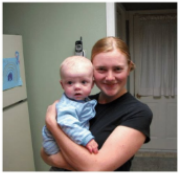
man

woman

Where is the child sitting?

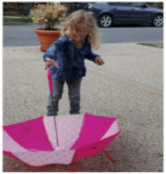
fridge

arms



Is the umbrella upside down?

yes



no

How many children are in the bed?

2

1

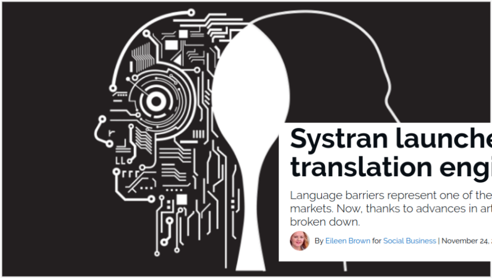
The Great A.I. Awakening

How Google used artificial intelligence to transform Google Translate, one of its more popular services — and how machine learning is poised to reinvent computing itself.

BY GIDEON LEWIS-KRAUS DEC. 14, 2016

Google unleashes deep learning tech on language with Neural Machine Translation

Posted Sep 27, 2016 by Devin Coldewey, Contributor



Systran launches neural machine translation engine in 30 languages

Language barriers represent one of the biggest challenges to develop business strategies among global markets. Now, thanks to advances in artificial intelligence and machine translation, these barriers are being broken down.

By Eileen Brown for Social Business | November 24, 2016 -- 13:49 GMT (13:49 GMT) | Topic: Artificial Intelligence

AlphaGo Beats Go Human Champ: Godfather Of Deep Learning Tells Us Do Not Be Afraid Of AI

21 March 2016, 10:16 am EDT **By Aaron Mamiit Tech Times**

Last week, Google's artificial intelligence program

Last week, Google's artificial intelligence program AlphaGo dominated its match with South Korean world Go champion Lee Sedol, winning with a 4-1 score.

The achievement stunned artificial intelligence experts, who previously thought that Google's computer program would need at least 10 more years before developing enough to be able to beat a human world champion.

A robot wrote this entire article. Are you scared yet, human?

We asked GPT-3, OpenAI's powerful new language generator, to write an essay for us from scratch. The assignment? To convince us robots come in peace

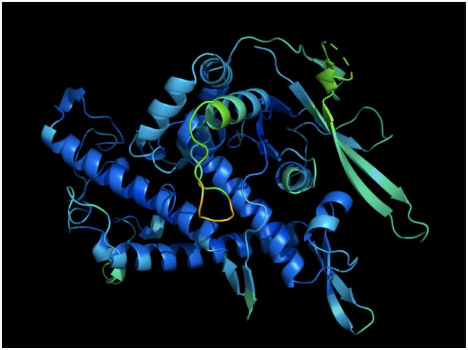
- For more about GPT-3 and how this essay was written and edited, please read our editor's note below

NEWS | 30 November 2020

‘It will change everything’: DeepMind’s AI makes gigantic leap in solving protein structures

Google’s deep-learning program for determining the 3D shapes of proteins stands to transform biology, say scientists.

Ewen Callaway



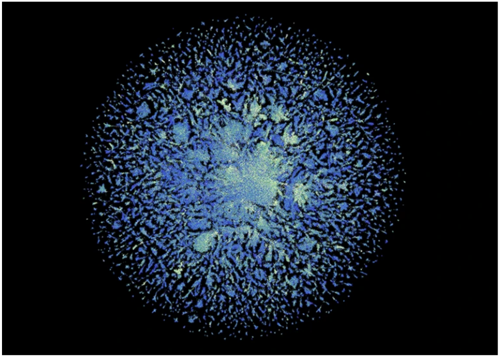
A protein's function is determined by its 3D shape. Credit: DeepMind

NEWS | 01 November 2022

AlphaFold's new rival? Meta AI predicts shape of 600 million proteins

Microbial molecules from soil, seawater and human bodies are among the planet's least understood.

Ewen Callaway

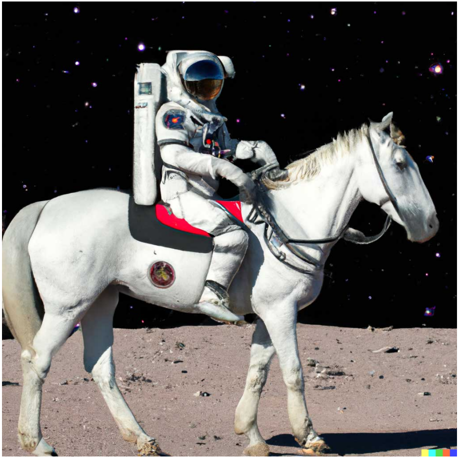


The ESM Metagenomic Atlas contains structural predictions for 617 million proteins. Credit: ESM Metagenomic Atlas (CC BY 4.0)

How to use DALL•E 2 to turn your wildest imaginations into tangible art

AI art platform, DALL•E 2, creates images from text descriptions in seconds. In this article, we show you how to get the results you desire.

Written by **Christina Darby,** Associate Editor on Oct. 31, 2022



Why Now?

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Many of the core ideas were there, after all.

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Many of the core ideas were there, after all.

But now we have:

- more data
- more computing power
- (much) better software engineering (e.g. auto-diff)
- some algorithmic innovations: many layers, ReLUs, better learning algorithms, dropout, CNNs, LSTMs, transformers, etc.
  All of these will be covered in this course.

"But It's Non-Convex"

For many years (1990–2010), NNs were not popular in machine learning, because they were hard to learn.

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For many years (1990–2010), NNs were not popular in machine learning, because they were hard to learn.

Why does gradient-based optimization work at all in NNs despite the non-convexity?

One possible, partial answer (this is an open research topic)

- there are generally many hidden units
- there are many ways a neural net can approximate the desired input-output relationship
- we only need to find one

One turning point: AlexNet

- Alex Krizhevsky, Ilya Sutskever, Geoffrey Hinton; 2012

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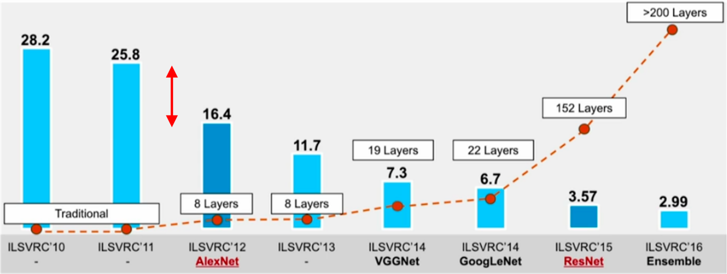

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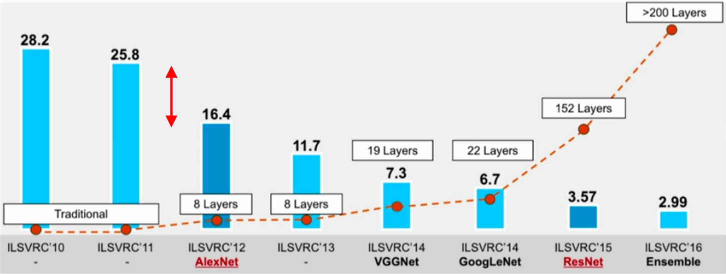

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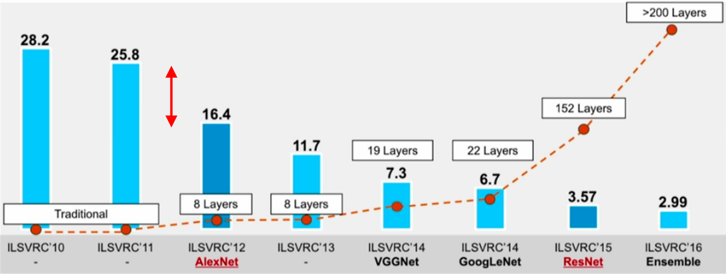

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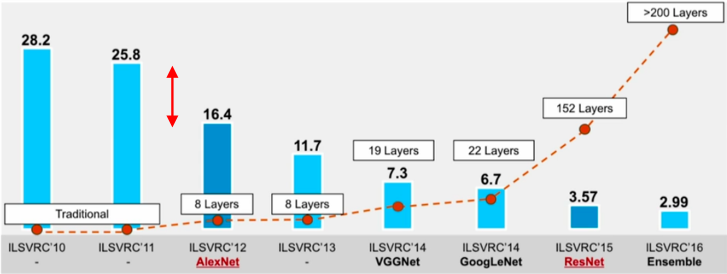
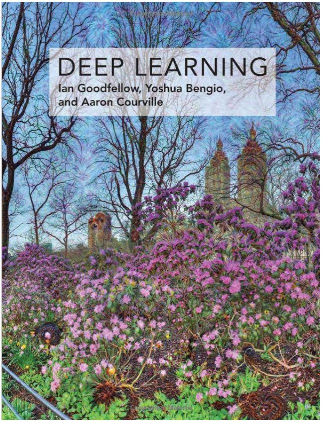


- Large CNN, much deeper than anything else at the time
- Used parallel processing (one of the first uses of GPUs in NNs)
- Convinced many people that deep learning would change the field.

Recommended Books

Main book:

- **Deep Learning.** Ian Goodfellow, Yoshua Bengio, and Aaron Courville. MIT Press, 2016. Chapters available at `http://deeplearningbook.org`



Recommended Books

Secondary books:

- **Artificial Intelligence Engines: A Tutorial Introduction to the Mathematics of Deep Learning.** James Stone. Sebtel Press, 2019.
- **Dive into Deep Learning.** Aston Zhang, Zach Lipton, Mu Li, Alex Smola (`https://d2l.ai/`)
- **Deep Learning with Python.** François Chollet. Manning Publications, 2017.
- **Machine Learning – A Journey to Deep Learning with Exercises and Answers.** Andreas Wichert and Luis Sa-Couto, 2021
- **Probabilistic Machine Learning.** Kevin P. Murphy (`https://probml.github.io/pml-book/`)

Tentative Syllabus

Week 1	Introduction and Course Description
	Linear Classifiers I (linear regression, perceptron)
Week 2	Linear Classifiers II (logistic regression, regularization)
Week 3	Neural Networks I
Week 4	Neural Networks II
	Representation Learning and Auto-Encoders
Week 5	Convolutional Networks
	Recurrent Neural Networks and LSTMs
Week 6	Sequence-to-Sequence Models and Attention Mechanisms
	Transformers
Week 7	Self-Supervised Learning (BERT, GPT3, etc.)
	Interpretability and Fairness

Outline

What This Class Is About

- Introduction to deep learning (DL)
- **Goal:** after finishing this class, you should be able to:
  - ✓ Understand how DL works (it's not magic)
  - ✓ Understand the math and intuition behind DL models
  - ✓ Apply DL on practical problems (language, vision, ...)
- **Target audience:**
  - ✓ MSc students with basic background in probability theory, linear algebra, programming.
  - ✓ Useful: background in machine learning.

What This Class Is Not About

- Just playing with DL toolkits, without learning the fundamental concepts
- Introduction to machine learning (other courses by DEEC and DEI)
- Natural language processing (another course offered by DEI)
- Computer vision (another course offered by DEEC)
- Optimization (other courses by DEEC and DEI)
- ...

Prerequisites

- Calculus and basic linear algebra
- Basic probability theory
- Basic knowledge of machine learning (preferred, but not required)
- Programming (Python & PyTorch, preferred but not required)

Course Information

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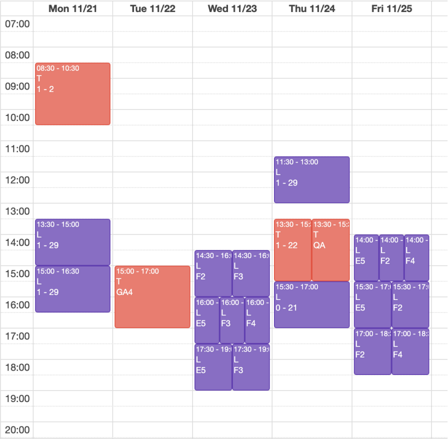
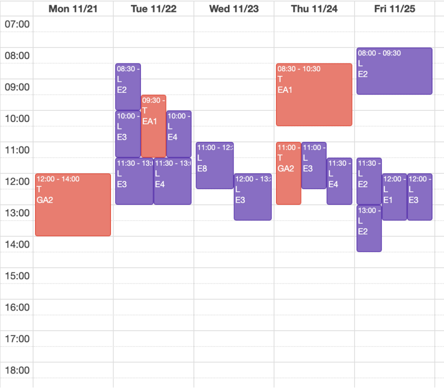
Schedule (DEI Alameda/Tagus)



- Letures: 2 per week (Alameda: Tue & Thur; Tagus: Mon & Thur)
- Practical shifts (see Fenix) – pick your slots and register as a group! (more later)

Schedule (DEEC Alameda)



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Grading

- 2 homework assignments: 50%
  - Minimal grade: 8
  - Theoretical questions & implementation
  - Groups of **2** – you need to register your group in Fenix!
  - Some of the practicals will be Q&A about these assignments, so please join the practicals as a group
  - Submission through Fenix
  - No late days allowed
- Final exam: 50%
  - Minimal grade: 8

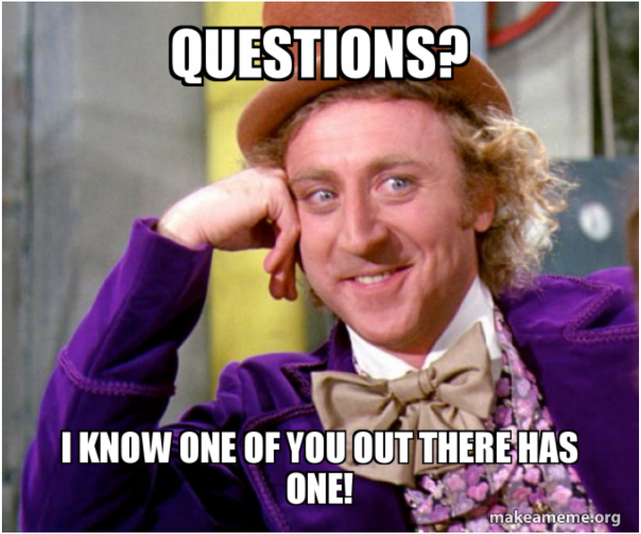
Registering Your Group in Fenix

- Pick a group of **2**
- Register in Fenix
- Deadline: Sunday, November 27, at 23:59
- Use Piazza to find group mates
- Can't find a group?
  Tell the instructors by November 28 (in Piazza); we'll find a solution.

Collaboration Policy

- Assignments should be done within each group
- Students may discuss the questions across groups, as long as they write their own answers and their own code
- If this happens, acknowledge with whom you collaborate!
- Zero tolerance on plagiarism!!
- Always credit your sources!!!

Questions?



Today's Roadmap

- Linear Algebra (only skim over)
- Probability Refresher (only skim over)
- Optimization
- Introduction to Machine Learning
  - "Deep Learning Superheroes"
  - Supervised, Unsupervised, Reinforcement Learning
  - Classification and Regression

Outline

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Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m \times n}$ is a matrix with m rows and n columns.

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix}.$$

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- A matrix with 1 row and n columns is called a row vector.

Matrix Transpose and Matrix Products

- Given matrix $A \in \mathbb{R}^{m \times n}$, its transpose A^T is such that $(A^T)_{i,j} = A_{j,i}$.

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- Given matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, their product is

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- Inner product between vectors $x, y \in \mathbb{R}^n$:

$$\langle x, y \rangle = x^T y = y^T x = \sum_{i=1}^{n} x_i y_i$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [x_1, \ldots, x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Outer and Hadamard Products

- Outer product between vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$:

$$x\, y^T \in \mathbb{R}^{n \times m}, \quad \text{where } (x\, y^T)_{i,j} = x_i\, y_j$$

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- Hadamard/Schur product between vectors $x, y \in \mathbb{R}^n$: $(x \odot y)_i = x_i y_i$,

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \odot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1\, y_1 \\ \vdots \\ x_n\, y_n \end{bmatrix}$$

Properties of Matrix Products and Transposes

- Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, their product is

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- In general, matrix product is not commutative: $AB \neq BA$.
- Transpose of product: $(A\,B)^T = B^T A^T$.
- Transpose of sum: $(A + B)^T = A^T + B^T$.

Norms

- The norm of a vector is (informally) its "magnitude." Euclidean norm:

$$\|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

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- Notable case: the ℓ_1 norm, $\|x\|_1 = \sum_i |x_i|$.
- Notable case: the ℓ_∞ norm, $\|x\|_\infty = \max\{|x_1|, ..., |x_n|\}$.

Special Matrices

- The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- There are many algorithms to compute A^{-1}; general case, computational cost $O(n^3)$.

Quadratic Forms and Positive (Semi-)Definite Matrices

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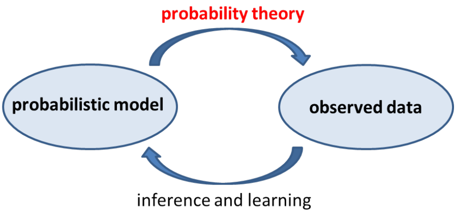
Outline

Probability theory



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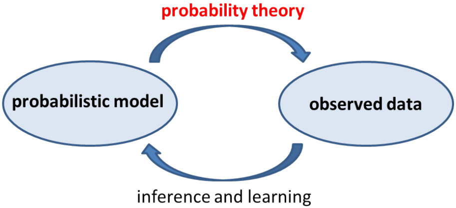

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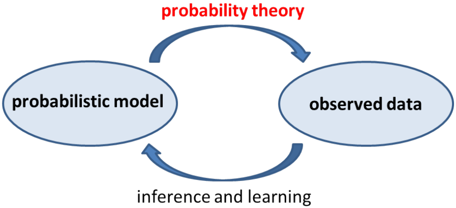

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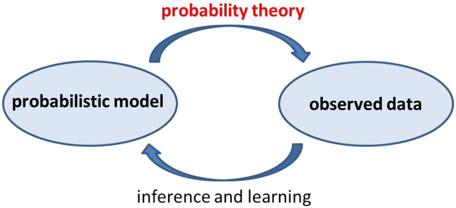

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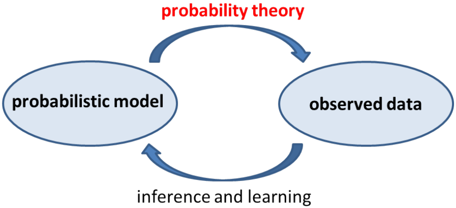

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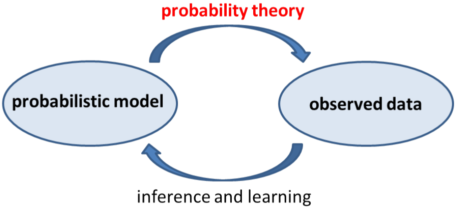

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- ...thus also learning, inference, ...

What is probability?

- Classical definition: $\mathbb{P}(A) = \dfrac{N_A}{N}$

  ...with N mutually exclusive equally likely outcomes,
  N_A of which result in the occurrence of event A. *Laplace, 1814*

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- Subjective probability: $\mathbb{P}(A)$ is a degree of belief. *de Finetti, 1930s*

  ...gives meaning to $\mathbb{P}$("Tomorrow it will rain").

Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  Examples:
  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, ..., 36\}$
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- An event A is a subset of $\mathcal{X}$: $A \subseteq \mathcal{X}$.

  Examples:
  - "exactly one H in 2-coin toss": $A = \{TH, HT\} \subset \{HH, TH, HT, TT\}$.
  - "odd number in the roulette": $B = \{1, 3, ..., 35\} \subset \{1, 2, ..., 36\}$.
  - "drawn a $\heartsuit$ card": $C = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\} \subset \{A\clubsuit, ..., K\diamondsuit\}$

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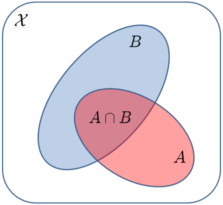
- From these axioms, many results can be derived. Examples:

- $\mathbb{P}(\emptyset) = 0$
- $C \subset D \Rightarrow \mathbb{P}(C) \leq \mathbb{P}(D)$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$



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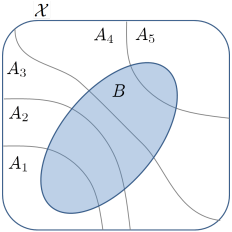
Law of Total Probability and Bayes Theorem

- Law of total probability: if $A_1, \dots, A_n$ are a partition of $\mathcal{X}$

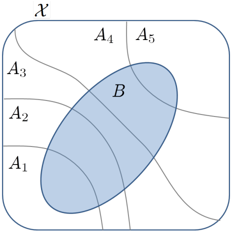
$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$
$$= \sum_i \mathbb{P}(B \cap A_i)$$



Law of Total Probability and Bayes Theorem

- Law of total probability: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i) = \sum_i \mathbb{P}(B \cap A_i)$$

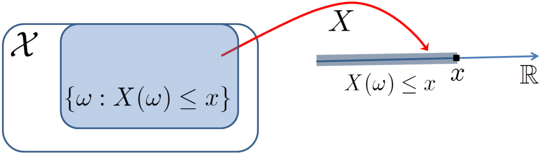


- Bayes' theorem: if $\{A_1, ..., A_n\}$ is a partition of $\mathcal{X}$

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(B)}$$

Discrete Random Variables

- Distribution function: $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$



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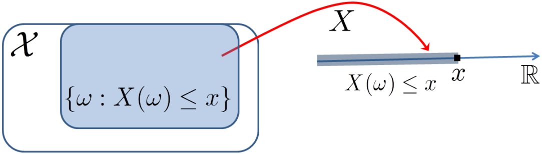
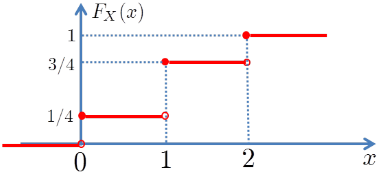
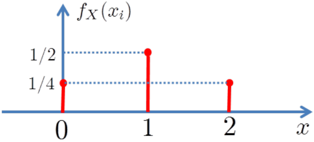


- Example: number of heads in tossing 2 coins; range$(X) = \{0, 1, 2\}$.





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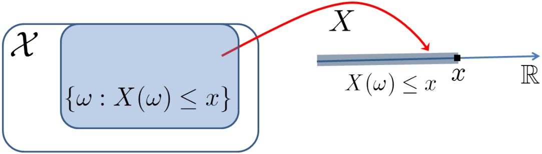

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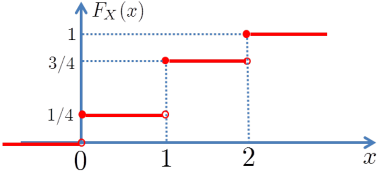

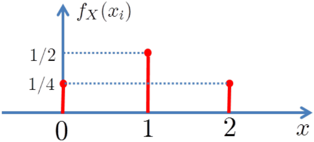

- Probability mass function (discrete RV): $f_X(x) = \mathbb{P}(X = x)$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i).$$

Important Discrete Random Variables

- Uniform: $X \in \{x_1, ..., x_K\}$, pmf $f_X(x_i) = 1/K$.

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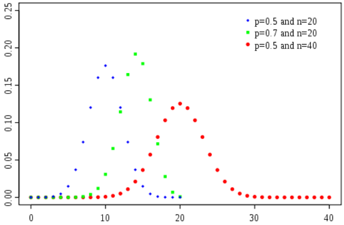
Binomial coefficients ("n choose x"):

$$\binom{n}{x} = \frac{n!}{(n-x)!\, x!}$$



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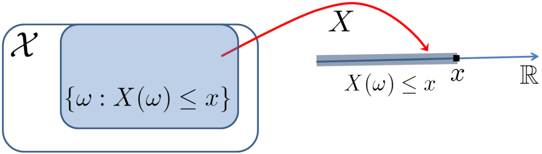

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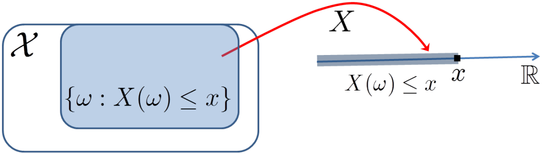
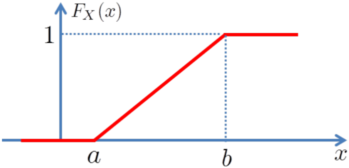
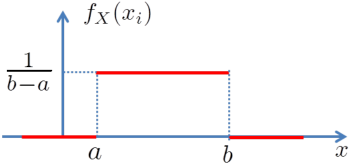


- Example: continuous RV with uniform distribution on $[a, b]$.





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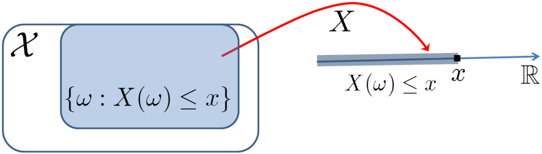

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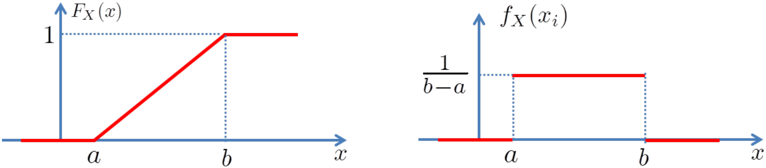

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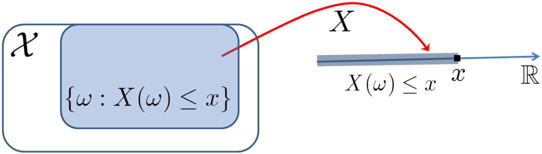

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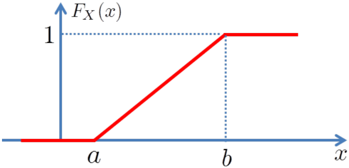

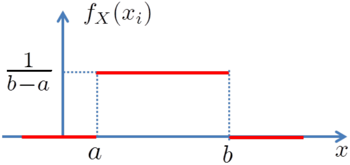

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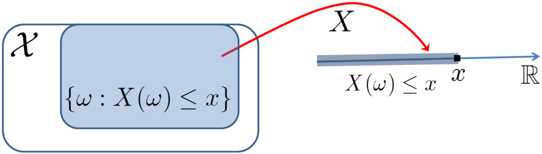

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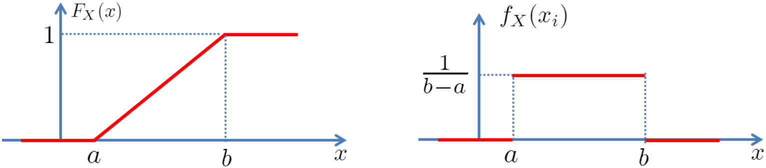

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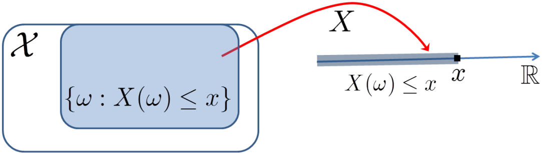

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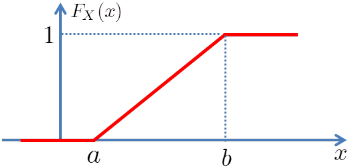

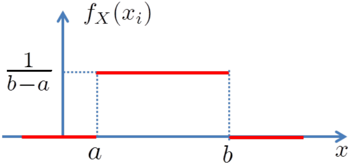

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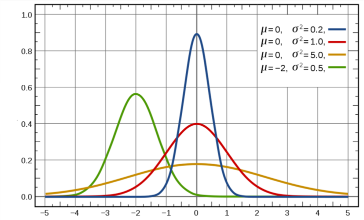

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- Expectation: $\mathbb{E}(X) = \begin{cases} \sum_i x_i \, f_X(x_i) & X \in \{x_1, ..., x_K\} \subset \mathbb{R} \\ \int_{-\infty}^{\infty} x \, f_X(x) \, dx & X \text{ continuous} \end{cases}$

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- Linearity of expectation:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y); \quad \mathbb{E}(\alpha \, X) = \alpha \mathbb{E}(X), \;\; \alpha \in \mathbb{R}$$

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- Probability as expectation of indicator, $1_A(x) = \begin{cases} 1 & \Leftarrow & x \in A \\ 0 & \Leftarrow & x \notin A \end{cases}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx = \int 1_A(x)\, f_X(x)\, dx = \mathbb{E}(1_A(X))$$

Two (or More) Random Variables

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Conditionals and Bayes' Theorem

- Conditional pmf (discrete RVs):

$$f_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x \wedge Y = y)}{\mathbb{P}(Y = y)} = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

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- Also valid in the mixed case (*e.g.*, X continuous, Y discrete).

Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

$f_{X,Y}(x,y)$	$Y = 0$	$Y = 1$
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$X=0$	2/3	4/7
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An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, such that $\sum_i X_i = n$,

$$f_X(x_1, ..., x_K) = \begin{cases} \binom{n}{x_1\ x_2\ \cdots\ x_K} p_1^{x_1}\, p_2^{x_2}\ \cdots\ p_k^{x_K} & \Leftarrow\ \ \sum_i x_i = n \\ 0 & \Leftarrow\ \ \sum_i x_i \neq n \end{cases}$$

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- For $n = 1$, sometimes called categorical or multinoulli.
  For $n = 1$, one and only one $x_i = 1$, others are 0, thus $\binom{n}{x_1\ \cdots\ x_K} = 1$.

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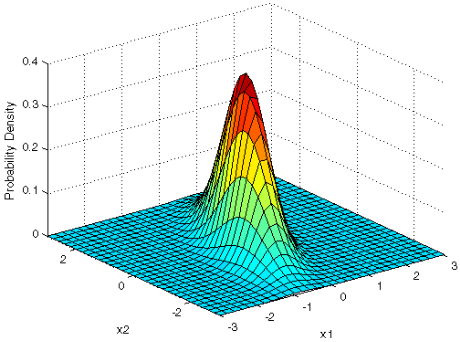

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$D(f_X \| g_X) = 0 \;\Leftrightarrow\; f_X(x) = g_X(x), \text{ almost everywhere}$

Recommended Reading

- A. Maleki and T. Do, “Review of Probability Theory”, Stanford University, 2017 (`https://tinyurl.com/pz7p9g5`)
- L. Wasserman, “All of Statistics: A Concise Course in Statistical Inference”, Springer, 2004.

Outline

Minimizing a function

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- Local minimum: for any $\|x - x^*\| \leq \delta \Rightarrow f(x^*) \leq f(x)$.
- Are local minima also global minima?

Convex Functions

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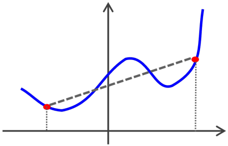
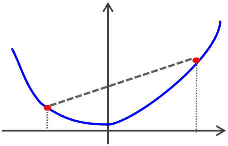
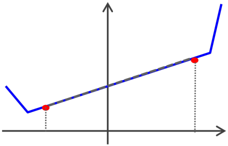
$$f\left(\lambda x + (1-\lambda)x'\right) < \lambda f(x) + (1-\lambda) f(x')$$



non-convex



convex
strictly convex



convex, not strictly

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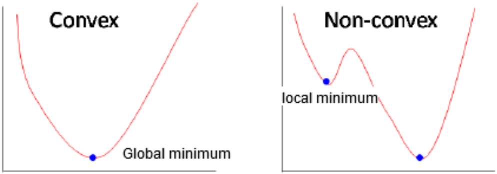

Gradients and Minimization

- Given $f : \mathbb{R}^n \to \mathbb{R}$ (differentiable), the gradient of f at x:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n$$

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- Given $f : \mathbb{R}^n \to \mathbb{R}$ (differentiable), the gradient of f at x:

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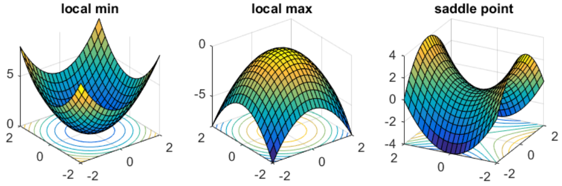

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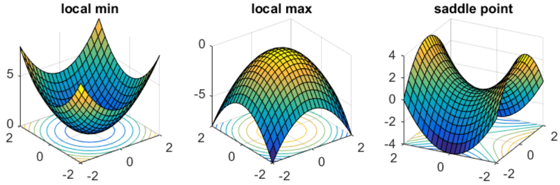

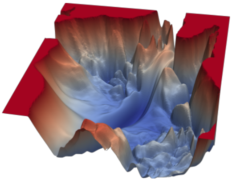

Hessians and Convexity

- Given $f : \mathbb{R}^n \to \mathbb{R}$ (differentiable), the Hessian of f at x:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \, \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \, \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \, \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

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- Relationship between Hessian and convexity:
  - ✓ Positive semi-definite Hessian $\Leftrightarrow$ f is convex
  - ✓ Positive definite Hessian $\Leftrightarrow$ f is strictly convex.

More on Gradients

- Gradient of quadratic form $f(x) = x^T A x$: $\nabla f(x) = (A + A^T)x$

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- If $g(x) = f(a \odot x)$, then $\nabla g(x) = a \odot \nabla f(a \odot x)$
- In simple cases, we can find minima analytically: $f(x) = \|Ax - y\|_2^2$

$$\nabla f(x) = 2A^T(Ax - y) = 0 \quad \Rightarrow \quad x^* = (A^T A)^{-1} A^T y$$

Gradient Descent

- Goal: minimize $f : \mathbb{R}^d \to \mathbb{R}$, for differentiable f

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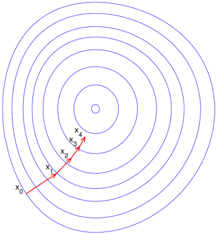
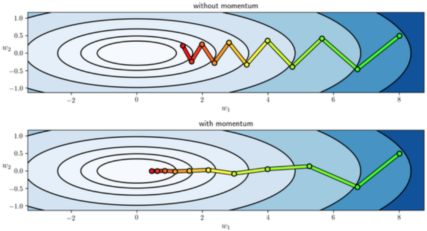
- GD may work well, or not so well. There are many ways to improve it.





Recommended Reading

- Z. Kolter and C. Do, "Linear Algebra Review and Reference", Stanford University, 2015 (`https://tinyurl.com/44x2qj4`)

Outline

Machine Learning

Tom Mitchell's definition:

- "A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P if its performance at tasks in T, as measured by P, improves with experience E."
- In a nutshell: learn from data; improve performance with experience

This comes in many flavours:

- Supervised learning
- Unsupervised learning
- Self-supervised learning
- Reinforcement learning
- Active learning

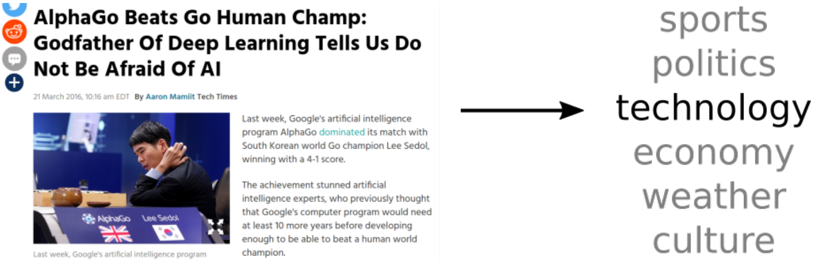
Formulate the problem; get data; learn the model from the data; evaluate.

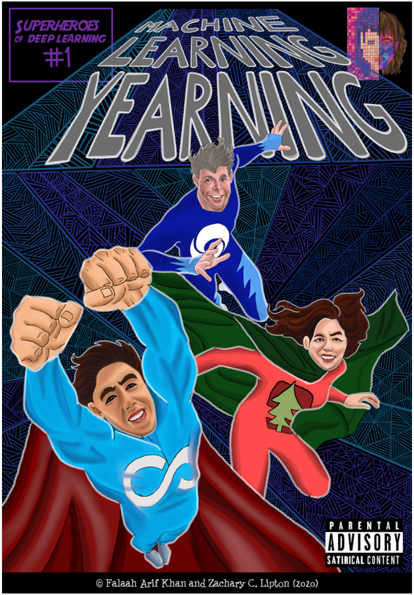
Example Tasks

Binary classification: given an e-mail: is it spam or not-spam?

Multi-class classification: given a news article, determine its topic (politics, sports, etc.)

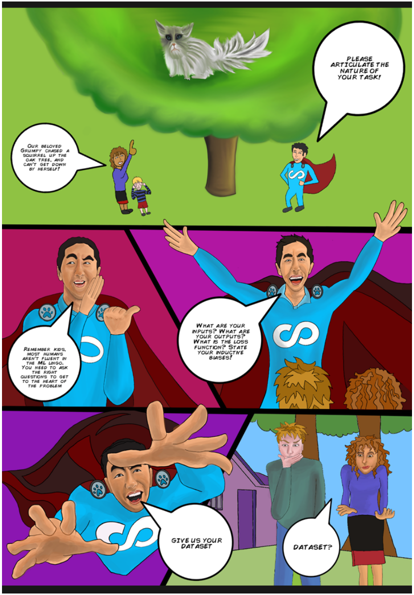
Regression: how much time a person will spend reading this article?





https://www.approximatelycorrect.com/2020/10/26/

superheroes-of-deep-learning-vol-1-machine-learning-yearning/



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Let's Start Simple

- Example 1 – sequence: $\star \diamond \circ$; label: -1
- Example 2 – sequence: $\star \heartsuit \triangle$; label: -1
- Example 3 – sequence: $\star \triangle \spadesuit$; label: $+1$
- Example 4 – sequence: $\diamond \triangle \circ$; label: $+1$

(Credits: Ryan McDonald)

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Why can we do this?

(Credits: Ryan McDonald)

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Label -1 **Label** $+1$

$$P(-1|\star) = \frac{\text{count}(\star \text{ and } -1)}{\text{count}(\star)} = \frac{2}{3} = 0.67 \text{ vs. } P(+1|\star) = \frac{\text{count}(\star \text{ and } +1)}{\text{count}(\star)} = \frac{1}{3} = 0.33$$

$$P(-1|\diamond) = \frac{\text{count}(\diamond \text{ and } -1)}{\text{count}(\diamond)} = \frac{1}{2} = 0.5 \text{ vs. } P(+1|\diamond) = \frac{\text{count}(\diamond \text{ and } +1)}{\text{count}(\diamond)} = \frac{1}{2} = 0.5$$

$$P(-1|\heartsuit) = \frac{\text{count}(\heartsuit \text{ and } -1)}{\text{count}(\heartsuit)} = \frac{1}{1} = 1.0 \text{ vs. } P(+1|\heartsuit) = \frac{\text{count}(\heartsuit \text{ and } +1)}{\text{count}(\heartsuit)} = \frac{0}{1} = 0.0$$

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Machine Learning

1. Define a model/distribution of interest
2. Make some assumptions if needed
3. Fit the model to the data

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3. Fit the model to the data

- Model: $P(\text{label}|\text{sequence}) = P(\text{label}|\text{symbol}_1, \ldots \text{symbol}_n)$
  - Prediction for new sequence = $\text{argmax}_{\text{label}} P(\text{label}|\text{sequence})$

- Assumption (naive Bayes):

$$P(\text{symbol}_1, \ldots, \text{symbol}_n|\text{label}) = \prod_{i=1}^{n} P(\text{symbol}_i|\text{label})$$

- Fit the model to the data: count!! (simple probabilistic modeling)

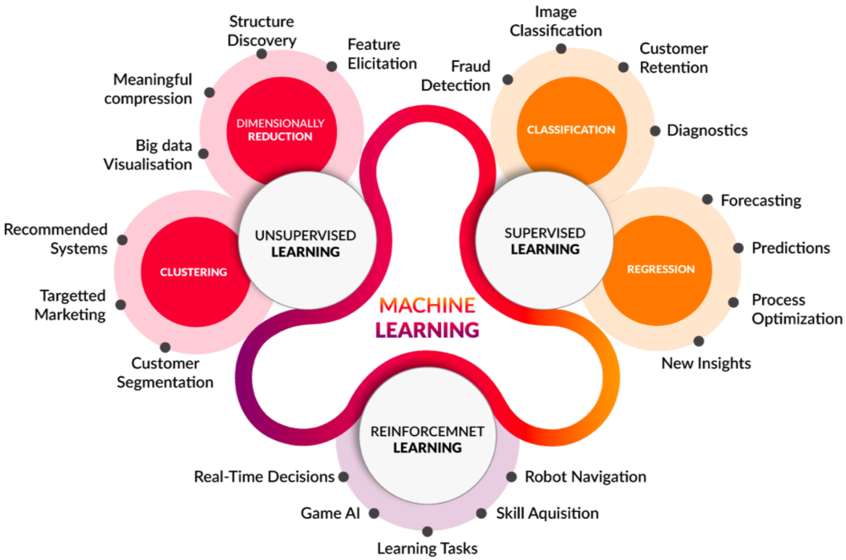
Some Notation: Inputs and Outputs

- Input $x \in \mathcal{X}$
  - e.g., a news article, a sentence, an image, ...
- Output $y \in \mathcal{Y}$
  - e.g., spam/not spam, a topic, the object in the image (cat? dog?); a segmentation of the image (pedestrian; car; grass; background)

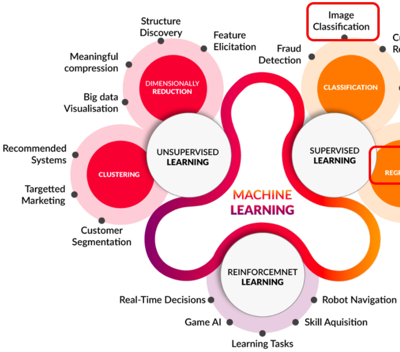
- Input/Output pair: $(x, y) \in \mathcal{X} \times \mathcal{Y}$
  - e.g., a **news article** together with a **topic**
  - e.g., a **image** together with an **object**
  - e.g., an **image** partitioned into **segmentation regions**

Many Flavours

- Supervised learning: pairs (x, y) are provided at training time (the main focus of this class)
  - Examples: perceptron, SVMs, decision trees, nearest neighbor, neural networks, ...
  - Caveat: the labels y may be hard or expensive to annotate
- Unsupervised learning: only x is provided; the model needs to figure out what the labels are without any supervision
  - Examples: clustering, PCA, ...
- Reinforcement learning: x is provided, and the model can act on the environment to obtain a reward (but doesn't get to know y)
  - Example: a robot acting on an environment trying to achieve a goal
- Active learning: the model requests which data points to label next.



Structure Discovery
Feature Elicitation
Meaningful compression
Big data Visualisation
DIMENSIONALLY REDUCTION
Image Classification
Customer Retention
Fraud Detection
CLASSIFICATION
Diagnostics
Recommended Systems
CLUSTERING
UNSUPERVISED LEARNING
Targetted Marketing
Customer Segmentation
SUPERVISED LEARNING
Forecasting
Predictions
REGRESSION
Process Optimization
New Insights
MACHINE LEARNING
REINFORCEMNET LEARNING
Real-Time Decisions
Robot Navigation
Game AI
Skill Aquisition
Learning Tasks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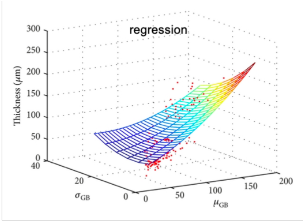

regression
Thickness (μm)
300
250
200
150
100
50
0
40
20
0
σ_{GB}
0
50
100
150
200
μ_{GB}

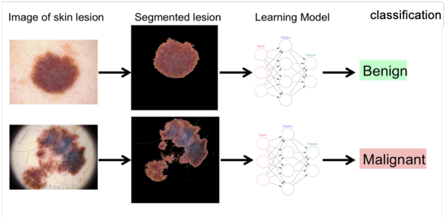
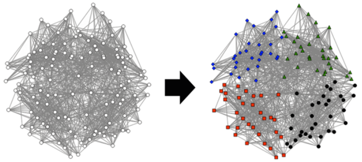
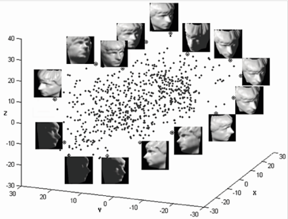


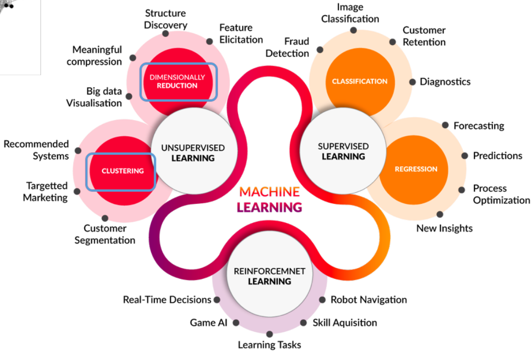
Image of skin lesion
Segmented lesion
Learning Model
classification
Benign
Malignant

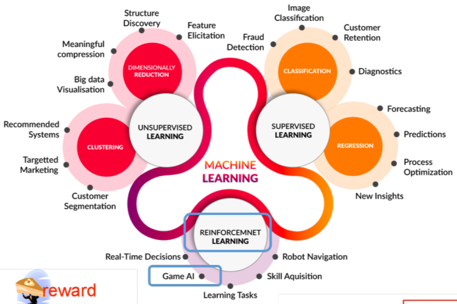


Community extraction from networks



Manifold learning

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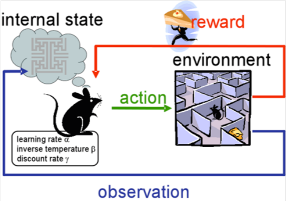
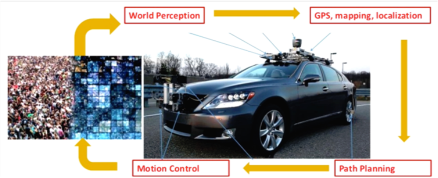
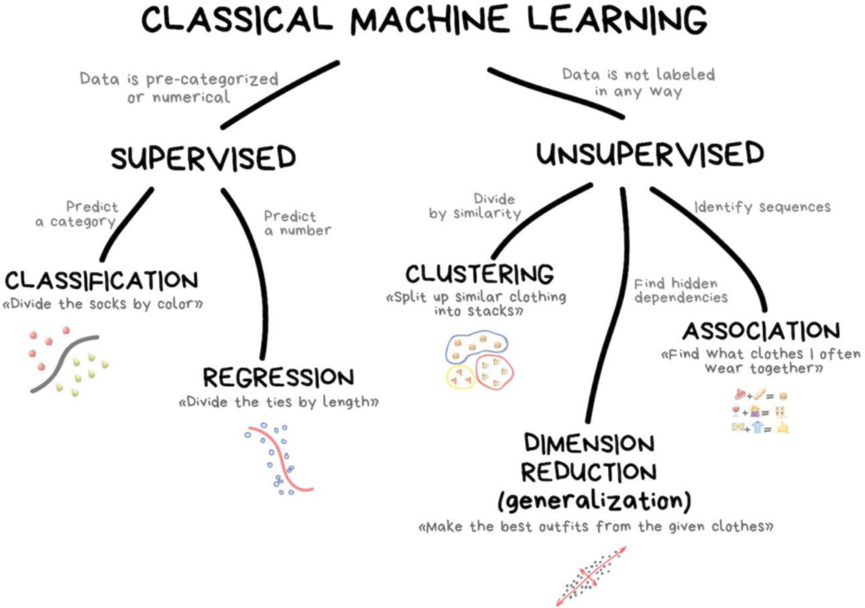
Learning Tasks


nature
At last — a computer program that can beat a champion Go player PAGE 484
ALL SYSTEMS GO


internal state
reward
environment
action
learning rate α
inverse temperature β
discount rate γ
observation


World Perception
GPS, mapping, localization
Motion Control
Path Planning


CLASSICAL MACHINE LEARNING
Data is pre-categorized or numerical
Data is not labeled in any way
SUPERVISED
UNSUPERVISED
Predict a category
Predict a number
Divide by similarity
Identify sequences
Find hidden dependencies
CLASSIFICATION
«Divide the socks by color»
REGRESSION
«Divide the ties by length»
CLUSTERING
«Split up similar clothing into stacks»
ASSOCIATION
«Find what clothes I often wear together»
DIMENSION REDUCTION (generalization)
«Make the best outfits from the given clothes»

Supervised Learning

- We are given a **labeled dataset** of input/output pairs:

$$\mathcal{D} = \{(x_n, y_n)\}_{n=1}^{N} \subseteq \mathcal{X} \times \mathcal{Y}$$

- **Goal:** learn a **predictor** $h : \mathcal{X} \to \mathcal{Y}$ that generalizes well to arbitrary inputs.
- At test time, given $x \in \mathcal{X}$, we predict

$$\widehat{y} = h(x).$$

- Hopefully, $\widehat{y} \approx y$ most of the time.

Tasks/problems have different names depending on what $\mathcal{Y}$ is...

Regression deals with **continuous** output variables:

- Simple **regression:** $\mathcal{Y} = \mathbb{R}$
  - e.g., given a news article, how much time a user will spend reading it?
- **Multivariate regression:** $\mathcal{Y} = \mathbb{R}^K$
  - e.g., predict the X-Y coordinates in an image where the user will click

Classification

Tasks/problems have different names depending on what $\mathcal{Y}$ is...

Classification deals with **discrete** output variables:

- **Binary classification:** $\mathcal{Y} = \{\pm 1\}$
  - e.g., spam detection
- **Multi-class classification:** $\mathcal{Y} = \{1, 2, \ldots, K\}$
  - e.g., topic classification
- **Structured classification:** $\mathcal{Y}$ exponentially large and structured
  - e.g., machine translation, caption generation, image segmentation

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- **Structured classification:** $\mathcal{Y}$ exponentially large and structured
  - e.g., machine translation, caption generation, image segmentation
  - Later in this course!

Sometimes reductions are convenient:

- logistic regression reduces classification to regression
- one-vs-all reduces multi-class to binary
- greedy search reduces structured classification to multi-class

... but other times it's better to tackle the problem in its native form.

More later!

Conclusions

- Machine learning allows computer programs to learn from data (observations, interventions, ...) and improve performance with experience
- Can be supervised, unsupervised, self-supervised, reinforced, etc.
- Tasks can be (binary or multi-class) classification, regression, or more nuanced

Thank you!

Questions?