Information and Communication Theory: Second Mini-Test

October 20, 2022

Name: ____ Number: _________

Duration: 45 minutes. Part I scores: correct answer = 3/2 point; wrong answer = –3/4 points.

Useful facts: $\log_a b = (\log_c b)/(\log_c a)$; $\log_2 3 \simeq 1.585$. Unless indicated otherwise, all logarithms are base-2.

Part I

1. Let $Y \in \{1, 2, 3, 4, 5\}$ be a source with $\mathbf{f}_Y = (1/3,\ 1/3,\ 1/6,\ 1/9,\ 1/18))$. The expected length of a Huffman **binary** code for this source is

   a) < 2 bits/symbol; ☐
   b) $= 2$ bits/symbol; ☐
   c) > 2 bits/symbol. ■

2. Let $X \in \{a, b, c, d\}$ be a source such that $\mathbb{P}(X = a) = 0.4 - 3\,\alpha$, $\mathbb{P}(X = b) = \alpha + 0.1$, $\mathbb{P}(X = c) = \alpha + 0.2$, and $\mathbb{P}(X = d) = \alpha + 0.3$, where $\alpha \in [0, 0.1]$. Consider the binary code $\{C(a) = 00,\ C(b) = 01,\ C(c) = 10,\ C(d) = 11\}$.

   a) C is a Huffman code for any $\alpha \in [0, 0.1]$; ☐
   b) C is not a Huffman code for $\alpha \in [0, 0.1]$; ☐
   c) C is a Huffman code for some, but not all, values $\alpha \in [0, 0.1]$. ■

3. Let X be the source defined in question 2, for $\alpha = 0$. The expected length of a Huffman **ternary** code for this source is

   a) < 1.2 trits/symbol; ☐
   b) $= 1.2$ trits/symbol; ☐
   c) > 1.2 trits/symbol. ■

4. Let $Z \in \{1, 2, 3\}$ be a source with $\mathbf{f}_Z = (0.96, 0.02, 0.02))$, which has entropy $H(Z) \simeq 0.282$ bits/symbol. The expected length of an optimal **binary** code for the second-order extension of this source is

   a) < 0.8 bits/symbol; ■
   b) $= 0.8$ bits/symbol; ☐
   c) > 0.8 bits/symbol. ☐

5. Let $X \in [-1, 1]$ be a random variable with uniform probability density function and $Y = 2 - X$ another random variable. Then, the corresponding differential entropies satisfy

   a) $h(X) < h(Y)$; ☐
   b) $h(X) = h(Y)$; ■
   c) $h(X) > h(Y)$. ☐

6. Let $X, Y \in [-1, 1]$ be two random variables with probability density functions (p.d.f.) $f_X(x) = (3/2)x^2$ and $f_Y(y) = (3/2)(1 - |x|)^2$, respectively. Then, the corresponding differential entropies satisfy (hint: draw de densities)

   a) $h(X) < h(Y)$; ☐
   b) $h(X) = h(Y)$; ■
   c) $h(X) > h(Y)$. ☐

7. Let $X \in [-1, 1]$ be the random variable defined in the previous question. Let Z be the random variable given by $Z = 0$, if $X \leq 0$, and $Z = 1$, if $X > 0$. Then,

a) $I(X; Z) < 1$ bit/symbol; ☐

b) $I(X; Z) = 1$ bit/symbol; ■

c) $I(X; Z) > 1$ bit/symbol. ☐

8. The binary Elias delta codeword for the natural number 21 is

a) 000010101; ☐

b) 0010110101; ☐

c) 001010101. ■

Part II

1. Consider a real-values source X with probability density function $f_X(x) = 2 - 2x$, for $x \in [0,\, 1]$, and $f_X(x) = 0$, for $x \notin [0,\, 1]$, which has differential entropy $h(X) = (1/2)\log(e/4)$.

   a) Find the differential entropy of the variable $Z = 2 - 2\,X$.

   b) Find the differential entropy of the variable T, with probability density function given by

$$f_T(t) = \begin{cases} (2/3)(t+1) & \Leftarrow \quad t \in [-1,\, 0] \\ (2/3) & \Leftarrow \quad t \in [0,\, 1] \\ 0 & \Leftarrow \quad |t| > 1. \end{cases}$$

   Important hint: draw the probability density function.

   c) Let U be given by: $U = 1$, if $T \leq 0$, and $U = 2$, if $T > 0$. Find the mutual information $I(U;T)$.