Department of Mathematics, IST — Probability and Statistics Unit

Probability Theory

2nd. Test 1st. Semester — **2014/15**
Duration: **1h30m** **2015/01/10** — **8:00 AM**, Room **P1**

- Please justify your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group I — Independence **3.5 points**

In each minute of a basketball game, player A (resp. player B) commits a single foul with probability $\alpha^{(A)}$ (resp. $\alpha^{(B)}$) and no foul with probability $1-\alpha^{(A)}$ (resp. $1-\alpha^{(B)}$), independently of what may have happened in the past minutes.

(a) After having admitted that players A and B commit fouls in an independent fashion, find the p.f. and the expected value of the time until the sixth one-minute-period with at least one foul committed by these two players. **(1.5)**

- **Two independent Bernoulli process**
  $\{X_i^{(*)},\, i \in \mathbb{N}\} \stackrel{i.i.d.}{\sim} \text{Bernoulli}\left(\alpha^{(*)}\right),\ * = A, B$
  $X_i^{(*)} = \begin{cases} 1, & \text{if player } * \text{ commits a foul during the } i^{th} \text{ minute} \\ 0, & \text{otherwise} \end{cases}$
- **Merging independent Bernoulli process**
  Since we admitted that players A and B commit fouls in an independent fashion, we can invoke Prop. 3.91 and conclude that $\{X_i,\, i \in \mathbb{N}\}$, where
  $$X_i = \begin{cases} 1, & \text{if at least one of these two players commits a foul} \\ & \text{during the } i^{th} \text{ minute} \\ 0, & \text{otherwise,} \end{cases}$$
  is also a Bernoulli process with parameter $\alpha = P(X_i = 1) = \alpha^{(A)} + \alpha^{(B)} - \alpha^{(A)} \times \alpha^{(B)}$.
- **R.v. and distribution**
  T_6 = number of minutes until the sixth one-minute-period with at least one foul committed by these two players
  $T_6 \stackrel{Prop.\,3.83}{\sim} \text{NegativeBinomial}(6, \alpha)$
- **Requested p.f. and expected value**
  $P(T_6 = x) = \binom{x-1}{6-1}(1-\alpha)^{x-6}\,\alpha^6,\ x = 6, 7, \ldots$
  $E(T_6) \stackrel{Exercise\,4.146}{=} \frac{6}{\alpha}$.

(b) Player A will foul out of the game once he/she commits his/her sixth foul and will play at most 20 minutes if he/she does not foul out. **(2.0)**

Derive the p.f. of player A's playing time; and determine the probability that he/she plays for 20 minutes when $\alpha^{(A)} = 0.1$.

- **R.v., distribution and range**
  $T_6^{(A)}$ = number of minutes until player A commits six fouls
  $T_6^{(A)} \stackrel{Prop.\,3.83}{\sim} \text{NegativeBinomial}(6, \alpha^{(A)})$
  $\mathbb{R}_{T_6^{(A)}} = \{6, 7, \ldots\}$
- **New r.v.**
  $Z = \min\left\{T_6^{(A)}, 20\right\}$ = player A's playing time
- **Range**
  $\mathbb{R}_Z = \{6, 7, \ldots, 19, 20\}$
- **Requested p.f.**
  For $z = 6, 7, \ldots, 19$,
  $$\begin{aligned} P(Z = z) &= P(Y = z) \\ &= \binom{z-1}{6-1}\left[1-\alpha^{(A)}\right]^{z-6}\left[\alpha^{(A)}\right]^6; \end{aligned}$$
  moreover,
  $$\begin{aligned} P(Z = 20) &= 1 - \sum_{x=6}^{19} P(Z = x) \\ &= 1 - \sum_{x=6}^{19} \binom{x-1}{6-1}\left[1-\alpha^{(A)}\right]^{z-6}\left[\alpha^{(A)}\right]^6. \end{aligned}$$
- **Requested probability**
  $$\begin{aligned} P(Z = 20) &\stackrel{\alpha^{(A)}=0.1}{=} 1 - \sum_{x=6}^{19}\binom{x-1}{6-1}(1-0.1)^{x-6}\,0.1^6 \\ &= 1 - F_{NegativeBinomial(6,0.1)}(19) \\ &\stackrel{Exercise\,2.68}{=} 1 - \left[1 - F_{Binomial(19,0.1)}(6-1)\right] \\ &\stackrel{table}{=} 0.9914. \end{aligned}$$

Group II — Expectation **10.0 points**

1. Let X have a Cauchy distribution with p.d.f. given by $f_X(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < +\infty$.

   (a) Consider $Y = -X$ and show that X and Y are r.v. such that the expectation of the sum is not equal to sum of the expectations. **(1.5)**

   - **R.v. and f.d.p.**
     $X \sim \text{Cauchy}(0, 1)$
     $f_X(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < +\infty$
   - **New r.v.**
     $Y = -X$
     $X + Y = X - X = 0$
   - **Expectation of the sum**
     $$\begin{aligned} E(X+Y) &= E(0) \\ &= 0 \end{aligned}$$

- **Sum of the expectations**
  First, note that

$$\begin{aligned} E(X^+) &= E(\max\{X,0\}) \\ &= \int_0^{+\infty} x \times \frac{1}{\pi(1+x^2)}\,dx \\ &= \frac{1}{2\pi}\ln(1+x^2)\Big|_0^{+\infty} \\ &= +\infty \\ E(X^-) &= E(-\min\{X,0\}) \\ &= \int_{-\infty}^{0} (-x) \times \frac{1}{\pi(1+x^2)}\,dx \\ &= -\frac{1}{2\pi}\ln(1+x^2)\Big|_{-\infty}^{0} \\ &= +\infty. \end{aligned}$$

  Secondly, since $E(X^-) = +\infty$ and $E(X^+) = +\infty$ then $E(X)$ does not exist (see Remark 4.40); in fact

$$\begin{aligned} E(|X|) &= E(X^+ + X^-) \\ &= \int_{-\infty}^{+\infty} |x| \times \frac{1}{\pi(1+x^2)}\,dx \\ &= \int_{-\infty}^{0} (-x) \times \frac{1}{\pi(1+x^2)}\,dx + \int_{-\infty}^{0} (-x) \times \frac{1}{\pi(1+x^2)}\,dx \\ &\not< +\infty \end{aligned}$$

  that is, X is not an integrable r.v.
  Thirdly, since $E(X)$ does not exist, $E(Y)$ and the sum $E(X) + E(Y)$ do not exist either.
  Unsurpringly, the expectation of the sum is not equal to sum of the expectations because the latter does not even exist.

(b) Does this result contradict the linearity of expectation? Justify your answer. **(0.5)**

- **Comment**
  No, the result does not contradict the linearity of expectation, because this property is stated for integrable r.v. (see Theorem 4.41) unlike X and Y.

2. Suppose it is known that the number of widgets produced in a factory during an hour is a r.v. with expected value 500.

(a) Obtain an upper bound for the probability that an hour's production will be of at least 550 widgets? **(1.0)**

- **R.v.**
  $X =$ number of widgets produced in a factory during an hour
  $E(X) = 500$
- **Requested upper bound**
  Since $E(X)$ is finite, we can apply Markov's inequality for L^1 r.v. and obtain:

$$\begin{aligned} P(X \geq 550) &\overset{X\geq 0}{=} P(|X| \geq 550) \\ &\leq \frac{E(|X|)}{550} \\ &= \frac{500}{550} \\ &\simeq 0.909091. \end{aligned}$$

(b) Recalculate this upper bound assuming additionally that the variance is equal to 100. Comment. **(1.5)**

- **Recalculating the upper bound**
  By additionally assuming that the variance is finite and equal to 100, we can invoke the one-sided Chebyshev's inequality for L^2 r.v. and get

$$\begin{aligned} P(X \geq 550) &= P[X - E(X) \geq 550 - E(X)] \\ &\overset{E(X)=500,\,V(X)=100}{=} P[X - E(X) \geq 5 \times \sqrt{V(X)}] \\ &\leq \frac{1}{1+5^2} \\ &\simeq 0.038462. \end{aligned}$$

  [Another possibility would be using Markov's inequality for L^2 r.v. This would lead to a less stringent upper bound:

$$\begin{aligned} P(X \geq 550) &= P(|X| \geq 550) \\ &\leq \frac{E(X^2)}{550^2} \\ &= \frac{V(X) + E^2(X)}{550^2} \\ &= \frac{100 + 500^2}{550^2} \\ &\simeq 0.826777.] \end{aligned}$$

- **Comment**
  Generally, the stronger the assumptions you make, the tighter the bounds you get!
  Expectedly, knowing the variance (substantially!) improved the upper bound for $P(X \geq 550)$. [Moreover, Markov's inequality applied in (a) is not one-sided unlike the inequality used in (b).]

3. Let $X = \cos(\Theta)$ and $Y = \sin(\Theta)$ be two r.v., where $\Theta \sim \text{Uniform}(0, 2\pi)$.

(a) Show X and Y are uncorrelated r.v. **(1.5)**
Note: It might be useful to recall in (a) and (b) that: $2\sin(\theta)\cos(\theta) = \sin(2\theta)$; $2\cos^2(\theta) = 1 + \cos(2\theta)$; $2\sin^2(\theta) = 1 - \cos^2(2\theta)$.

- **R.v. and p.d.f.**
  $\Theta \sim \text{Uniform}(0, 2\pi)$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

- **Other r.v.**
  $X = \cos(\Theta)$

$Y = \sin(\Theta)$

- **Checking whether X and Y are uncorrelated**
  By Corollary 4.81,

$$\begin{aligned} E(X) &= E[\cos(\Theta)] \\ &= \int_0^{2\pi} \cos(\theta) \times \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{2\pi}\, \sin(\theta)\Big|_0^{2\pi} \\ &= 0 \\ E(Y) &= E[\sin(\Theta)] \\ &= \int_0^{2\pi} \sin(\theta) \times \frac{1}{2\pi}\, d\theta \\ &= -\frac{1}{2\pi}\, \cos(\theta)\Big|_0^{2\pi} \\ &= 0 \\ E(XY) &= E[\cos(\Theta)\, \sin(\Theta)] \\ &= \int_0^{2\pi} \cos(\theta)\, \sin(\theta) \times \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} 2\cos(\theta)\, \sin(\theta) d\theta \\ &\overset{hint}{=} \frac{1}{4\pi} \int_0^{2\pi} \sin(2\theta) d\theta \\ &= -\frac{1}{8\pi}\, \cos(2\theta)\Big|_0^{2\pi} \\ &= 0. \end{aligned}$$

  Hence, $cov(X, Y) = E(XY) - E(X) \times E(Y) = 0$ and

  $corr(X, Y) = 0$,

  that is, X and Y are uncorrelated r.v.

(b) Prove that $E(X^2Y^2) \neq E(X^2) \times E(Y^2)$. What can you conclude? **(2.5)**

- **To prove**
  $E(X^2Y^2) \neq E(X^2) \times E(Y^2)$
- **Proof**

$$\begin{aligned} E(X^2) &= \int_0^{2\pi} \cos^2(\theta) \times \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} 2\cos^2(\theta)\, d\theta \\ &\overset{hint}{=} \frac{1}{4\pi} \int_0^{2\pi} [1 + \cos(2\theta)]\, d\theta \\ &= \left[\frac{\theta}{4\pi} + \frac{1}{8\pi} \sin(2\theta)\right]\Big|_0^{2\pi} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_0^{2\pi} \sin^2(\theta) \times \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} 2\sin^2(\theta)\, d\theta \\ &\overset{hint}{=} \frac{1}{4\pi} \int_0^{2\pi} [1 - \cos(2\theta)]\, d\theta \\ &= \left[\frac{\theta}{4\pi} - \frac{1}{8\pi} \sin(2\theta)\right]\Big|_0^{2\pi} \\ &= \frac{1}{2} \\ E(X^2Y^2) &= \int_0^{2\pi} \cos^2(\theta)\, \sin^2(\theta) \times \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{8\pi} \int_0^{2\pi} 2\cos^2(\theta) \times 2\sin^2(\theta)\, d\theta \\ &\overset{hint}{=} \frac{1}{8\pi} \int_0^{2\pi} [1 + \cos(2\theta)] \times [1 - \cos(2\theta)]\, d\theta \\ &= \frac{1}{8\pi} \int_0^{2\pi} [1 - \cos^2(2\theta)]\, d\theta \\ &= \frac{1}{16\pi} \int_0^{2\pi} [2 - 2\cos^2(2\theta)]\, d\theta \\ &\overset{hint}{=} \frac{1}{16\pi} \int_0^{2\pi} \{2 - [1 + \cos(4\theta)]\}\, d\theta \\ &= \left[\frac{\theta}{16\pi} - \frac{1}{64\pi} \sin(4\theta)\right]\Big|_0^{2\pi} \\ &= \frac{1}{8} \\ &\neq E(X^2) \times E(Y^2) \\ &= \frac{1}{4}. \end{aligned}$$

QED

- **Conclusion**
  Since $E(X^2Y^2) \neq E(X^2) \times E(Y^2)$, Corollary 4.86 (or Theorem 3.6, disjoint blocks theorem) leads us to state that X and Y are not independent r.v.

4. Suppose X (resp. Y) represent the height of the woman (resp. man) in a opposite sex married couple. Admit that the random vector (X, Y) has a bivariate normal distribution with parameters $\mu_X = 66.8$ inches, $\mu_Y = 70$ inches, $\sigma_X = \sigma_Y = 2$ inches and correlation coefficient $\rho = 0.68$.

   Obtain the probability that the wife is taller than her husband. **(1.5)**

   - **Random vector (X, Y)**
     X = wife's height (in inches)
     Y = husband's height (in inches)
     $(X, Y) \sim \text{Normal}_2(\underline{\mu}, \mathbf{\Sigma})$, where:

$$\underline{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} 66.8 \\ 70 \end{bmatrix};$$
$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_X^2 & \rho\,\sigma_X\,\sigma_Y \\ \rho\,\sigma_X\,\sigma_Y & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} 2^2 & 0.68\times 2\times 2 \\ 0.68\times 2\times 2 & 2^2 \end{bmatrix}.$$

- **Requested probability**
  $P(X > Y) = P(X - Y > 0)$
- **New r.v.**
  $W = X - Y = \mathbf{C} \times \begin{bmatrix} X \\ Y \end{bmatrix} + \underline{b}$, where $\mathbf{C} = [1 \quad -1]$ and $\underline{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- **Distribution of W**
  According to Theorem 4.190, $W = X - Y \sim \text{Normal}(E(W), V(W))$, where:

$$\begin{aligned} E(W) &= \mathbf{C}\,\underline{\mu} + \underline{b} \\ &= \mu_X - \mu_Y \\ &= 66.8 - 70 \\ &= -3.2; \\ V(W) &= \mathbf{C}\,\mathbf{\Sigma}\,\mathbf{C}^\top \\ &= V(X) - 2cov(X,Y) + V(Y) \\ &= \sigma_X^2 - 2\times\rho\,\sigma_X\,\sigma_Y + \sigma_Y^2 \\ &= 2^2 - 2\times 0.68\times 2\times 2 + 2^2 \\ &= 2.56. \end{aligned}$$

- **Requested probability** (cont'd)

$$\begin{aligned} P(W > 0) &= 1 - \Phi\left[\frac{0 - E(W)}{\sqrt{V(W)}}\right] \\ &= 1 - \Phi\left[\frac{0-(-3.2)}{\sqrt{2.56}}\right] \\ &\simeq 1 - \Phi(2) \\ &\stackrel{table}{=} 1 - 0.9772 \\ &= 0.0228. \end{aligned}$$

Group III — Convergence of sequences of r.v. **6.5 points**

1. Let $\{X_1, X_2, \dots\}$ be a sequence of r.v. defined by the p.f. **(2.5)**

$$P(X_n = x) = \begin{cases} \frac{1}{n+4}, & x = -n-4 \\ 1 - \frac{4}{n+4}, & x = -1 \\ \frac{3}{n+4}, & x = n+4 \end{cases}$$

Prove that $X_n \stackrel{P}{\rightarrow} X$ and $E(X_n) \not\rightarrow E(X)$, where $X \stackrel{d}{=} -1$. Comment.

- **Sequence of independent r.v.**
  $\{X_1, X_2, \dots\}$
- **P.f. and c.d.f.**

$$P(X_n = x) = \begin{cases} \frac{1}{n+4}, & x = -n-4 \\ 1 - \frac{4}{n+4}, & x = -1 \\ \frac{3}{n+4}, & x = n+4 \end{cases}$$

$$F_{X_n}(x) = \begin{cases} 0, & x < -n-4 \\ \frac{1}{n+4}, & -n-4 \le x < -1 \\ 1 - \frac{3}{n+4}, & -1 \le x < n+4 \\ 1, & x \ge n+4 \end{cases}$$

- **Limiting r.v.**
  $X \stackrel{d}{=} -1$
- **Proving that $X_n \stackrel{P}{\rightarrow} X$**
  Note that, for all $\epsilon > 0$,

$$\begin{aligned} P(|X_n - X| > \epsilon) &\stackrel{X\stackrel{d}{=}-1}{=} 1 - P(-\epsilon - 1 \le X_n \le \epsilon - 1) \\ &= 1 - F_{X_n}(\epsilon - 1) + F_{X_n}[(-\epsilon-1)^-]. \end{aligned}$$

Since -1 is not a continuity point of $F_{X_n}(x)$ for any $n \in \mathbb{N}$, we can add:

- $\epsilon - 1 > -1$, thus $F_{X_n}(\epsilon - 1) > F_{X_n}(-1) = 1 - \frac{3}{n+4}$;
- $-\epsilon - 1 < -1$, hence $F_{X_n}[(-\epsilon-1)^-] \le F_{X_n}[(-1)^-] = \frac{1}{n+4}$.

Consequently, for all $\epsilon > 0$,

$$\begin{aligned} P(|X_n - X| > \epsilon) &= 1 - F_{X_n}(\epsilon - 1) + F_{X_n}[(-\epsilon-1)^-] \\ &\le 1 - \left(1 - \frac{3}{n+4}\right) + \frac{1}{n+4} \\ &= \frac{4}{n+4} \\ &\rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$.
Hence, for all $\epsilon > 0$, $\lim_{n\rightarrow+\infty} P(|X_n - X| > \epsilon) = 0$, i.e., $X_n \stackrel{P}{\rightarrow} X$. QED

- **Proving that $E(X_n) \not\rightarrow E(X)$**
  X_n is a simple r.v., therefore its expectation is given by

$$\begin{aligned} E(X_n) &= (-n-4)\times\frac{1}{n+4} + (-1)\times\left(1 - \frac{4}{n+4}\right) + (n+4)\times\frac{3}{n+4} \\ &= -1 - 1 + \frac{4}{n+4} + 3 \\ &= 1 + \frac{4}{n+4} \\ &\rightarrow 1, \end{aligned}$$

as $n \rightarrow +\infty$. However, $E(X) = E(-1) = -1$. Hence, $E(X_n) \not\rightarrow E(X)$. QED

- **Comment**

We have just shown that

$$X_n \xrightarrow{P} X \;\not\Rightarrow\; E(X_n) \to E(X)$$

that is, the convergence in probability of a sequence of r.v., $\{X_1, X_2, \dots\}$, to X is not enough to ensure that the corresponding sequence of expections $\{E(X_1), E(X_2), \dots\}$, converges to $E(X)$.

2. Let X_n take values 1 and n^{-1} with probabilities n^{-1} and $(1-n^{-1})$, respectively, and $g(x) = I_{(0,+\infty)}(x)$. **(2.0)**

Show that $X_n \xrightarrow{P} X$ and yet $g(X_n) \not\xrightarrow{P} g(X)$, where $X \stackrel{d}{=} 0$, thus, illustrating that convergence in probability may not be preserved under non continuous mappings.[1]

- **Sequence of r.v.**

$\{X_1, X_2, \dots\}$

- **Common p.f. and c.d.f.**

$$P(X_n = x) = \begin{cases} 1-n^{-1}, & x = n^{-1} \\ n^{-1}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_1(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases} \qquad F_n(x) = \begin{cases} 0, & x < n^{-1} \\ 1-n^{-1}, & n^{-1} \leq x < 1 \\ 1, & x \geq 1 \end{cases} \qquad (n = 2, 3, \dots)$$

- **Proving that $X_n \xrightarrow{P} X$**

Since $X \stackrel{d}{=} 0$, for all $\epsilon > 0$:

$$\begin{aligned} P(|X_n - X| > \epsilon) &= 1 - P(-\epsilon \leq X_n \leq \epsilon) \\ &= 1 - F_{X_n}(\epsilon) + F_{X_n}(-\epsilon^-) \\ &\stackrel{X_n \geq 0}{=} 1 - F_{X_n}(\epsilon) \\ &= \begin{cases} 1, & 0 < \epsilon < n^{-1} \\ n^{-1}, & n^{-1} \leq \epsilon < 1 \\ 0, & \epsilon \geq 1; \end{cases} \\ \lim_{n\to+\infty} P(|X_n - X| > \epsilon) &= \begin{cases} 0, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \\ &= 0. \end{aligned}$$

That is, $X_n \xrightarrow{P} 0$.

- **Verifying that $g(X_n) \not\xrightarrow{P} g(X)$**

For each n, $g(X_n) = I_{(0,+\infty)}(X_n) = 1$ because X_n is a non negative r.v. However, $g(X) = g(0) = 0$ and hence $g(X_n)$ cannot converge in any reasonable sense to $g(X)$.

[1] I.e., the continuity of the mapping is an essential condition in Theorem 5.81.

3. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the expected lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for at least 2000 hours is at least 0.95? **(2.0)**

Hint: It may be useful to invoke the Lindeberg-Lévy Central Limit Theorem.

- **R.v.**

X_i = duration of the i^{th} component
$X_i \stackrel{i.i.d.}{\sim} X$, $i = 1, 2, \dots$
$E(X) = \mu = 100$, $V(X) = \sigma^2 = 30^2 < \infty$;

- **Another r.v.**

$S_n = \sum_{i=1}^n X_i$
= continual duration of the electrical system when $(n-1)$components are in stock

- **Approximate distribution of S_n**

Let $\{Z_1, Z_2, \dots\}$ be the sequence of the standardized partial sums, where $Z_n = \frac{S_n - E(S_n)}{\sqrt{V(S_n)}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$. Then, according to the Lindeberg-Lévy CLT, $Z_n \xrightarrow{d}$ Normal$(0,1)$. Moreover, for a sufficiently large value of n,

$$P(S_n \leq s) \simeq \Phi\left(\frac{s - n\mu}{\sqrt{n\sigma^2}}\right).$$

- **Requested number of components in stock**

This number can be obtained approximately:

$$\begin{aligned} (n-1) \in \mathbb{N} \;:\; & P(S_n > 2000) \geq 0.95 \\ & 1 - \Phi\left(\frac{2000 - n \times 100}{\sqrt{n \times 30^2}}\right) \geq 0.95 \\ & -\Phi^{-1}(0.95) = -1.6449 \geq \frac{2000 - 100n}{30\sqrt{n}} \\ & 100 \times \sqrt{n}^2 - 1.6449 \times 30 \times \sqrt{n} - 2000 \geq 0. \end{aligned}$$

Now, note that: the largest solution of $100 \times x^2 - 1.6449 \times 30 \times x - 2000 = 0$ is

$$\frac{1.6449 \times 30 + \sqrt{(1.6449 \times 30)^2 - 4 \times 100 \times (-2000)}}{2 \times 100} \simeq 4.725672;$$

the ceiling of its square is 23; and since

$$\begin{aligned} 1 - \Phi\left(\frac{2000 - 100 \times 23}{30\sqrt{23}}\right) &\simeq 1 - \Phi(-2.09) \\ &= \Phi(2.09) \\ &= 0.9817 \\ &\geq 0.95, \end{aligned}$$

the approximate number of components in stock is $n - 1 = 22$.
[Note that $1 - \Phi\left(\frac{2000-100\times 22}{30\sqrt{22}}\right) \simeq 0.92 \not\geq 0.95$.]