Department of Mathematics, IST — Probability and Statistics Unit

Probability Theory

1st. Test ("Recurso") 1st. Semester — **2014/15**

Duration: **1h30m** **2015/01/24** — **8:00 AM**, Room P1

- Please justify your answers.
- This test has TWO PAGES and FOUR GROUPS. The total of points is 20.0.

Group I — Warm up **2.5 points**

Suppose:

- X_n represents the deviation from a target of the accumulated sales of a store at day n;
- X_n is modelled by a symmetric random walk;
- the accounting period starts with a deviation from target equal to $X_0 = 0$;
- R_{2n} is the time of the last return (of the deviation from target) to 0 during the first $2n$ days.

Then

$$\lim_{n\to+\infty} P\left(a \leq \frac{R_{2n}}{2n} \leq b\right) = \int_a^b \frac{1}{\pi\sqrt{y(1-y)}}\,dy, \qquad (1)$$

for $0 \leq a < b \leq 1$.

(a) Interpret $\frac{R_{2n}}{2n}$ and relate (1) to Proposition 0.15 of your lecture notes. **(1.0)**

- **Process**
  Symmetric random walk (SRW)
- **R.v.**
  $Y_i =$ size of the i^{th} step
  $Y_i \stackrel{i.i.d.}{\sim} Y,\ i \in \mathbb{N}$
  $$P(Y=y) = \begin{cases} \frac{1}{2}, & y=-1 \text{ (unitary increase in the deviation from target)} \\ \frac{1}{2}, & y=1 \text{ (unitary decrease in the deviation from target)} \\ 0, & \text{otherwise} \end{cases}$$
  $X_n = \sum_{i=1}^n Y_i$
  $=$ deviation from target of the accumulated sales of a store at day n ($n \in \mathbb{N}$)
- **Initial condition**
  $X_0 = 0$
- **Requested interpretation**
  The r.v. $\frac{R_{2n}}{2n}$ takes values in $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ and therefore corresponds to the fraction of those first $2n$ days it took the last return to the origin to occur.
  [$\frac{R_{2n}}{2n}$ gives us an idea of how early (or how late) the last return to the origin occurred in those first $2n$ days.]

- **Related result**
  According to Prop. 0.15,
  $$\lim_{n\to+\infty} P\left(a \leq \frac{R_{2n}}{2n} \leq b\right) = \int_a^b \frac{1}{\pi\sqrt{y(1-y)}}\,dy = \frac{2}{\pi}\times[\arcsin(b)-\arcsin(a)],$$
  for $0 \leq a < b \leq 1$. Consequently (and interestingly), result (1) coincides with the limit law of the fraction of time spent positive of a SRW, the *arc sine law.*

(b) Consider 100 days and obtain an approximate value for the probability that the last return to 0 occurs not later than the 20^{th} day. Compare it with the approximate probability that the last return to 0 occurs after the 39^{th} day but not later than the 60^{th} day. Comment. **(1.5)**

- **Requested approximate probabilities**
  Since
  $$\lim_{n\to+\infty} P\left(a \leq \frac{R_{2n}}{2n} \leq b\right) = \int_a^b \frac{1}{\pi\sqrt{y(1-y)}}\,dy = \frac{2}{\pi}\times[\arcsin(b)-\arcsin(a)],$$
  for $0 \leq a < b \leq 1$, we can add that, for $2n = 100$ an approximate value for the probability that the last return to 0 occurs:
  - not later than the 20^{th} day is given by
    $$P\left(0 \leq \frac{R_{100}}{100} \leq \frac{20}{100}\right) \simeq \frac{2}{\pi}\times[\arcsin(0.2)-\arcsin(0)] \simeq 0.295167;$$
  - after the 39^{th} day but not later than the 60^{th} day is
    $$P\left(\frac{40}{100} \leq \frac{R_{100}}{100} \leq \frac{60}{100}\right) \simeq \frac{2}{\pi}\times[\arcsin(0.6)-\arcsin(0.4)] \simeq 0.128188.$$
- **Comment**
  Even though the (approximate) probabilities refer to periods with the same range (20 days), it is reasonable to state that
  $$P\left(0 \leq \frac{R_{100}}{100} \leq \frac{20}{100}\right) \geq P\left(\frac{40}{100} \leq \frac{R_{100}}{100} \leq \frac{60}{100}\right).$$
  This result illustrates that in the long-run the most probable scenarios are those where the last visit to the origin occurs very early (or very late) rather than in the midst of the first $2n$ days period.
  This is a consequence of the fact that the limit law of $\frac{R_{2n}}{2n}$ is an U–shaped density, thus, in the long-run this r.v. is more likely to be near 0 or 1 than near 1/2, as mentioned in Prop. 0.15.

Group II — Probability spaces **4.5 points**

1. Consider the sample space Ω and let $\mathcal{F}$ be a $\sigma-$algebra on Ω and $A, B \in \mathcal{F}$.

(a) Prove that $\mathcal{F}$ contains the sets $A\backslash B$ and $A\Delta B$. **(1.5)**

- **$\sigma-$algebra on Ω**
  $\mathcal{F}$
- **To prove**
  $A\backslash B,\ A\Delta B \in \mathcal{F}$
- **Proof**
  According to Def. 1.38, a minimal set of postulates for $\mathcal{F}$ to be a $\sigma-$algebra on Ω is:
  (i) $\Omega \in \mathcal{F}$;
  (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (i.e., $\mathcal{F}$ is closed under complementation);
  (iii) $A_i \in \mathcal{F},\ i \in I$ (countable set) $\Rightarrow \cup_{i\in I} A_i \in \mathcal{F}$ (i.e., $\mathcal{F}$ is closed under countable union).
  Consequently,

$$\begin{aligned} A\backslash B &= A \cap B^c \\ &= (A^c \cup B)^c \\ &\in \mathcal{F} \\ A\Delta B &= (A\backslash B) \cup (B\backslash A) \\ &= (A^c \cup B)^c \cup (A \cup B^c)^c \\ &\in \mathcal{F}. \end{aligned}$$

(b) Let $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Justify that $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$. **(1.5)**

- **Events and probabilities**
  A, B
  $P(A) = \frac{3}{4},\ P(B) = \frac{1}{3}$
- **To prove**
  $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$
- **Proof**
  If we recall the addition rule and that another elementary property of p.f. is monotonicity (Prop. 1.54), we can add that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ P(A \cap B) &\overset{P(A\cup B)\leq P(\Omega)=1}{\geq} P(A) + P(B) - 1 \\ &= \frac{3}{4} + \frac{1}{3} - 1 \\ &= \frac{1}{12}. \end{aligned}$$

Using once again the fact that a p.f. P is a non-decreasing set function and capitalizing on

$$\begin{aligned} A \cap B &\subseteq A \\ A \cap B &\subseteq B, \end{aligned}$$

we can conclude that

$$\begin{aligned} P(A \cap B) &\leq P(A) \\ P(A \cap B) &\leq P(B). \end{aligned}$$

Hence,

$$\begin{aligned} P(A \cap B) &\leq \min\{P(A), P(B)\} \\ &= \min\left\{\frac{3}{4}, \frac{1}{3}\right\} \\ &= \frac{1}{3}. \end{aligned}$$

QED

2. A fair coin is tossed repeatedly (and independently). Prove that the event *no head will ever turn up* has probability zero. **(1.5)**

- **To prove**
  $P(\text{no head ever}) = 0$
- **Key events**
  $A_n = \{\text{no head in the first } n \text{ tosses}\},\ n \in \mathbb{N}$
- **Proof**
  $A_1 \supset A_2 \supset \ldots$, i.e., $\{A_1, A_2, \ldots\}$ is a decreasing sequence of events. As a consequence,

$$\begin{aligned} \lim_{n\to+\infty} A_n &= \bigcap_{n=1}^{+\infty} A_n \\ &= \{\text{no head ever}\} \\ &= A. \end{aligned}$$

Furthermore, invoking the fact that the fair coin is tossed repeatedly/independently and the monotone continuity of p.f. (Prop. 1.64), we successively get:

$$\begin{aligned} P(\text{head at toss } i) &= 2^{-1},\ i \in \mathbb{N} \\ P(A_n) &= \prod_{i=1}^{n} P(\text{head at toss } i) \\ &= 2^{-n} \\ P(A) &= P\left(\lim_{n\to+\infty} A_n\right) \\ &= \lim_{n\to+\infty} P(A_n) \\ &= \lim_{n\to+\infty} 2^{-n} \\ &= 0. \end{aligned}$$

QED

Group III — Random variables and independence **8.5 points**

1. The radius (X) of a circle is roughly measured. Admit that X is uniformly distributed over the interval (a,b), where $0 < a < b$.

   (a) Is the area of the circle $Y = \pi X^2 : \Omega \to \mathbb{R}$ also a r.v.? Justify. **(2.0)**

   - **Important**
     Let:
     - X be a real r.v.;
     - $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable spaces.

     Then, by Def. 2.13, $X : \Omega \to \mathbb{R}$ and

     $$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R}),$$

     in particular,

     $$X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R},$$

     according to Prop. 2.16.
   - **To prove**
     $Y(\omega) = h[X(\omega)] = \pi[X(\omega)]^2$ is also a r.v.
   - **Proof**
     Firsty, recall that a function $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff

     $$h^{-1}(B) \in \mathcal{B}(\mathbb{R}), \forall B \in \mathcal{B}(\mathbb{R}),$$

     according to Def. 2.36, with $n = m = 1$. Furthermore, in order that $h : \mathbb{R} \to \mathbb{R}$ be Borel measurable it suffices that $h^{-1}((-\infty, y]) \in \mathcal{B}(\mathbb{R})$, $\forall y \in I\!R$, according to Remark 2.47 (with $n = 1$).
     Secondly, since $X(\omega) \in [a, b]$, we get

     $$Y(\omega) = h[X(\omega)] = \pi[X(\omega)]^2 \in \left[\pi a^2,\, \pi b^2\right].$$

     Finally, $h^{-1}((-\infty, y])$ equals:
     - for $y < \pi a^2$,
       $$\begin{aligned}\{x \in \mathbb{R} : h(x) \leq y\} &= \emptyset \\ &\in \mathcal{B}(\mathbb{R});\end{aligned}$$
     - for $\pi a^2 \leq y < \pi b^2$,
       $$\begin{aligned}\{x \in \mathbb{R} : h(x) \leq y\} &= \left\{x \in \mathbb{R} : \pi x^2 \leq y\right\} \\ &= \left[a,\, \sqrt{y/\pi}\right] \\ &\in \mathcal{B}(\mathbb{R});\end{aligned}$$
     - for $y \geq \pi b^2$,
       $$\begin{aligned}\{x \in \mathbb{R} : h(x) \leq y\} &= \mathbb{R} \\ &\in \mathcal{B}(\mathbb{R}).\end{aligned}$$

     As a result, we can state that h is indeed a Borel measurable function and therefore $Y = h(X)$ is a r.v., by Corollary 2.40. QED

   (b) Find the c.d.f. and p.d.f. of Y. **(2.0)**

   - **R.v.**
     X = rough measure of the radius of a circle
     $X \sim \text{Uniforme}(a, b)$, $0 < a < b$
   - **P.d.f. of X**
     $$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{c.c.} \end{cases}$$
   - **Range of X**
     $\mathbb{R}_X = [a, b]$
   - **C.d.f. of X**
     $$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases} \end{aligned}$$
   - **Transformation**
     $Y = h(X) = \pi X^2$
   - **Range of Y**
     $\mathbb{R}_Y = h(\mathbb{R}_X) = [\pi a^2,\, \pi b^2]$
   - **C.d.f. of Y**
     Since $y = h(x) = \pi x^2$ is a strictly increasing function in $\mathbb{R}_X$, we can apply Prop. 2.71 and obtain $F_Y(y)$ in terms of $F_X(x)$:
     $$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\left[\pi X^2 \leq y\right] \\ &\overset{(a)}{=} \begin{cases} P(\emptyset) = 0, & y < a^2 \\ P\left(a \leq X \leq \sqrt{y/\pi}\right) = F_X(\sqrt{y/\pi}) - F_X(a) & \\ \qquad = \frac{\sqrt{y/\pi}-a}{b-a}, & \pi a^2 \leq y < \pi b^2 \\ P(\Omega) = 1, & y \geq \pi b^2. \end{cases} \end{aligned}$$
   - **P.d.f. of Y**
     $$\begin{aligned} f_Y(y) &= \frac{d\,F_Y(y)}{dy} \\ &= \begin{cases} \frac{d}{dy}\left(\frac{\sqrt{y/\pi}-a}{b-a}\right) = \frac{1}{b-a} \times \frac{1}{2\sqrt{\pi y}}, & \pi a^2 \leq y \leq \pi b^2 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

2. Let $X \sim \text{Exponential}(1)$ and find the transformation $Y = g(X)$ such that $f_Y(y) = \frac{1}{2\sqrt{y}} \times I_{(0,1)}(y)$. **(2.0)**

- **R.v.**
  $X \sim \text{Exponential}(1)$
  Y has p.d.f. $f_Y(y) = \frac{1}{2\sqrt{y}} \times I_{(0,1)}(y)$.
- **C.d.f. of X and Y**

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases}$$

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(z)\, dz \\ &= \begin{cases} 0, & y < 0 \\ \int_0^y \frac{1}{2\sqrt{z}}\, dz = \sqrt{y}, & 0 \leq y < 1 \\ 0, & y \geq 1 \end{cases} \end{aligned}$$

- **Important**
  By Prop. 2.140, if $U \sim \text{Uniform}(0,1)$ then $F_Z^{-1}(U) \stackrel{d}{=} Z$, $Z = X, Y$. Consequently,

$$\begin{aligned} F_X(X) &= 1 - e^{-X} \\ &\sim \text{Uniform}(0,1) \\ F_Y(Y) &= \sqrt{Y} \\ &\sim \text{Uniform}(0,1). \end{aligned}$$

  By setting $F_Y(Y) = F_X(X)$, we obtain

$$\begin{aligned} \sqrt{Y} &= 1 - e^{-X} \\ Y &= \left(1 - e^{-X}\right)^2 \\ &= g(X). \end{aligned}$$

3. Two people agreed to meet each other on a particular day, between 5 and 6PM. They arrive independently and uniformly in the interval between 5 and 6PM, and wait for not more than 15 minutes. What is the probability that they meet each other? **(2.5)**

- **R.v.**
  X_i = arrival time (in hours) of person i, $i = 1, 2$
  $X_i \stackrel{i.i.d.}{\sim} \text{Uniform}(5,6)$, $i = 1, 2$
  $f_{X_i}(x_i) = I_{[5,6]}(x_i)$, $i = 1, 2$
- **Joint p.d.f.**

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &\stackrel{X_i\ indep}{=} f_{X_1}(x_1) \times f_{X_2}(x_2) \\ &= I_{[5,6]\times[5,6]}(x_1,x_2) \end{aligned}$$

- **Requested probability**
  Considering that the two people arrive independently and uniformly in the interval between 5 and 6PM, and wait for not more than 15 minutes, we have

$$\begin{aligned} P\left(|X_1 - X_2| \leq \frac{1}{4}\right) &= P(-0.25 \leq X_1 - X_2 \leq 0.25) \\ &= \int_{\{(x_1,x_2)\in\mathbb{R}^2: -0.25 \leq x_1 - x_2 \leq 0.25\}} f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2 \\ &\stackrel{(x_1,x_2)\in[5,6]^2,\, etc.}{=} \int_5^6 \int_{\max\{5, x_2-0.25\}}^{\min\{6, x_2+0.25\}} dx_1\, dx_2 \\ &= \int_5^6 [\min\{6, x_2+0.25\} - \max\{5, x_2-0.25\}]\, dx_2 \\ &= \left[\int_5^{5.75} (x_2+0.25)\, dx_2 + \int_{5.75}^6 6\, dx_2\right] \\ &\quad - \left[\int_5^{5.25} 5\, dx_2 + \int_{5.25}^6 (x_2 - 0.25)\, dx_2\right] \\ &= \left[\left(\frac{x_2^2}{2} + 0.25x_2\right)\Big|_5^{5.75} + 6x_2|_{5.75}^6\right] \\ &\quad - \left[5x_2|_5^{5.25} + \left(\frac{x_2^2}{2} - 0.25x_2\right)\Big|_{5.25}^6\right] \\ &= 0.4375. \end{aligned}$$

Group IV — Independence **4.5 points**

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, A_2, B \in \mathcal{F}$ with $P(B) > 0$. Then A_1 and A_2 are *conditionally independent* given event B iff **(2.5)**

$$P[A_1 \cap A_2) \mid B] = P(A_1 \mid B) \times P(A_2 \mid B).$$

Suppose we have at our disposal two coins, say a and b, and select a coin at random and toss it twice. Let:

- p_a and p_b ($p_a \neq p_b$) be the probabilities of heads for a and b, respectively;
- A_1 = {head at the first tossing} and A_2 = {head at the second tossing} be two events;
- B = {coin a is selected} a third event.

Prove that A_1 and A_2 are *conditionally independent* given event B, but A_1 and A_2 are dependent events.

- **Events**
  A_1 = {head at the first tossing}
  A_2 = {head at the second tossing}
  B = {coin a is selected}
- **To prove**
  $P[A_1 \cap A_2) \mid B] = P(A_1 \mid B) \times P(A_2 \mid B)$ (*conditional independence* given event B)
  BUT

$P(A_1 \cap A_2) \neq P(A_1) \times P(A_2)$ (dependence)

- **Proof**

On the one hand,

$$\begin{aligned} P(A_i \mid B) &= p_a,\ i = 1,2 \\ P[(A_1 \cap A_2) \mid B] &= p_a \times p_a \\ &= P(A_1 \mid B) \times P(A_2 \mid B), \end{aligned}$$

that is, A_1 and A_2 are *conditionally independent* given event B (and B^c).[1]

On the other hand, using the total probability law and the fact that a coin is selected at random (out of two possible coins, a and b), we conclude that:

$$\begin{aligned} P(A_i) &= P(A_i \mid B) \times P(B) + P(A_i \mid B^c) \times P(B^c) \\ &= p_a \times \frac{1}{2} + p_b \times \frac{1}{2} \\ &= \frac{p_a + p_b}{2},\ i = 1,2; \\ P(A_1 \cap A_2) &= P[(A_1 \cap A_2) \mid B] \times P(B) + P[(A_1 \cap A_2) \mid B^c] \times P(B^c) \\ &\overset{cond.\,indep.}{=} P(A_1 \mid B) \times P(A_2 \mid B) \times P(B) \\ &\quad + P(A_1 \mid B^c) \times P(A_2 \mid B^c) \times P(B^c) \\ &= p_a \times p_a \times \frac{1}{2} + p_b \times p_b \times \frac{1}{2} \\ &= \frac{p_a^2 + p_b^2}{2}. \end{aligned}$$

Moreover, since $p_a \neq p_b$ the equality

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1) \times P(A_2) \\ \frac{p_a^2 + p_b^2}{2} &= \left(\frac{p_a + p_b}{2}\right)^2 \end{aligned}$$

is not fulfilled and we conclude that A_1 and A_2 are dependent.

2. Let X, Y and Z represent the coordinates of a particle bound to be positioned on $(\mathbb{R}^+)^3$. Suppose X, Y and Z have the joint p.d.f.

$$f_{X,Y,Z}(x,y,z) = \begin{cases} e^{-(x+y+z)}, & x,y,z > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Are X, Y and Z *mutually* independent r.v.? **(2.0)**

- **Random vector**

(X,Y,Z) = coordinates of a particle bound to be positioned on $(\mathbb{R}^+)^3$

- **Joint p.d.f.**

$$f_{X,Y,Z}(x,y,z) = \begin{cases} e^{-(x+y+z)}, & x,y,z > 0 \\ 0, & \text{otherwise} \end{cases}$$

[1]In fact, $P(A_i \mid B^c) = p_b$, $i = 1,2$, and $P[(A_1 \cap A_2) \mid B^c] = p_b \times p_b = P(A_1 \mid B^c) \times P(A_2 \mid B^c)$, i.e., A_1 and A_2 are also *conditionally independent* given event B

- **Checking whether X, Y and Z independent r.v.**

According to the independence criterion for absolutely continuous r.v. (Theo. 3.38), X, Y and Z *mutually* independent r.v. iff

$$f_{X,Y,Z}(x,y,z) = f_X(x) \times f_Y(y) \times f_Z(z),\ x,y,z \in \mathbb{R}^3. \tag{2}$$

The marginal p.d.f. can be obtained as follows. For $x > 0$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y,Z}(x,y,z)\, dy\, dz \\ &= \int_0^{+\infty} \int_0^{+\infty} e^{-(x+y+z)}\, dy\, dz \\ &= e^{-x} \int_0^{+\infty} e^{-z} \left(-e^{-y}\right)\Big|_0^{+\infty} dz \\ &= e^{-x} \int_0^{+\infty} e^{-z}\, dz \\ &= e^{-x} \left(-e^{-z}\right)\Big|_0^{+\infty} \\ &= e^{-x}; \end{aligned}$$

and, for $x \leq 0$, $f_X(x) = 0$. Similarly, $f_Y(y) = e^{-y} \times I_{(0,+\infty)}(y)$ and $f_Z(z) = e^{-z} \times I_{(0,+\infty)}(z)$.

Finally, we can conclude that (2) holds, and we can state that X, Y and Z are *mutually* independent r.v.