Department of Mathematics, IST — Probability and Statistics Unit

Probability Theory

1st. Test 1st. Semester — **2014/15**

Duration: **1h30m** **2014/11/19** — **8:00 AM**, Room P9

- Please justify your answers.
- This test has TWO PAGES and FOUR GROUPS. The total of points is 20.0.

Group I — Warm up **2.0 points**

Random walk models are often found in physics, from particle motion to a simple description of a polymer. A student of physics assumes that the position of a particle at time n, X_n, is governed by an ASYMMETRIC random walk — starting at 0 and with probability of an upward (resp. downward) step equal to p (resp. $1-p$), where $p \in (0,1)\backslash\{\frac{1}{2}\}$.

(a) Derive the p.f. of the position at time n, X_n. (1.5)

- **Process**
  ASYMMETRIC random walk
- **R.v.**
  $Y_n =$ size of the n^{th} step
  $Y_n \stackrel{i.i.d.}{\sim} Y$, $n \in \mathbb{N}$
  $P(Y=y) = \begin{cases} p, & y=1 \\ 1-p, & y=-1 \\ 0, & \text{otherwise} \end{cases}$
  $X_n = \sum_{i=1}^n Y_i =$ position of the particle at time n ($n \in \mathbb{N}$)
- **Initial condition**
  $X_0 = 0$
- **Requested probability**
  According to Prop. 0.3, if $n \in I\!N$, $k \in \{-n,\dots,0,\dots,n\}$, and $\frac{n+k}{2}$ is an integer (i.e., $n \mod 2 = k \mod 2$) then the event $\{X_n = k\}$ occurs if $\frac{n+k}{2}$ of the steps $Y_1,\dots,Y_n$ are equal to 1 and the remainder are equal to -1. In fact, $X_n = k$ if we observe a steps up and b steps down where

$$(a,b): \begin{cases} a,b \in \{0,\dots,n\} \\ a+b=n \\ a-b=k \end{cases}$$

  that is, $a = \frac{n+k}{2}$ and a has to be an integer in $\{0,\dots,n\}$.
  Invoking the independence between steps and their common p.f. we get

$$P(X_n = k) = \binom{n}{\frac{n+k}{2}} \times p^{\frac{n+k}{2}} \times (1-p)^{\frac{n-k}{2}},$$

  for $n \in \mathbb{N}$, $k \in \{-n,\dots,0,\dots,n\}$, $\frac{n+k}{2} \in \{0,1,\dots,n\}$.

(b) Identify all possible values of the first passage time $T^0 = \inf\{n \in \mathbb{N} : X_n = 0\}$. (0.5)

- **First passage time**
  $T^0 = \inf\{n \in \mathbb{N} : X_n = 0\}$ (first return time to 0)
- **Range**
  $\{0,2,4,\dots\} \cup \{\infty\}$
  $\{0,2,4,\dots\}$ — Since we started at 0 we shall return to the origin after an even number of steps.
  $\{\infty\}$ — Only in the symmetric case $p = \frac{1}{2}$ will the random walk return to the point where it started from (with probability one). Otherwise, with some positive probability, it will never ever return to that point, thus T^0 can also take value ∞.

Group II — Probability spaces **5.0 points**

1. Consider the sample space Ω and let $\mathcal{F}$ be a $\sigma-$algebra on Ω and $B \in \mathcal{F}$.

(a) Show that $\mathcal{G} = \{D \cap B : D \in \mathcal{F}\}$ is a $\sigma-$algebra on B. (1.5)

- **$\sigma-$algebra on Ω**
  $\mathcal{F}$
- **To prove**
  $\mathcal{G} = \{D \cap B : D \in \mathcal{F}\}$ is a $\sigma-$algebra on B, where $B \in \mathcal{F}$
- **Proof**
  According to Def. 1.38, a minimal set of postulates for $\mathcal{G}$ to be a $\sigma-$algebra on B is:
  (i) $B \in \mathcal{G}$;
  (ii) $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$ (i.e., $\mathcal{G}$ is closed under complementation);
  (iii) $A_i \in \mathcal{G}$, $i \in I$ (countable set) $\Rightarrow \cup_{i\in I} A_i \in \mathcal{G}$ (i.e., $\mathcal{G}$ is closed under countable union).

  Note that $\mathcal{G}$ is a collection of subsets of B resulting from the intersection between sets from $\mathcal{F}$ and B also a set from $\mathcal{F}$. As a consequence:
  (i) $D = B$ [or Ω] $\in \mathcal{F} \Rightarrow D \cap B = B \in \mathcal{G}$; ✓
  (ii) if $A \in \mathcal{G}$ then $\exists D \in \mathcal{F} : D \cap B = A$; moreover, capitalizing on the fact that $D \in \mathcal{F}$ we successively get $D^c \in \mathcal{F}$, $D^c \cap B \in \mathcal{G}$ and $D^c \cap B = B\backslash D = B\backslash(D \cap B) = B\backslash A = A^c \cap B \stackrel{A\subseteq B}{=} A^c \in \mathcal{G}$; ✓
  (iii) $A_1, A_2, \dots \in \mathcal{G}$ then $\exists D_i \in \mathcal{F} : D_i \cap B = A_i$, $i \in I$ (countable set); in addition, the fact that $D_i \in \mathcal{F}$ leads to $\cup_{i\in I} D_i \in \mathcal{F}$ and $(\cup_{i\in I} D_i) \cap B \in \mathcal{G}$. Now, note that $(\cup_{i\in I} D_i) \cap B = \cup_{i\in I}(D_i \cap B) = \cup_{i\in I} A_i$ therefore it belongs to $\mathcal{G}$. ✓

(b) Is $\mathcal{G}$ a $\sigma-$algebra on Ω ? (0.5)

- **Checking whether $\mathcal{G}$ a $\sigma-$algebra on Ω**
  NO, because $\Omega \notin \mathcal{G}$ (unless $B = \Omega$).

2. Let Ω be the set of all non negative integers and $\mathcal{F}$ be the class of all subsets of Ω. In each of the following cases does P define a p.f. on $(\Omega, \mathcal{F})$:

(a) $P(A) = \sum_{x\in A} e^{-\lambda} \frac{\lambda^x}{x!}$, for $A \in \mathcal{F}$, where $\lambda > 0$? (2.0)

- **Sample space**
  $\Omega = \mathbb{N}_0$
- **$\sigma-$algebra on Ω**
  $\mathcal{F}$ be the class of all subsets of Ω (or *power set*).
- **Definition of a p.f.** (Def. 1.48)
  A probability on $(\Omega, \mathcal{F})$ is a function $P : \Omega \to I\!R$ if
  (i) $P(A) \geq 0$, $\forall A \in \mathcal{F}$;
  (ii) $P(\Omega) = 1$;
  (iii) whenever $A_1, A_2, \dots$ are (pairwise) disjoint events in $\mathcal{F}$, $P\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} P(A_i)$.
- **Checking whether "$P(A) = \sum_{x\in A} e^{-\lambda} \frac{\lambda^x}{x!}$, for $A \in \mathcal{F}$, where $\lambda > 0$", is a p.f. on $(\Omega, \mathcal{F})$**
  (i) Since $\lambda > 0$, we get $e^{-\lambda} \frac{\lambda^x}{x!} \geq 0$, $\forall x \in \Omega$, and therefore $P(A) = \sum_{x\in A} e^{-\lambda} \frac{\lambda^x}{x!} \geq 0$, $\forall A \in \mathcal{F}$. ✓
  (ii) $P(\Omega) = \sum_{x\in\mathbb{N}_0} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{+\infty} \frac{\lambda^x}{x!} = e^{-\lambda} \times e^{\lambda} = 1$. ✓
  (iii) If $A_1, A_2, \dots$ are (pairwise) disjoint events in $\mathcal{F}$ then $\bigcup_{i=1}^{+\infty} A_i \subseteq \Omega$ and $P\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{x\in\cup_{i=1}^{+\infty} A_i} e^{-\lambda} \frac{\lambda^x}{x!} \overset{P(\Omega)=1<\infty}{=} \sum_{i=1}^{+\infty} \sum_{x\in A_i} e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{i=1}^{+\infty} P(A_i)$. ✓

(b) for $A \in \mathcal{F}$, let $P(A) = 1$ if A has a finite number of elements, and $P(A) = 0$ otherwise ? (1.0)

- **Checking whether "$P(A) = 1$ if A has a finite number of elements, and $P(A) = 0$ otherwise" is a p.f. on $(\Omega, \mathcal{F})$**
  (i) Since $P(A) = 1$ or $P(A) = 0$, we have $P(A) \geq 0$, $\forall A \in \mathcal{F}$. ✓
  (ii) Since $\Omega = \mathbb{N}_0$ and has an infinite number of elements, this function leads to $P(\Omega) = 0 \neq 1$. ×

  Thus, we are not dealing with a p.f. in this case.

Group III — Random variables and independence **8.5 points**

1. If a projectile is fired from the origin at an angle X from the earth with a speed v, then the point Y at which it returns to the earth can be expressed as $Y = \frac{v^2}{g} \sin(2X)$, where g is the gravitational constant (equal to 980 centimeters per second squared). Assume that X is a r.v. with p.d.f.

$$f_X(x) = \begin{cases} \frac{12}{\pi}, & \frac{\pi}{6} \leq x \leq \frac{\pi}{4} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that $Y : \Omega \to \mathbb{R}$ is also a r.v. (2.0)

- **Important**
  Let:
  - X be a real r.v.;
  - $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable spaces.

  Then, by Def. 2.13, $X : \Omega \to \mathbb{R}$ and
  $$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}, \ \forall B \in \mathcal{B}(\mathbb{R}),$$
  in particular,
  $$X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}, \ \forall x \in \mathbb{R},$$
  according to Prop. 2.16.
- **To prove**
  $Y(\omega) = \frac{v^2}{g} \sin[2X(\omega)]$ is also a r.v.
- **Proof**
  Firsty, let us remind the reader that a function $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff
  $$h^{-1}(B) \in \mathcal{B}(\mathbb{R}), \ \forall B \in \mathcal{B}(\mathbb{R}),$$
  according to Def. 2.36, with $n = m = 1$. Furthermore, in order that $h : \mathbb{R} \to \mathbb{R}$ be Borel measurable it suffices that $h^{-1}((-\infty, y]) \in \mathcal{B}(\mathbb{R})$, $\forall y \in I\!R$, according to Remark 2.47 (with $n = 1$).
  Secondly, since $X(\omega) \in \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, we get
  $$Y(\omega) = \frac{v^2}{g} \sin[2X(\omega)] \in \left[\frac{\sqrt{3}}{2}\frac{v^2}{g}, \frac{v^2}{g}\right].$$
  Finally, $h^{-1}((-\infty, y])$ equals:
  - for $y < \frac{\sqrt{3}}{2}\frac{v^2}{g}$,
    $$\begin{aligned}\{x \in \mathbb{R} : h(x) \leq y\} &= \emptyset \\ &\in \mathcal{B}(\mathbb{R});\end{aligned}$$
  - for $\frac{\sqrt{3}}{2}\frac{v^2}{g} \leq y < \frac{v^2}{g}$,
    $$\begin{aligned}\{x \in \mathbb{R} : h(x) \leq y\} &= \left\{x \in \mathbb{R} : \frac{v^2}{g}\sin(2x) \leq y\right\} \\ &= \left(-\infty, \frac{1}{2}\arcsin\left(\frac{g}{v^2}y\right)\right] \\ &\in \mathcal{B}(\mathbb{R});\end{aligned}$$
  - for $y \geq \frac{v^2}{g}$,
    $$\begin{aligned}\{x \in \mathbb{R} : h(x) \leq y\} &= \mathbb{R} \\ &\in \mathcal{B}(\mathbb{R}).\end{aligned}$$

  As a result, we can state that h is indeed a Borel measurable function and therefore Y is a r.v., by Corollary 2.40. ✓

(b) Find the c.d.f. and the p.d.f. of Y. (2.5)

- **R.v.**
  $X =$ angle from the earth

- **P.d.f. of X**

$$f_X(x) = \begin{cases} \frac{12}{\pi}, & \frac{\pi}{6} \le x \le \frac{\pi}{4} \\ 0, & \text{otherwise.} \end{cases}$$

- **C.d.f. of X**

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= \begin{cases} 0, & x < \frac{\pi}{6} \\ \frac{12}{\pi}\left(x - \frac{\pi}{6}\right) = \frac{12x}{\pi} - 2, & \frac{\pi}{6} \le x < \frac{\pi}{4} \\ 1, & x \ge \frac{\pi}{4}. \end{cases} \end{aligned}$$

- **Range of X**
  $\mathbb{R}_X = \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$
- **Transformation**
  $Y = h(X) = \frac{v^2}{g}\sin(2X)$
- **Range of Y**
  $\mathbb{R}_Y = h(\mathbb{R}_X) = \left[\frac{\sqrt{3}}{2}\frac{v^2}{g}, \frac{v^2}{g}\right]$
- **C.d.f. of Y**
  Since $y = h(x) = \frac{v^2}{g}\sin(2x)$ is a strictly increasing function in $\mathbb{R}_X$, we can apply Prop. 2.71 and obtain $F_Y(y)$ in terms of $F_X(x)$:

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P\left[\frac{v^2}{g}\sin(2X) \le y\right] \\ &\overset{(a)}{=} P\left[X \le \frac{1}{2}\arcsin\left(\frac{g}{v^2}y\right)\right] \\ &= \begin{cases} 0, & y < \frac{\sqrt{3}}{2}\frac{v^2}{g} \\ F_X\left[\frac{1}{2}\arcsin\left(\frac{g}{v^2}y\right)\right] = \frac{12}{\pi} \times \frac{1}{2}\arcsin\left(\frac{g}{v^2}y\right) - 2, & \\ \quad = \frac{6}{\pi}\arcsin\left(\frac{g}{v^2}y\right) - 2, & \frac{\sqrt{3}}{2}\frac{v^2}{g} \le y < \frac{v^2}{g} \\ 1, & y \ge \frac{v^2}{g}. \end{cases} \end{aligned}$$

- **P.d.f. of Y**

$$\begin{aligned} f_Y(y) &= \frac{d\,F_Y(y)}{dy} \\ &= \begin{cases} \frac{d}{dy}\left[\frac{6}{\pi}\arcsin\left(\frac{g}{v^2}y\right) - 2\right] = \frac{6}{\pi}\frac{\frac{g}{v^2}}{\sqrt{1-\left(\frac{g}{v^2}\right)^2 y^2}}, & \frac{\sqrt{3}}{2}\frac{v^2}{g} \le y \le \frac{v^2}{g} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Describe two methods to generate (pseudo-)random numbers from the distribution of Y. (1.5)

- **Quantile function and transformation — 1st. method**
  Let $u \in (0,1)$ then the quantile function of X can be derived as follows:

$$\begin{aligned} F_X(x) &= u \\ \frac{12x}{\pi} - 2 &= u \\ F_X^{-1}(u) &= x \\ &= \frac{\pi}{12}(u+2). \end{aligned}$$

By Prop. 2.140, if

$$U \sim \text{Uniform}(0,1)$$

then $F_X^{-1}(U) \overset{d}{=} X$, i.e.,

$$\frac{\pi}{12}(U+2) \overset{d}{=} X.$$

Hence, if we want to generate (pseudo-)random numbers from the distribution of $Y = \frac{v^2}{g}\sin(2X)$ then we have to:

- ∘ generate u from a Uniform$(0,1)$ distribution;
- ∘ assign

$$x = \frac{\pi}{12}(u+2)$$

and

$$y = \frac{v^2}{g}\sin(2x).$$

- **Quantile function and transformation — 2nd. method**
  Let $u \in (0,1)$ then the quantile function of Y is given by:

$$\begin{aligned} F_Y(y) &= u \\ \frac{6}{\pi}\arcsin\left(\frac{g}{v^2}y\right) - 2 &= u \\ F_Y^{-1}(u) &= y \\ &= \frac{v^2}{g}\sin\left[\frac{\pi}{6}(u+2)\right]. \end{aligned}$$

Invoking once again Prop. 2.140, if $U \sim \text{Uniform}(0,1)$ then $F_Y^{-1}(U) \overset{d}{=} Y$, i.e.,

$$\frac{v^2}{g}\sin\left[\frac{\pi}{6}(U+2)\right] \overset{d}{=} Y.$$

Thus, to generate (pseudo-)random numbers from the distribution of Y we have to:

- ∘ generate u from a Uniform$(0,1)$ distribution;
- ∘ assign $y = \frac{v^2}{g}\sin\left[\frac{\pi}{6}(u+2)\right]$.

2. Suppose that $X \sim \text{Exponential}(\lambda)$ and refers to a measurement. When this measurement is read aloud by someone there is an independent added error $Y \sim \text{Normal}(0, \sigma^2)$. Let $Z = X + Y$ be the value actually read aloud. (2.5)

Prove that the p.d.f. of Z is given by

$$f_Z(z) = \lambda\, e^{-\lambda z + \frac{\lambda^2\sigma^2}{2}} \times \Phi\left(\frac{z}{\sigma} - \lambda\sigma\right),\ z \in \mathbb{R}.$$

- **R.v.**
  $X =$ measurement
  $Y =$ added error
- **Distributions**
  $X \sim \text{Exponential}(\lambda)$
  $\perp\!\!\!\perp$
  $Y \sim \text{Normal}(0, \sigma^2)$
- **Other r.v.**
  $Z = X + Y =$ measurement read aloud
- **P.d.f. of Z**
  Since X and Y are independent r.v., we can apply Cor. 2.125 and obtain

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_X(z-y) \times f_Y(y)\, dy \\
&\overset{z-y>0}{=} \int_{-\infty}^{z} \lambda e^{-\lambda(z-y)} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}\, dy \\
&= \lambda e^{-\lambda z + \frac{\lambda^2\sigma^2}{2}} \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2 - 2\lambda\sigma^2 y + \lambda^2\sigma^4}{2\sigma^2}}\, dy \\
&= \lambda e^{-\lambda z + \frac{\lambda^2\sigma^2}{2}} \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\lambda\sigma^2)^2}{2\sigma^2}}\, dy \\
&= \lambda e^{-\lambda z + \frac{\lambda^2\sigma^2}{2}} F_{\text{Normal}(\lambda\sigma^2, \sigma^2)}(z) \\
&= \lambda e^{-\lambda z + \frac{\lambda^2\sigma^2}{2}} \Phi\left(\frac{z - \lambda\sigma^2}{\sigma}\right) \\
&= \lambda\, e^{-\lambda z + \frac{\lambda^2\sigma^2}{2}} \times \Phi\left(\frac{z}{\sigma} - \lambda\sigma\right),\ z \in \mathbb{R}
\end{aligned}$$

✓

Group IV — Independence **4.5 points**

1. (a) Show that A and B are independent events iff A and B^c are independent. (1.0)

- **Events**
  A, B
- **To prove**
  $A \perp\!\!\!\perp B \ \Leftrightarrow\ A \perp\!\!\!\perp B^c$

- **Proof** ($\Rightarrow$)

$$\begin{aligned}
A \perp\!\!\!\perp B &\Leftrightarrow P(A \cap B) = P(A) \times P(B) \\
&\Rightarrow \\
P(A \cap B^c) &= P(A \backslash B) \\
&= P(A) - P(A \cap B) \\
P(A \cap B^c) &= P(A) - P(A) \times P(B) \\
&= P(A) \times [1 - P(B)] \\
&= P(A) \times P(B^c)
\end{aligned}$$

- **Proof** ($\Leftarrow$)

$$\begin{aligned}
A \perp\!\!\!\perp B^c &\Leftrightarrow P(A \cap B^c) = P(A) \times P(B^c) \\
&\Rightarrow \\
P(A) &= P(A \backslash B) + P(A \cap B) \\
&= P(A \cap B^c) + P(A \cap B) \\
P(A \cap B) &= P(A) - P(A \cap B^c) \\
&= P(A) - P(A) \times P(B^c) \\
&= P(A) \times [1 - P(B^c)] \\
&= P(A) \times P(B).
\end{aligned}$$

✓

(b) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A, B, C \in \mathcal{F}$ with $P(B), P(C) > 0$. Prove that if B and C are independent then (1.5)

$$P[A \mid (B \cap C)] \times P(C) + P[A \mid (B \cap C^c)] \times P(C^c) = P(A \mid B).$$

- **Events**
  $A, B, C \in \mathcal{F}$
- **To prove**
  $B \perp\!\!\!\perp C,\ P(B), P(C) > 0$
  $\Rightarrow\ P[A \mid (B \cap C)] \times P(C) + P[A \mid (B \cap C^c)] \times P(C^c) = P(A \mid B)$
- **Proof**
  Since $P(B), P(C) > 0$ and $B \perp\!\!\!\perp C$, we can conclude that: B and C are not disjoint events; $P(B \cap C) > 0$; $P(B \cap C^c) > 0$; $B \perp\!\!\!\perp C^c$ from (a); and

$$\begin{aligned}
&P[A \mid (B \cap C)] \times P(C) + P[A \mid (B \cap C^c)] \times P(C^c) \\
&= \tfrac{P(A \cap B \cap C)}{P(B \cap C)} \times P(C) + \tfrac{P(A \cap B \cap C^c)}{P(B \cap C^c)} \times P(C^c) \\
&\overset{B \perp\!\!\!\perp C,\, B \perp\!\!\!\perp C^c}{=} \tfrac{P(A \cap B \cap C)}{P(B) \times P(C)} \times P(C) + \tfrac{P(A \cap B \cap C^c)}{P(B) \times P(C^c)} \times P(C^c) \\
&= \tfrac{P(A \cap B \cap C)}{P(B)} + \tfrac{P(A \cap B \cap C^c)}{P(B)} \\
&= \tfrac{1}{P(B)} \times [P(A \cap B \cap C) + P(A \cap B \cap C^c)] \\
&= \tfrac{1}{P(B)} \times P(A \cap B) \\
&= P(A \mid B).
\end{aligned}$$

✓

2. Let X, Y and Z represent the coordinates of a particle bound to be positioned on one of the 8 vertices of a unit cube. Suppose X, Y and Z have the joint p.f.

$$P(X=x,Y=y,Z=z) = \begin{cases} \frac{3}{16}, & (x,y,z) \in \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\} \\ \frac{1}{16}, & (x,y,z) \in \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}. \end{cases}$$

(a) Are X, Y and Z *mutually* independent r.v.? (0.5)

- **Random vector**
  (X,Y,Z) = coordinates of a particle bound to be positioned on one of the 8 vertices of a unit cube

- **Joint p.d.f.**

$$P(X=x,Y=y,Z=z) = \begin{cases} \frac{3}{16}, & (x,y,z) \in \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\} \\ \frac{1}{16}, & (x,y,z) \in \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\} \end{cases}$$

- **Checking whether X, Y and Z *mutually* independent r.v.**
  According to the independence criterion for discrete r.v. (Theo. 3.35), X, Y and Z *mutually* independent discrete r.v. iff

$$P(X=x,Y=y,Z=z) = P(X=x) \times P(Y=y) \times P(Z=z),$$

for all $x,y,z \in \{0,1\}$.
However, an inspection of the joint p.f. leads to

$$\begin{aligned} P(X=0) &= \sum_{y=0}^{1}\sum_{z=0}^{1} P(X=0,Y=y,Z=z) \\ &= \frac{1}{2} \\ P(Y=0) &= \sum_{x=0}^{1}\sum_{z=0}^{1} P(X=x,Y=0,Z=z) \\ &= \frac{1}{2} \\ P(Z=0) &= \sum_{x=0}^{1}\sum_{y=0}^{1} P(X=x,Y=y,Z=0) \\ &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} P(X=0,Y=0,Z=0) &= \frac{3}{16} \\ &\neq P(X=0) \times P(Y=0) \times P(Z=0) \\ &= \frac{1}{8}, \end{aligned}$$

therefore X, Y and Z are NOT *mutually* independent r.v.

(b) Show that X, Y and Z are *pairwise* independent r.v. Comment on this result in light of (a). (1.5)

- **Checking whether X, Y and Z are *pairwise* independent r.v.**
  Another look at the joint p.f. yields

$$\begin{aligned} P(X=x,Y=y) &= \sum_{z=0}^{1} P(X=x,Y=y,Z=z) \\ &= \frac{1}{4},\ (x,y) \in \{(0,0),(0,1),(1,0),(1,1)\} \\ P(X=x,Z=z) &= \sum_{y=0}^{1} P(X=x,Y=y,Z=z) \\ &= \frac{1}{4},\ (x,z) \in \{(0,0),(0,1),(1,0),(1,1)\} \\ P(Y=y,Z=z) &= \sum_{x=0}^{1} P(X=x,Y=y,Z=z) \\ &= \frac{1}{4},\ (y,z) \in \{(0,0),(0,1),(1,0),(1,1)\}. \end{aligned}$$

Now, capitalizing on (a), we can add that X, Y and Z have p.f.

$$\begin{aligned} P(X=x) &= P(Y=x) \\ &= P(Z=x) \\ &= \frac{1}{2},\ x \in \{0,1\}. \end{aligned}$$

Thus,

$$\begin{aligned} P(X=x,Y=y) &= P(X=x) \times P(Y=y) \\ P(X=x,Z=z) &= P(X=x) \times P(Z=z) \\ P(Y=y,Z=z) &= P(Y=y) \times P(Z=z), \end{aligned}$$

for all $x,y,z \in \{0,1\}$, that is, X, Y and Z are *pairwise* independent discrete r.v., according to the criterion mentioned in (a).

- **Comment**
  This exercise illustrates the fact that *pairwise* independence DOES NOT IMPLY *mutual* independence[, even though *mutual* independence implies *pairwise* independence].