Department of Mathematics, IST — Probability and Statistics Unit

Probability Theory

2nd. Test — 1st. Semester — **2013/14**

Duration: **1h30m** — **2014/01/25 — 9:45 AM**, Room P12

- Please justify your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group I — Independence, Poisson processes and expectation **3.0 points**

A certain model supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when exactly 196 such mistakes have occurred. Assuming this model is valid:

(a) find the mean and variance of the lifetime of an individual; (1.5)

- **Stochastic process**
  $\{N(t) : t \geq 0\} \sim PP(\lambda)$
  $N(t) =$ number of mistakes in cell division up to time t (in years)
  $\lambda = 2.5$ mistakes per year
- **R.v.**
  $S_{196} =$ lifetime of an individual
  $=$ time of the occurrence of the 196^{th} mistake in cell division
- **Distribution of S_{196}**
  According to Prop. 3.107,
  $$S_{196} \sim \text{Gamma}(196, 2.5).$$
- **Requested mean and variance**
  $E(S_{196}) \stackrel{form.}{=} \frac{196}{2.5} = 78.4$
  $V(S_{196}) \stackrel{form.}{=} \frac{196}{2.5^2} = 31.36$

(b) obtain an approximate value for the probability that an individual dies after her/his 90^{th} anniversary. (1.5)

- **Requested probability**
  $P(S_{196} > 90)$
- **Requested probability — approximate value**
  The Lindeberg-Lévy CLT allows us to add that, when we deal with a sufficiently large number n of i.i.d. r.v. $X_1, \ldots, X_n$, with common mean μ and common positive and finite variance σ^2, the c.d.f. of the sum of these r.v. can be approximate as follows:

$$\begin{aligned} P(S_n \leq s) &= P\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq \frac{s - n\mu}{\sqrt{n\sigma^2}}\right) \\ &\stackrel{CLT}{\simeq} \Phi\left(\frac{s - n\mu}{\sqrt{n\sigma^2}}\right). \end{aligned}$$

In this case, we have $n = 196$, $X_i \sim \text{Exponential}(2.5)$, $\mu = \frac{1}{2.5}$ and $\sigma^2 = \frac{1}{2.5^2}$, therefore:

$$\begin{aligned} P(S_{196} > 90) &\simeq 1 - \Phi\left(\frac{90 - 196 \times \frac{1}{2.5}}{\sqrt{196 \times \frac{1}{2.5^2}}}\right) \\ &\simeq 1 - \Phi(2.07) \\ &\stackrel{table}{=} 1 - 0.9808 \\ &= 0.0192. \end{aligned}$$

- **[Exact value and another approximate value**

$$\begin{aligned} P(S_{196} > 90) &= 1 - F_{Gamma(196,2.5)}(90) \\ &\stackrel{Mathematica}{=} 0.022707; \\ P(S_{196} > 90) &\stackrel{form.}{=} 1 - F_{\chi^2_{2\times 196}}(2 \times 2.5 \times 90) \\ &\stackrel{Mathematica}{=} 0.022707; \\ P(S_{196} > 90) &\stackrel{form.}{=} F_{Poisson(2.5\times 90)}(196 - 1) \\ &= F_{Poisson(225)}(195) \\ &\stackrel{CLT}{\simeq} \Phi\left(\frac{195 - 225}{\sqrt{225}}\right) \\ &= \Phi(-2) \\ &= 1 - \Phi(2) \\ &\stackrel{table}{=} 1 - 0.9772 \\ &= 0.0228.] \end{aligned}$$

Group II — Expectation **9.0 points**

1. The alternating current (AC) generated by a machine has a Uniform$(-2, 2)$ distribution. This current — after having been converted by a half-wave rectifier[1] — can be represented by the r.v. $Y = \max\{0, X\}$.

   After obtaining the c.d.f. of Y, use it to find the expectation of this r.v. (2.0)

   - **R.v., distribution, range, p.d.f. and c.d.f.**
     $X =$ alternating current
     $X \sim \text{Uniform}(-2, 2)$

[1] A rectifier is an electrical device that converts alternating current (AC) to direct current (DC).

$I\!R_X = [-2, 2]$

$$f_X(x) = \begin{cases} \frac{1}{4}, & -2 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt = \begin{cases} 0, & x < -2 \\ \int_{-2}^{x} \frac{1}{4}\, dt = \frac{x+2}{4}, & -2 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

- **Another r.v., range, c.d.f.**

  $Y =$ direct current $= \max\{0, X\}$

  $I\!R_Y = [0, 2]$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= \begin{cases} 0, & y < 0 \\ P(Y = 0) = P(X \leq 0) = F_X(0) = \frac{0+2}{4} = \frac{1}{2}, & y = 0 \\ P(X \leq y) = F_X(y) = \frac{y+2}{4}, & 0 < y \leq 2 \\ 1, & y > 2 \end{cases} \end{aligned}$$

  [We are dealing with a non negative mixed r.v.!]

- **Requested expected value**

$$\begin{aligned} E(Y) &\overset{Y \geq 0,\, Theo.\, 4.65}{=} \int_0^{+\infty} [1 - F_Y(y)]\, dy \\ &= \int_0^2 \left(1 - \frac{y+2}{4}\right) dy \\ &= \int_0^2 \frac{2-y}{4}\, dy \\ &= \left(\frac{y}{2} - \frac{y^2}{8}\right)\Big|_0^2 \\ &= \frac{1}{2}. \end{aligned}$$

- **[Requested expected value — without using the c.d.f. of Y**

  Since $Y = \max\{0, X\}$ is a (non negative) Borel measurable function and X is an absolutely continuous r.v., we can invoke Corollary 4.81 and get:

$$\begin{aligned} E(Y) &= E[\max\{0, X\}] \\ &= \int_{-2}^{2} \max\{0, x\} \times \frac{1}{4}\, dx \\ &= \int_{-2}^{0} 0 \times \frac{1}{4}\, dx + \int_0^2 x \times \frac{1}{4}\, dx \\ &= 0 + \frac{x^2}{8}\Big|_0^2 \\ &= \frac{1}{2}.] \end{aligned}$$

2. Without evaluating the expectation, show that if $X \sim$ Uniform (0,2) then $E[X\, \ln(X)] \geq 0$. (1.0)

   - **R.v.**

     $X \sim$ Uniform (0,2)

   - **Requested lower limit**

     Since $E(X) \overset{form.}{=} \frac{0+2}{2} = 1$ and $g(x) = x \ln(x)$ is convex function (because $\frac{d^2\, x \ln(x)}{dx^2} = \frac{d\,[\ln(x) + x/x]}{dx} = \frac{1}{x} > 0,\ x \in (0, 2)$), we can apply Jensen's inequality to obtain:

$$\begin{aligned} E[g(X)] &= E[X\, \ln(X)] \\ &\geq g[E(X)] \\ &= g(1) = 1 \times \ln(1) \\ &= 0. \end{aligned}$$

3. Let X and Y be two r.v. in L^2 such that $E(X) = E(Y) = 0$, $V(X) = V(Y) = 1$ and $corr(X, Y) = \rho \in (0, 1)$. Now define (1.5)

$$X_\theta = X\, \cos(\theta) - Y\, \sin(\theta) \qquad Y_\theta = X\, \sin(\theta) + Y\, \cos(\theta),$$

   where θ is a constant in $[0, 2\,\pi]$.

   Determine all the values of θ for which X_θ and Y_θ are uncorrelated.

   - **Random vector**

$$(X, Y): \begin{cases} E(X) = E(Y) = 0 \\ V(X) = V(Y) = 1 \\ corr(X, Y) = \frac{cov(X,Y)}{\sqrt{V(X) \times V(Y)}} = cov(X, Y) = \rho \in (0, 1) \end{cases}$$

   - **Determining θ**

     Capitalizing on the properties of this random vector and taking advantage of the fact that the covariance is a bilinear operator (Proposition 4.159), we get

$$\begin{aligned} \theta \in [0, 2\,\pi] \ :\ & corr(X_\theta, Y_\theta) = 0 \\ & cov\left(X \cos(\theta) - Y \sin(\theta), X \sin(\theta) + Y \cos(\theta)\right) = 0 \\ & \sin(\theta) \cos(\theta)\, V(X) + \cos^2(\theta)\, cov(X, Y) - \sin^2(\theta)\, cov(X, Y) \\ & \qquad - \sin(\theta) \cos(\theta)\, V(Y) = 0 \\ & \left[\cos^2(\theta) - \sin^2(\theta)\right] \times \rho = 0 \\ & \tan(\theta) = \pm 1 \\ & \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \end{aligned}$$

4. (a) Let: (2.5)

- $\underline{N} = (N_1, \ldots, N_d) \sim \text{Multinomial}_{d-1}(n, \underline{p})$;
- $\underline{N}_{-j} = (N_1, \ldots, N_{j-1}, N_{j+1}, \ldots, N_d)$.

Prove that

$$\underline{N}_{-j}|N_j = n_j \sim \text{Multinomial}_{d-2}(n - n_j, \underline{p}_{-j|j}),$$

where

$$\underline{p}_{-j|j} = \left(\frac{p_1}{1-p_j}, \ldots, \frac{p_{j-1}}{1-p_j}, \frac{p_{j+1}}{1-p_j}, \ldots, \frac{p_{d-1}}{1-p_j}, \frac{p_d}{1-p_j}\right).$$

- **Random vector/distribution**
  $\underline{N} = (N_1, \ldots, N_d) \sim \text{Multinomial}_{d-1}(n, \underline{p})$
- **Joint p.f.**

$$\begin{aligned} P(\underline{N} = \underline{n}) &= P(N_1 = n_1, \ldots, N_d = n_d) \\ &\overset{Def.\,4.208}{=} \frac{n!}{\prod_{i=1}^d n_i!} \times \prod_{i=1}^d p_i^{n_i}, \end{aligned}$$

  for $n_i \in \{0, \ldots, n\}$, $i = 1, \ldots, d$, such that $\sum_{i=1}^d n_i = n$.
- **Another random vector**
  $\underline{N}_{-j} = (N_1, \ldots, N_{j-1}, N_{j+1}, \ldots, N_d)$
- **Distribution of N_j**
  $N_j \sim \text{Binomial}(n, p_j)$, according to Prop. 4.220.
- **P.f. of $\underline{N}_{-j}$ conditional to $N_j = n_j$**

$$\begin{aligned} P(\underline{N}_{-j} = \underline{n}_{-j}|N_j = n_j) &= \frac{P(\underline{N}_{-j} = \underline{n}_{-j}, N_j = n_j)}{P(N_j = n_j)} \\ &= \frac{P(\underline{N} = \underline{n})}{P(N_j = n_j)} \\ &= \frac{\frac{n!}{\prod_{i=1}^d n_i!} \times \prod_{i=1}^d p_i^{n_i}}{\binom{n}{n_j} p_j^{n_j}(1-p_j)^{n-n_j}} \\ &= \frac{(n-n_j)!}{\prod_{i=1,\, i\neq j}^d n_i!} \times \prod_{i=1,\, i\neq j}^d \left(\frac{p_i}{1-p_j}\right)^{n_i}, \end{aligned}$$

  for $n_i \in \{0, \ldots, n\}$, $i = 1, \ldots, d$, $i \neq j$, such that $\sum_{i=1, i\neq j}^d n_i = n - n_j$ and $n_j \in \{0, \ldots, n\}$.
- **Conclusion**
  $\underline{N}_{-j}|N_j = n_j \sim \text{Multinomial}_{d-2}(n - n_j, \underline{p}_{-j|j})$, where $\underline{p}_{-j|j} = \left(\frac{p_1}{1-p_j}, \ldots, \frac{p_{j-1}}{1-p_j}, \frac{p_{j+1}}{1-p_j}, \ldots, \frac{p_{d-1}}{1-p_j}, \frac{p_d}{1-p_j}\right)$.

(b) Suppose that two chess players had played numerous games and it was determined that the probability that Player A would win is 0.40, the probability that Player B would win is 0.35, and the probability that the game would end in a draw is 0.25. (2.0)

If these two chess players played 12 games, what is the probability that Player A would win 7 games, given that 3 games ended in draw?

- **Random vector**
  $\underline{N} = (N_1, N_2, N_3)$
  N_1 = number of times player A wins in n games
  N_2 = number of times player B wins in n games
  N_3 = number of draws in n games
- **Distribution**
  $\underline{N} \sim \text{Multinomial}_{d-1}(n, \underline{p})$
- **Parameters**
  $d = 3$
  $n = 12$
  $\underline{p} = (p_1, p_2, p_3) = (0.40, 0.35, 0.25)$
- **Joint p.f.**

$$\begin{aligned} P(\underline{N} = \underline{n}) &= P(N_1 = n_1, N_2 = n_2, N_3 = n_3) \\ &\overset{Def.\,4.208}{=} \frac{12!}{\prod_{i=1}^3 n_i!} \times 0.40^{n_1} \times 0.35^{n_2} \times 0.25^{n_3}, \end{aligned}$$

  for $n_i \in \mathbb{N}_0$, $i = 1, 2, 3$ such that $\sum_{i=1}^3 n_i = 12$.
- **Distribution and p.f. of N_3**
  $N_3 \sim \text{Binomial}(12, 0.25)$, according to Prop. 4.220.
  $P(N_3 = n_3) = \binom{12}{n_3} \times 0.25^{n_3} \times (1-0.25)^{12-n_3}$, $n_3 = 0, 1, \ldots, 12$
- **Requested probability**

$$\begin{aligned} P(N_1 = 7, N_2 = 12 - (7+3)|N_3 = 3) &= \frac{P(N_1 = 7, N_2 = 2, N_3 = 3)}{P(N_3 = 3)} \\ &= \frac{\frac{12!}{7!\,2!\,3!} \times 0.40^7 \times 0.35^2 \times 0.25^3}{\frac{12!}{3!\,(12-3)!} \times 0.25^3 \times (1-0.25)^{12-3}} \\ &\simeq \frac{0.0248371}{0.258104} \\ &\simeq 0.096229 \end{aligned}$$

  [It coincides with

$$\frac{(12-3)!}{7!\,2!} \times \left(\frac{0.40}{1-0.25}\right)^7 \left(\frac{0.35}{1-0.25}\right)^2,$$

  because $\underline{N}_{-3}|N_3 = n_3 \sim \text{Multinomial}_{d-2}(n - n_3, \underline{p}_{-3|3})$ where $\underline{p}_{-3|3} = \left(\frac{p_1}{1-p_3}, \frac{p_2}{1-p_3}\right)$.[2]]

[2] This distributional result can be read as follows: $((N_1, N_2)|N_3 = n_3) \overset{d}{=} (N_1|N_3 = n_3) \sim \text{Binomial}(n - n_3, \frac{p_1}{1-p_3})$.

Group III — Convergence of sequences of r.v. **8.0 points**

1. (a) Let $\{X_1, X_2, \dots\}$ be a strictly decreasing sequence of positive r.v. Prove that if $X_n \stackrel{P}{\to} 0$ then $X_n \stackrel{a.s.}{\to} 0$. (1.5)

   - **To prove**
     If $\{X_1, X_2, \dots\}$ such that $0 > X_1 \geq X_2 \geq \dots$ and $X_n \stackrel{P}{\to} 0$ then $X_n \stackrel{a.s.}{\to} 0$.
   - **Proof**
     Since $X_n \stackrel{P}{\to} 0$ means that

     $$\forall \epsilon > 0, \lim_{n\to+\infty} P(|X_n - 0| > \epsilon) = 0$$

     and $0 > X_1 \geq X_2 \geq \dots$ implies that

     $$\sup_{k\geq n} |X_k - 0| = |X_n - 0| \quad [= X_n],$$

     we can add that

     $$\begin{aligned}\forall \epsilon > 0, \lim_{n\to+\infty} P(\sup_{k\geq n} |X_k - 0| > \epsilon) &= \lim_{n\to+\infty} P(|X_n - 0| > \epsilon)\\ &= 0,\end{aligned}$$

     i.e., $X_n \stackrel{a.s.}{\to} 0$, according to Proposition 5.37.

   (b) Comment this convergence result. (0.5)

   - **Comment on the convergence result**
     Let us remind the reader that $\stackrel{a.s.}{\to} \Rightarrow \stackrel{P}{\to}$ but $\stackrel{P}{\to} \not\Rightarrow \stackrel{a.s.}{\to}$.
     The result which we have just proved is an implication of restricted validity stating that if we combine the strictly decreasing and positive character of a sequence of r.v. with the convergence in probability to zero of that same sequence, then we end up with the almost sure convergence to zero (see Exercise 5.80).

2. The Cauchy distribution, named after Augustin Cauchy, is often used in statistics as the canonical example of a *pathological* distribution — both its mean and its variance are undefined. (2.5)

   Let:

   - $\{X_1, X_2, \dots\}$ be a sequence of i.i.d. r.v. with a standard Cauchy distribution, i.e., with c.d.f. given by $F_X(x) = 1 - \frac{1}{\pi}\arctan(\frac{1}{x})$, $x \in \mathbb{R}$;
   - $Y_n = \frac{1}{n} \times \max_{i=1,\dots,n} X_i$ be the maximum of the n first terms of the sequence.

   Show that $Y_n \stackrel{d}{\to} Y$, where $F_Y(y) = \begin{cases} e^{-\frac{1}{\pi y}}, & y > 0 \\ 0, & \text{otherwise.}\end{cases}$

   Hint: It may be helpful to know that $\arctan(y) = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \dots$.

- **R.v.**
  $X_i \stackrel{i.i.d.}{\sim} X$, $i \in I\!N$
  $X \sim \text{Cauchy}(0,1)$
  $F_X(x) = 1 - \frac{1}{\pi}\arctan(\frac{1}{x})$
- **Another r.v.**
  $Y_n = \frac{1}{n} \times X_{(n)}$
- **C.d.f. of Y_n**

  $$\begin{aligned}F_{Y_n}(y) &= P(Y \leq y)\\ &= P[X_{(n)} \leq ny]\\ &= [F_X(ny)]^n\\ &= \left[1 - \frac{1}{\pi}\arctan\left(\frac{1}{ny}\right)\right]^n, \; y \in \mathbb{R}\end{aligned}$$

- **Note**
  $F_{Y_n}(y)$ ought to belong to $[0,1]$, regardless of the value of n or y. Moreover: $\lim_{n\to+\infty} F_{Y_n}(y) \in [0,1]$, for any y; $e^{-\frac{1}{\pi y}} \notin (0,1)$, $y \leq 0$. Consequently, $e^{-\frac{1}{\pi y}}$ cannot be the limit of $F_{Y_n}(y)$ when $y \leq 0$. In fact, $\lim_{n\to+\infty} F_{Y_n}(y) = 0$, for $y \leq 0$.
- **Checking the convergence in distribution**
  We have:

  (i) for $y > 0$,

  $$\begin{aligned}\lim_{n\to+\infty} F_{Y_n}(y) &= \lim_{n\to+\infty}\left[1 - \frac{1}{\pi}\arctan\left(\frac{1}{ny}\right)\right]^n\\ &= \lim_{n\to+\infty}\left\{1 - \frac{1}{\pi}\left[\frac{1}{ny} - \frac{1}{3(ny)^3} + \frac{1}{5(ny)^5} - \dots\right]\right\}^n\\ &= \lim_{n\to+\infty}\left(1 + \frac{-\frac{1}{\pi y}}{n}\right)^n\\ &= e^{-\frac{1}{\pi y}};\end{aligned}$$

  (ii) $F_Y(y) = \begin{cases} e^{-\frac{1}{\pi y}}, & y > 0 \\ 0, & \text{otherwise,}\end{cases}$
  is the c.d.f. of a(n absolutely continuous) r.v., say Y;[3]

  (iii) $\lim_{n\to+\infty} F_{Y_n}(y) = F_Y(y)$, for all the continuity points of the c.d.f. of Y, $\mathbb{R}^+$.

  Therefore

  $$Y_n \stackrel{d}{\to} Y.$$

[3] After all, this is mentioned in the text. Furthermore, $F_Y(y)$ is non-decreasing, right-continuous and has the following limits: $\lim_{y\to 0^+} F_Y(y) = 0$, $\lim_{y\to+\infty} F_Y(y) = 1$.

3. Let:

- $\{X_1, X_2, \dots\}$ and $\{Y_1, Y_2, \dots\}$ be two independent sequences of i.i.d. r.v. to $X \sim \text{Poisson}(\lambda_X)$ and $Y \sim \text{Poisson}(\lambda_Y)$, respectively;
- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ be the mean of the first n terms of $\{X_1, X_2, \dots\}$ ($\bar{Y}_n$ is defined similarly).

(a) Show that $\frac{(\bar{X}_n-\bar{Y}_n)-(\lambda_X-\lambda_Y)}{\sqrt{\frac{\bar{X}_n}{n}+\frac{\bar{Y}_n}{n}}} \xrightarrow{d} \text{Normal}(0,1)$. (3.0)

- **Sequence of r.v.**
  $\{X_1, X_2, \dots\}$, where $X_i \overset{i.i.d.}{\sim} X \sim \text{Poisson}(\lambda_X)$, $i \in I\!N$
  $\perp\!\!\!\perp$
  $\{Y_1, Y_2, \dots\}$, where $Y_i \overset{i.i.d.}{\sim} Y \sim \text{Poisson}(\lambda_Y)$, $i \in I\!N$
- **Other r.v.**
  $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, $n \in I\!N$
  $\perp\!\!\!\perp$
  $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$, $n \in I\!N$
- **Another sequence of r.v.**
  $\{Z_n, n \in I\!N\}$, where $Z_n = \frac{(\bar{X}_n-\bar{Y}_n)-(\lambda_X-\lambda_Y)}{\sqrt{\frac{\bar{X}_n}{n}+\frac{\bar{Y}_n}{n}}}$.
- **Asymptotic distribution of Z_n**
  To prove that $Z_n \xrightarrow{d} \text{Normal}(0,1)$ we need to:
  - rewrite Z_n as follows,
  $$Z_n = \frac{U_n}{V_n},$$
  where
  $$U_n = \frac{(\bar{X}_n-\bar{Y}_n)-(\lambda_X-\lambda_Y)}{\sqrt{\frac{\lambda_X}{n}+\frac{\lambda_Y}{n}}}$$
  $$V_n = \sqrt{\frac{\frac{\bar{X}_n}{n}+\frac{\bar{Y}_n}{n}}{\frac{\lambda_X}{n}+\frac{\lambda_Y}{n}}} = \sqrt{\frac{\bar{X}_n+\bar{Y}_n}{\lambda_X+\lambda_Y}};$$
  - show that $U_n \xrightarrow{d} \text{Normal}(0,1)$;
  - prove that $V_n \xrightarrow{P} 1$;
  - apply Slutsky's Theorem.
- **Auxiliary results**
  - $E(\bar{X}_n) = \lambda_X$
  - $E(\bar{Y}_n) = \lambda_Y$
  - $E(\bar{X}_n - \bar{Y}_n) = \lambda_X - \lambda_Y$
  - $V(\bar{X}_n) = \frac{\lambda_X}{n}$
  - $V(\bar{Y}_n) = \frac{\lambda_Y}{n}$
  - $V(\bar{X}_n - \bar{Y}_n) \overset{\bar{X}_n \perp\!\!\!\perp \bar{Y}_n}{=} \frac{\lambda_X}{n} + \frac{\lambda_Y}{n}$
- **Convergence of U_n**
  Combining the auxiliary results and the mere application of Lindeberg-Lévy CLT (Theorem 5.186) leads to the conclusion that
  $$U_n = \frac{(\bar{X}_n-\bar{Y}_n)-(\lambda_X-\lambda_Y)}{\sqrt{\frac{\lambda_X}{n}+\frac{\lambda_Y}{n}}} = \frac{(\bar{X}_n-\bar{Y}_n)-E(\bar{X}_n-\bar{Y}_n)}{\sqrt{V(\bar{X}_n-\bar{Y}_n)}} \xrightarrow{d} \text{Normal}(0,1).$$
- **Convergence of V_n**
  We can invoke the WLLN for i.i.d. r.v. in L^2 (Theorem 5.129)[4] and state the following convergences in probability:
  $$\bar{X}_n \xrightarrow{P} \lambda_X$$
  $$\bar{Y}_n \xrightarrow{P} \lambda_Y.$$
  Invoking now the closure of convergence in probability under addition (Theorem 5.83) and under continuous mappings (Theorem 5.81), we get:
  $$\bar{X}_n + \bar{Y}_n \xrightarrow{P} \lambda_X + \lambda_Y;$$
  $$V_n = \sqrt{\frac{\bar{X}_n+\bar{Y}_n}{\lambda_X+\lambda_Y}} \xrightarrow{P} 1.$$
- **Conclusion**
  Finally, we apply Slutsky's theorem to justify the preservation of the convergence in distribution under (restricted) division (Remark 5.95) to obtain the desired result:
  $$Z_n = \frac{U_n}{V_n} \xrightarrow{d} \text{Normal}(0,1).$$

(b) Discuss the pertinence of the previous result. (0.5)

- **Pertinence of the convergence result** $\frac{(\bar{X}_n-\bar{Y}_n)-(\lambda_X-\lambda_Y)}{\sqrt{\frac{\bar{X}_n}{n}+\frac{\bar{Y}_n}{n}}} \xrightarrow{d} \text{Normal}(0,1)$
  This result is extremely relevant because it provides a pivotal quantity which is essential to derive asymptotic confidence intervals for the difference between the unknown expected values of two independent r.v. $X \sim \text{Poisson}(\lambda_X)$ and $Y \sim \text{Poisson}(\lambda_Y)$.

[4] We could have also invoked Markov's Theorem (Theorem 5.136) because $\lim_{n\to+\infty} V(\bar{X}_n) = 0$ and $\lim_{n\to+\infty} V(\bar{Y}_n) = 0$.