Department of Mathematics, IST — Probability and Statistics Unit

Probability Theory

1st. Test 1st. Semester — **2013/14**

Duration: **1h30m** **2013/11/13** — **8:00 AM**, Room C11

- Please justify your answers.
- This test has TWO PAGES and FOUR GROUPS. The total of points is 20.0.

Group I — Warm up **3.0 points**

During World War II, random walk was used to model the distance that an escaped prisoner of war would travel in a given time.

A military expert assumes that the position of a prisoner at time n ($n = 1, 2, \dots$), X_n, is governed by a **symmetric** *random walk — starting at "0" (the prison camp) and with probability of an northward (resp. southward) "step" (of one km in one time unit) equal to $\frac{1}{2}$ (resp. $\frac{1}{2}$).*

(a) *Obtain an approximate value for the probability that the fraction of time the prisoner is north of the prison camp does not exceed 25%.* (2.0)

- **Process**
  Symmetric random walk (SRW)
- **R.v.**
  Y_i = size of the i^{th} step
  $Y_i \overset{i.i.d.}{\sim} Y$, $i \in \mathbb{N}$
  $$P(Y = y) = \begin{cases} \frac{1}{2}, & y = -1 \text{ (step southward)} \\ \frac{1}{2}, & y = 1 \text{ (step northward)} \\ 0, & \text{otherwise} \end{cases}$$
  $X_n = \sum_{i=1}^{n} Y_i$ = position of the prisoner at time n ($n \in \mathbb{N}$)
- **Initial condition**
  $X_0 = 0$ (prisoner starts at the prison camp!)
- **New r.v.**
  $$\begin{aligned} \frac{W_n}{n} &\overset{Prop.\,0.15}{=} \frac{1}{n}\sum_{i=1}^{n} I_{\mathbb{N}}(X_i + X_{i-1}) \\ &= \text{fraction of time the prisoner is north of the prison camp} \end{aligned}$$
- **Requested probability (approximate value)**
  According to Prop. 0.15, the requested probability is approximately equal to
  $$\begin{aligned} \lim_{n\to+\infty} P\left(\frac{W_n}{n} \le 0.25\right) &= \frac{2}{\pi}\arcsin\left(\sqrt{0.25}\right) \\ &= \frac{2}{\pi}\arcsin(0.5) \end{aligned}$$

$$\begin{aligned} \lim_{n\to+\infty} P\left(\frac{W_n}{n} \le 0.25\right) &= \frac{2}{\pi} \times \frac{\pi}{6} \\ &= \frac{1}{3}. \end{aligned}$$

(b) *Admit that the prisoner has already taken exactly 25 steps.* (1.0)
Find an approximate value for the probability that the prisoner is not in a 3km neighbourhood of the prison camp.

- **Requested probability**
  According to Prop. 0.13 (equation (21)), for large values of n and $a < b$, we have
  $$P(a < X_n \le b) \simeq \Phi(b/\sqrt{n}) - \Phi(a/\sqrt{n}).$$
  Thus,
  $$\begin{aligned} P(|X_{25}| > 3) &= 1 - P(-3 \le X_{25} \le 3) \\ &= 1 - P(-4 < X_{25} \le 3) \\ &\simeq 1 - \Phi(3/\sqrt{25}) - \Phi(-4/\sqrt{25}) \\ &= 1 - [\Phi(0.6) - \Phi(-0.8)] \\ &= 1 - \{\Phi(0.6) - [1 - \Phi(0.8)]\} \\ &\overset{table}{=} 1 - [0.7257 - (1 - 0.7881)] \\ &= 1 - 0.5138 \\ &= 0.4862. \end{aligned}$$

Group II — Probability spaces and independence **5.0 points**

1. *Consider the sample space Ω and let $\mathcal{F}_i$ be a $\sigma -$ algebra on Ω, for each $i \in I$.*

   (a) *Show that $(\cap_{i\in I}\mathcal{F}_i)$ is also a $\sigma -$ algebra on Ω. What can be said about the index set I?* (1.5)

   - **$\sigma -$ algebras on Ω**
     $\mathcal{F}_i$, $i \in I$
   - **To prove**
     $(\cap_{i\in I}\mathcal{F}_i)$ is also a $\sigma -$ algebra on Ω
   - **Proof**
     According to Def. 1.38, a minimal set of postulates for $\mathcal{F}_i$ to be a $\sigma -$ algebra on Ω is:
     (i) $\Omega \in \mathcal{F}_i$;
     (ii) $A \in \mathcal{F}_i \Rightarrow A^c \in \mathcal{F}_i$ (i.e., $\mathcal{F}_i$ is closed under complementation);
     (iii) $A_j \in \mathcal{F}_i$, $j \in J$ (countable set) $\Rightarrow \cup_{j\in J} A_j \in \mathcal{F}_i$ (i.e., $\mathcal{F}_i$ is closed under countable union).
     Hence, we have to prove that these 3 postulates are true for $(\cap_{i\in I}\mathcal{F})$.
     Note that $(\cap_{i\in I}\mathcal{F})$ is the collection of subsets of Ω lying in all $\mathcal{F}_i$, $i \in I$. As a consequence:

(i) $\Omega \in \mathcal{F}_i,\ \forall i \in I \ \Rightarrow\ \Omega \in (\cap_{i\in I}\mathcal{F}_i)$;

(ii) $A \in (\cap_{i\in I}\mathcal{F}_i) \ \Rightarrow\ A \in \mathcal{F}_i,\ \forall i \in I \ \Rightarrow\ A^c \in \mathcal{F}_i,\ \forall i \in I \ \Rightarrow\ A^c \in (\cap_{i\in I}\mathcal{F}_i)$;

(iii) $A_1, A_2, \dots \in (\cap_{i\in I}\mathcal{F}_i) \ \Rightarrow\ A_1, A_2, \dots \in \mathcal{F}_i, \forall i \in I \ \Rightarrow\ \cup_{j=1}^{+\infty} A_j \in \mathcal{F}_i,\ \forall i \in I \ \Rightarrow\ \cup_{j=1}^{+\infty} A_j \in (\cap_{i\in I}\mathcal{F}_i)$.

That is, $(\cap_{i\in I}\mathcal{F}_i)$ is also a $\sigma-$algebra on Ω. QED

- **Characteristics of the index set I**
  It can be any non empty set (finite, countable or not countable!).

(b) *Discuss the pertinence of the result in (a).* (0.5)

- **Pertinence of the result**
  Note that the result proven in (a) allows us to consider an arbitrary family of subsets of Ω, say $\mathcal{A}$, and define $\sigma(\mathcal{A})$, the $\sigma-$algebra generated by $\mathcal{A}$, i.e., the intersection of all the $\sigma-$algebras containing $\mathcal{A}$.

2. (a) *Prove by induction that $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$.* (1.0)

- **To prove**

$$P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i) \qquad (1)$$

- **Proof**
  Inequality (1) is trivially true if $n = 1$: $P(A_1) \leq P(A_1)$.
  Now, assume inequality (1) holds for n (induction hypothesis!). Then

$$\begin{aligned} P\left(\cup_{i=1}^{n+1} A_i\right) &= P\left[(\cup_{i=1}^n A_i) \cup A_{n+1}\right] \\ &\stackrel{addition\,rule}{=} P\left(\cup_{i=1}^n A_i\right) + P(A_{n+1}) - P\left[(\cup_{i=1}^n A_i) \cap A_{n+1}\right] \\ &\stackrel{P(.)\geq 0}{\leq} P\left(\cup_{i=1}^n A_i\right) + P(A_{n+1}) \\ &\stackrel{induction\,hip.}{\leq} \sum_{i=1}^n P(A_i) + P(A_{n+1}) \\ &= \sum_{i=1}^{n+1} P(A_i). \end{aligned}$$

QED

(b) *Demonstrate that $P(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n P(A_i^c)$.* (0.5)

- **To prove**
  $P(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n P(A_i^c)$
- **Proof**

$$\begin{aligned} P(\cap_{i=1}^n A_i) &\stackrel{De\,Morgan's\,laws}{=} P\left[(\cup_{i=1}^n A_i^c)^c\right] \\ &= 1 - P\left(\cup_{i=1}^n A_i^c\right) \\ &\stackrel{(1)}{\geq} 1 - \sum_{i=1}^n P(A_i^c). \end{aligned}$$

QED

3. There are two roads from A to B and two other roads from B to C. Each of the four roads is blocked by snow with probability p, independently of the others. (1.5)

Find the probability that it is possible to go from A to B, given that it is not possible to go from A to C.

- **Events**
  $E \leftrightarrow F =$ it is possible to go from E to F
  $E \not\leftrightarrow F =$ it is not possible to go from E to F
- **Key probabilities**
  Since there are two roads from A to B — and also from B and C —, and each road is blocked by snow with probability p, independently of the other, we get

$$\begin{aligned} P(A \not\leftrightarrow B) &= P(B \not\leftrightarrow C) \\ &= P(\text{both roads blocked}) \\ &\stackrel{indep.}{=} p \times p \\ &= p^2 \\ P(A \leftrightarrow B) &= P(B \leftrightarrow C) \\ &= 1 - p^2. \end{aligned}$$

Moreover,

$$\begin{aligned} P(A \leftrightarrow C) &= P[(A \leftrightarrow B) \cap (B \leftrightarrow C)] \\ &\stackrel{indep.}{=} P(A \leftrightarrow B) \times P(B \leftrightarrow C) \\ &= (1 - p^2)^2. \end{aligned}$$

- **Requested probability**

$$\begin{aligned} P(A \leftrightarrow B | A \not\leftrightarrow C) &= \frac{P[(A \leftrightarrow B) \cap (A \not\leftrightarrow C)]}{P(A \not\leftrightarrow C)} \\ &= \frac{P\{(A \leftrightarrow B) \cap [(A \not\leftrightarrow B) \cup (B \not\leftrightarrow C)]\}}{P(A \not\leftrightarrow C)} \\ &= \frac{P[(A \leftrightarrow B) \cap (B \not\leftrightarrow C)]}{P(A \not\leftrightarrow C)} \\ &\stackrel{indep.}{=} \frac{P[(A \leftrightarrow B)] \times P[(B \not\leftrightarrow C)]}{1 - P(A \leftrightarrow C)} \\ &= \frac{(1-p^2) \times p^2}{1 - (1-p^2)^2}. \end{aligned}$$

Group III — Random variables and independence **7.5 points**

1. *The deviation from a target is a r.v. X.*

(a) *Show that $Y = X^2$ is a Borel measurable function, therefore also a r.v.* (1.5)

- **R.v.**
  X = deviation from a target
- **Important**
  Let:
  - X be a real r.v.;
  - $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable spaces.

  Then, by Def. 2.13, $X : \Omega \to \mathbb{R}$ and

  $$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R}), \qquad (2)$$

  in particular:

  $$X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R},$$

  according to Prop. 2.16; and

  $$X^{-1}((-\infty, x)) = \{\omega \in \Omega : X(\omega) < x\} \in \mathcal{F}, \forall x \in \mathbb{R}.$$

- **To prove**
  $Y = g(X) = X^2$ is a Borel measurable function, therefore also a r.v., by Corollary 2.40.
- **Proof**
  Firsty, let us remind the reader that a function $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff

  $$g^{-1}(B) \in \mathcal{B}(I\!R), \forall B \in \mathcal{B}(I\!R),$$

  according to Def. 2.36, with $n = m = 1$. Furthermore, in order that $g : \mathbb{R} \to \mathbb{R}$ be Borel measurable it suffices that $g^{-1}((-\infty, y]) \in \mathcal{B}(\mathbb{R})$, $\forall y \in I\!R$, according to Remark 2.47 (with $n = 1$).
  Secondly, $g^{-1}((-\infty, y]) = Y^{-1}((-\infty, y]) = \{\omega \in \Omega : Y(\omega) \leq y\}$ equals:
  - for $y < 0$,

    $$\{Y \leq y\} = \emptyset \in \mathcal{F};$$

  - for $y \geq 0$,

    $$\begin{aligned}\{Y \leq y\} &= \{X^2 \leq y\} \\ &= \{-\sqrt{y} \leq X \leq \sqrt{y}\} \\ &= \{X \leq \sqrt{y}\} \cap \overline{\{X < -\sqrt{y}\}} \\ &= X^{-1}\left((-\infty, \sqrt{y}]\right) \cap \overline{X^{-1}\left((-\infty, -\sqrt{y})\right)},\end{aligned}$$

    where

    $$X^{-1}\left((-\infty, \sqrt{y}]\right), X^{-1}\left((-\infty, -\sqrt{y})\right) \in \mathcal{F},$$

    by (2).

  Finally, since $\mathcal{F}$ is a $\sigma-$algebra — thus, closed under [countable union,] countable intersection and complementation — we can state that

  $$\overline{X^{-1}\left((-\infty, -\sqrt{y})\right)} \in \mathcal{F}$$

  and $\{Y \leq y\}$ is the intersection of two elements of $\mathcal{F}$, thus,

  $$\{Y \leq y\} \in \mathcal{F}.$$

  As a result, Y is a Borel measurable function and therefore a r.v., by Corollary 2.40.
  QED

(b) *Derive the p.d.f. of Y and sketch the graph of $f_Y(y)$, considering $X \sim Uniform(-1, 2)$.* (3.0)

- **R.v.**
  X = deviation from a target
- **Distribution**
  $X \sim \text{Uniform}(-1, 2)$
- **P.d.f. of X**

  $$f_X(x) = \begin{cases} \frac{1}{2-(-1)} = \frac{1}{3}, & -1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- **Range of X**
  $\mathbb{R}_X = [-1,\ 2]$
- **Transformation**
  $Y = g(X) = X^2$
- **Range of Y**
  $\mathbb{R}_Y = g(\mathbb{R}_X) = [0, 4]$
- **Monotonic restrictions**
  $y = g(x) = x^2$ is a function with two monotonic restrictions:

  $$g_1(x) = x^2, \quad x \in [-1, 0]$$
  $$g_2(x) = x^2, \quad x \in [0, 2].$$

- **Inverses**
  For $0 < y \leq 1$, one deals with $n(y) = 2$ inverses (schematics!):

  $$g_1^{-1}(y) = -\sqrt{y}$$
  $$g_2^{-1}(y) = \sqrt{y}.$$

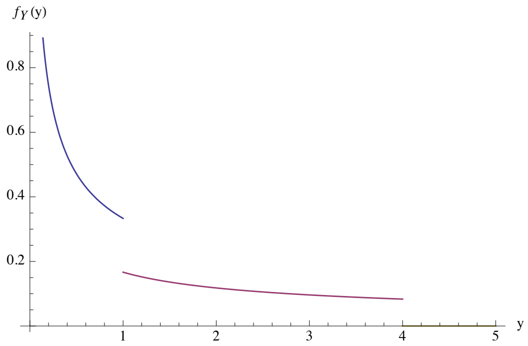
  For $1 < y \leq 4$, one deals with just one inverse: $g_2^{-1}(y) = \sqrt{y}$.
- **Derivatives**

  $$\frac{dg_1^{-1}(y)}{dy} = -\frac{1}{2\sqrt{y}}$$
  $$\frac{dg_2^{-1}(y)}{dy} = \frac{1}{2\sqrt{y}}$$

- **P.d.f. of Y**
  According to Theorem 2.98, we get:

$$
\begin{aligned}
f_Y(y) &= \sum_{k=1}^{n(y)} f_X[g_k^{-1}(y)] \times \left|\frac{dg_k^{-1}(y)}{dy}\right| \\
&= \begin{cases} \frac{1}{3} \times \left|-\frac{1}{2\sqrt{y}}\right| + \frac{1}{3} \times \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{3\sqrt{y}}, & 0 < y \leq 1 \\ \frac{1}{3} \times \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{6\sqrt{y}}, & 1 < y \leq 4 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

- **Graph**



- **[Alternatively...**

We could obtain $F_X(x)$, then $F_Y(y)$ in terms of $F_X(x)$ and differentiate $F_Y(y)$ to obtain $f_Y(y)$:

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= \int_{-\infty}^{x} f_X(t)\, dt \\
&= \begin{cases} 0, & x < -1 \\ \frac{1}{3} \times [x - (-1)] = \frac{x+1}{3}, & -1 \leq x \leq 2 \\ 1, & x > 2 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) &= P(X^2 \leq y) \\
&= \begin{cases} 0, & y \leq 0 \\ P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \\ \qquad = \frac{\sqrt{y}+1}{3} - \frac{-\sqrt{y}+1}{3} = \frac{2\sqrt{y}}{3}, & 0 < y \leq 1 \\ P(-1 \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-1) & \\ \qquad = \frac{\sqrt{y}+1}{3} - 0 = \frac{\sqrt{y}+1}{3}, & 1 < y \leq 4 \\ 1, & y > 4 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \frac{d\,F_Y(y)}{dy} \\
&= \begin{cases} \frac{d}{dy}\left(\frac{2\sqrt{y}}{3}\right) = \frac{1}{3\sqrt{y}}, & 0 < y \leq 1 \\ \frac{d}{dy}\left(\frac{\sqrt{y}+1}{3}\right) = \frac{1}{6\sqrt{y}}, & 1 < y \leq 4 \\ 0, & \text{otherwise.]} \end{cases}
\end{aligned}
$$

(c) Determine $P(Y < X)$. (1.0)

- **Requested probability**

$$
\begin{aligned}
P(Y < X) &= P(0 < X < 1) \\
&= \int_0^1 f_X(x)\, dx \\
&= \int_0^1 \frac{1}{3}\, dx \\
&= \frac{1}{3}.
\end{aligned}
$$

2. *An interval of unitary length, say $(0,1)$, is divided into 3 sub-intervals by randomly (and independently) choosing two points. Let X and Y be the abscissae of these two points.* (2.0)

What is the probability that the three segments form a triangle?

Hint: *The three segments form a triangle iff $(0 < X < \frac{1}{2} < Y < 1$ and $Y - X < \frac{1}{2})$ or $(0 < Y < \frac{1}{2} < X < 1$ and $X - Y < \frac{1}{2})$.*

- **Random vector**

(X, Y)

$X =$ abscissa of 1st. point

$Y =$ abscissa of 2nd. point

- **Distributions**

$X \sim Y \sim \text{Uniform}(0,1)$

$X \perp\!\!\!\perp Y$

- **Joint p.d.f.**

$$
\begin{aligned}
f_{X,Y}(x,y) &\overset{X \perp\!\!\!\perp Y}{=} f_X(x) \times f_Y(y) \\
&= \begin{cases} 1, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

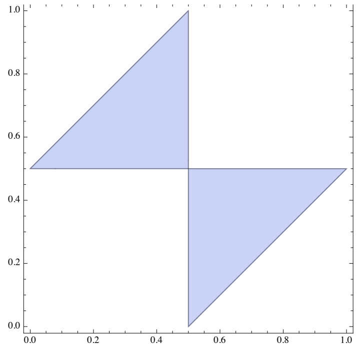
- **Requested probability**

The probability that the three segments form a triangle corresponds to the event mentioned in the hint,

$$
\left(0 < X < \frac{1}{2} < Y < 1 \text{ and } Y - X < \frac{1}{2}\right) \text{ or } \left(0 < Y < \frac{1}{2} < X < 1 \text{ and } X - Y < \frac{1}{2}\right).
$$

Furthermore, since $f_{X,Y}(x,y)$ is unitary, the probability of this event corresponds to the area of the shaded region pictured ahead. Thus,

$$
P(\text{three segments form a triangle}) = \frac{1}{4}.
$$



- **[Alternatively...**

  Capitalizing once again on the hint, we could have integrated the joint p.d.f. over the region

$$A = \left\{(x,y) \in \mathbb{R}^2 : \left(0 < x < \frac{1}{2} < y < 1 \text{ and } y - x < \frac{1}{2}\right) \text{ or } \left(0 < y < \frac{1}{2} < x < 1 \text{ and } x - y < \frac{1}{2}\right)\right\}$$

  and get:

$$\begin{aligned} P(\text{three segm. form a triangle}) &= \int_A \int f_{X,Y}(x,y)\, dy\, dx \\ &= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}+x} dy\, dx + \int_{\frac{1}{2}}^1 \int_{-\frac{1}{2}+x}^{\frac{1}{2}} dy\, dx \\ &= \int_0^{\frac{1}{2}} \left(\frac{1}{2} + x - \frac{1}{2}\right) dx + \int_{\frac{1}{2}}^1 \left[\frac{1}{2} - \left(-\frac{1}{2} + x\right)\right] dx \\ &= \left.\frac{x^2}{2}\right|_0^{\frac{1}{2}} + \left.\left(x - \frac{x^2}{2}\right)\right|_{\frac{1}{2}}^1 \\ &= \frac{1}{8} + \frac{1}{2} - \frac{3}{8} \\ &= \frac{1}{4}.] \end{aligned}$$

Group IV — Independence **4.5 points**

1. *Let $\{A_i,\, i = 1, \ldots, 5\}$ be a measurable partition of Ω such that $P(A_i) = \frac{15}{64}$, $i = 1, 2, 3$, $P(A_4) = \frac{1}{64}$ and $P(A_5) = \frac{18}{64}$. Define $B = A_1 \cup A_4$, $C = A_2 \cup A_4$ and $D = A_3 \cup A_4$.* (2.0)

   Check that $P(B \cap C \cap D) = P(B) \times P(C) \times P(D)$ but that B, C, D are not independent events.

- **Events/probabilities**

  $P(A_i) = \frac{15}{64}$, $i = 1, 2, 3$
  $P(A_4) = \frac{1}{64}$
  $P(A_5) = \frac{18}{64}$
  Note that these events form a partition of Ω, thus, are disjoint...

- **Other events**

  $B = A_1 \cup A_4$
  $C = A_2 \cup A_4$
  $D = A_3 \cup A_4$

- **Checking whether** $P(B \cap C \cap D) = P(B) \times P(C) \times P(D)$

  On one hand, we have

$$\begin{aligned} P(B \cap C \cap D) &= P[(A_1 \cup A_4) \cap (A_2 \cup A_4) \cap (A_3 \cup A_4)] \\ &= P(A_4) \\ &= \frac{1}{64}. \end{aligned}$$

  On the other hand,

$$\begin{aligned} P(B) \times P(C) \times P(D) &= P(A_1 \cup A_4) \times P(A_2 \cup A_4) \times P(A_3 \cup A_4) \\ &\overset{disj.ev.}{=} [P(A_1) + P(A_4)] \times [P(A_2) + P(A_4)] \times [P(A_3) + P(A_4)] \\ &= \left(\frac{15}{64} + \frac{1}{64}\right) \times \left(\frac{15}{64} + \frac{1}{64}\right) \times \left(\frac{15}{64} + \frac{1}{64}\right) \\ &= \left(\frac{1}{4}\right)^3 \\ &= \frac{1}{64}, \end{aligned}$$

  which is indeed equal to $P(B \cap C \cap D)$. QED

- **Checking whether** B**,** C**,** D **are not independent**

  According to Def. 3.6, the events B, C, D are independent not only if $P(B \cap C \cap D) = P(B) \times P(C) \times P(D)$ (as we have already checked), but also if:

  (i) $P(B \cap C) = P(B) \times P(C)$;
  (ii) $P(B \cap D) = P(B) \times P(D)$;
  (iii) $P(C \cap D) = P(C) \times P(D)$.

  Let us checked if any of these equalities fail. For instance, we have

$$\begin{aligned} P(B \cap C) &= P[(A_1 \cup A_4) \cap (A_2 \cup A_4)] \\ &\overset{disj.ev.}{=} P(A_4) \\ &= \frac{1}{64}, \end{aligned}$$

  whereas

$$\begin{aligned} P(B) \times P(C) &= P(A_1 \cup A_4) \times P(A_2 \cup A_4) \\ &\overset{disj.ev.}{=} [P(A_1) + P(A_4)] \times [P(A_2) + P(A_4)] \end{aligned}$$

$$\begin{aligned} P(B) \times P(C) &= \left(\frac{15}{64} + \frac{1}{64}\right) \times \left(\frac{15}{64} + \frac{1}{64}\right) \\ &= \left(\frac{1}{4}\right)^2 \\ &= \frac{1}{16}. \end{aligned}$$

Thus, B, C, D are certainly not independent events.

2. *The daily number of requests that arrive to a server is a r.v. $N \sim$ Geometric(p). The associated processing times are i.i.d. r.v. X_i, which are independent of N and exponentially distributed with mean λ^{-1}.*

   (a) *Derive the survival function of the shortest processing time in a day the server had to process $N = n$ ($n \in \mathbb{N}$) requests.* (1.0)

   - **R.v.**
     $N =$ daily number of requests
     $N \sim$ Geometric(p)
   - **Other r.v.**
     $X_i =$ processing time of request i, $i \in \mathbb{N}$
     $X_i \stackrel{i.i.d.}{\sim}$ Exponential(λ), $i \in \mathbb{N}$
   - **New r.v.**
     $Y|\{N = n\} =$ shortest processing time given that $\{N = n\}$
     $Y|\{N = n\} \stackrel{d}{=} \min_{i=1,\ldots,n} X_i$ because $X_i \perp\!\!\!\perp N$, $i \in \mathbb{N}$
   - **Distribution of $Y|\{N = n\}$**
     According to Prop. 3.60,

     $$Y|\{N = n\} \sim \text{Exponential}(n\lambda).$$

   - **Survival function of $Y|\{N = n\}$**

     $$\begin{aligned} S_{Y|\{N=n\}}(y) &= P[Y > y|\{N = n\}] \\ &= \begin{cases} 1, & y \leq 0 \\ e^{-n\lambda y}, & y > 0. \end{cases} \end{aligned}$$

   (b) *Determine the survival function of the shortest processing time in a randomly chosen day.* (1.5)

   - **R.v.**
     $Y =$ shortest processing time in a randomly chosen day
   - **Survival function of Y**
     By using the total probability law and the fact that

     $$P(N = n) = (1 - p)^{n-1}\, p,\ n \in \mathbb{N},$$

     we obtain, for $y > 0$ (and $\lambda > 0$):

$$\begin{aligned} S_Y(y) &= P(Y > y) \\ &= \sum_{n=1}^{+\infty} P[Y > y|\{N = n\}] \times P(N = n) \\ &= \sum_{n=1}^{+\infty} e^{-n\lambda y} \times (1 - p)^{n-1}\, p \\ &= \frac{p}{1-p} \sum_{n=1}^{+\infty} \left[e^{-\lambda y} \times (1 - p)\right]^n \\ &\stackrel{0<e^{-\lambda y}\times(1-p)<1}{=} \frac{p}{1-p} \times \frac{e^{-\lambda y} \times (1-p)}{1 - e^{-\lambda y} \times (1-p)} \\ &= \frac{p\, e^{-\lambda y}}{1 - e^{-\lambda y} \times (1-p)} \\ &= \frac{p}{e^{\lambda y} - (1-p)}. \end{aligned}$$