Department of Mathematics, IST — Probability and Statistics Unit

Probability Theory

1st. Test ("Recurso") 1st. Semester — **2013/14**

Duration: **1h30m** **2014/01/25** — **8:00 AM**, Room P12

- Please justify your answers.
- This test has TWO PAGES and FOUR GROUPS. The total of points is 20.0.

Group I — Warm up **3.0 points**

Suppose:

- X_n represents the deviation from a target of the accumulated sales of a store at day n;
- X_n is modelled by a symmetric random walk;
- the accounting period starts with a deviation from target equal to $X_0 = 0$.

(a) Compute the probability of the event $A = \{X_2 \geq 0,\ X_4 = -2,\ X_6 < 0,\ X_7 < 0\}$. (2.0)

Hint: Identify the paths comprising event A.

- **Process**

  Symmetric random walk (SRW)

- **R.v.**

  $Y_i =$ size of the i^{th} step

  $Y_i \stackrel{i.i.d.}{\sim} Y,\ i \in \mathbb{N}$

  $$P(Y = y) = \begin{cases} \frac{1}{2}, & y = -1 \text{ (unitary increase in the deviation from target)} \\ \frac{1}{2}, & y = 1 \text{ (unitary decrease in the deviation from target)} \\ 0, & \text{otherwise} \end{cases}$$

  $X_n = \sum_{i=1}^{n} Y_i =$ deviation from target of the accumulated sales of a store at day n $(n \in \mathbb{N})$

- **Initial condition**

  $X_0 = 0$

- **Event and associated paths**

  $A = \{X_2 \geq 0, X_4 = -2, X_6 < 0, X_7 < 0\}$

  The paths comprising this event have coordinates:

  (i) $(0,0) \to (1,\pm1) \to (2,0) \to (3,-1) \to (4,-2) \to (5,-3) \to (6,-4) \to (7,-5)$;

  (ii) $(0,0) \to (1,\pm1) \to (2,0) \to (3,-1) \to (4,-2) \to (5,-3) \to (6,-4) \to (7,-3)$;

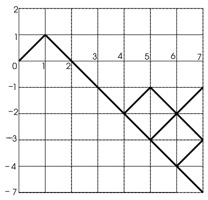
  (iii) $(0,0) \to (1,\pm1) \to (2,0) \to (3,-1) \to (4,-2) \to (5,-3) \to (6,-2) \to (7,-3)$;

  (iv) $(0,0) \to (1,\pm1) \to (2,0) \to (3,-1) \to (4,-2) \to (5,-3) \to (6,-2) \to (7,-1)$;

  (v) $(0,0) \to (1,\pm1) \to (2,0) \to (3,-1) \to (4,-2) \to (5,-1) \to (6,-2) \to (7,-1)$;

  (vi) $(0,0) \to (1,\pm1) \to (2,0) \to (3,-1) \to (4,-2) \to (5,-1) \to (6,-2) \to (7,-3)$.

  The paths with $X_1 = 1$ are shown in the picture below:[1]



- **Requested probability**

  Since we are dealing with a SRW, each of these 12 paths of 7 steps have probability $\frac{1}{2^7}$ (see Prop. 0.3, formula (2)) and:

  $$\begin{aligned} P(A) &= P(\{X_2 \geq 0, X_4 = -2, X_6 < 0, X_7 < 0\}) \\ &= 12 \times \frac{1}{2^7} \\ &= \frac{3}{32} \\ &= 0.09375. \end{aligned}$$

(b) Find the probability that the deviation from target remains non negative for $n = 10$ days. (1.0)

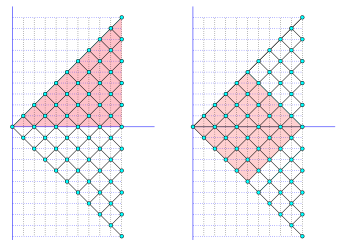
Hint: Identify the paths comprising this event and relate them to the ones associated with the event $\{X_n = 0\}$.

- **Event**

  $B = \{X_i \geq 0,\ i = 1, \dots, n\}$

- **Important**

  The number of paths comprising event B are shown in the shaded region of the picture below:[2]



The picture leads to the conclusion that it contains exactly the same number of

[1] Taken from Konstantopoulos, T. (2009). Introductory Lecture Notes on Markov Chains and Random Walks. (http://www2.math.uu.se/ takis/L/McRw/mcrw.pdf). Note that its smallest ordinate is -5 and not -7.

[2] This picture was taken from Konstantopoulos (2009, p. 83).

paths associated to the event

$$C = \{X_n = 0\}.$$

- **Requested probability**

$$\begin{aligned} P(B) &= P(C) \\ &= P(X_n = 0) \\ &\overset{Prop.\,0.3,\,formula\,(5)}{=} \binom{n}{\frac{n+0}{2}} \times \frac{1}{2^n} \\ &= \binom{10}{5} \times \frac{1}{2^{10}} \\ &\simeq 0.246094. \end{aligned}$$

Group II — Probability spaces **5.0 points**

1. Consider a sample space $\Omega = \{A, B, C\}$ and give an example of two $\sigma-$algebras on Ω, say $\mathcal{F}$ and $\mathcal{H}$, such that $\mathcal{F} \cup \mathcal{H}$ is not a $\sigma-$algebra on Ω.[3] (1.0)

- **Sample space**
  $\Omega = \{A, B, C\}$

- **Two $\sigma-$algebras on Ω**
  We ought to mention that, according to Def. 1.38, a minimal set of postulates for a non-empty class of subsets $\mathcal{A}$ of Ω to be a $\sigma-$algebra on Ω is:
  (i) $\Omega \in \mathcal{A}$;
  (ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$;
  (iii) $A_1, A_2, \dots \in \mathcal{A} \Rightarrow \cup_{i=1}^{+\infty} A_i \in \mathcal{A}$.
  These postulates are satisfied by:

$$\begin{aligned} \mathcal{F} &= \{\emptyset, \{A\}, \{B, C\}, \Omega\}; \\ \mathcal{H} &= \{\emptyset, \{A, B\}, \{C\}, \Omega\}. \end{aligned}$$

- **Checking whether $\mathcal{F} \cup \mathcal{H}$ is $\sigma-$algebra**
  We have to prove that at least one of these 3 postulates is not true for

$$\mathcal{A} = \mathcal{F} \cup \mathcal{H} = \{\emptyset, \{A\}, \{C\}, \{A, B\}, \{B, C\}, \Omega\}.$$

  The following disjoint events

$$\begin{aligned} D_1 &= \{A\} \\ D_2 &= \{C\} \\ D_3 &= D_4 = \dots = \emptyset \end{aligned}$$

[3] Recall that $\mathcal{F} \cup \mathcal{H}$ is the collection of subsets of lying in either $\mathcal{F}$ and $\mathcal{H}$.

yield

$$\begin{aligned} \cup_{i=1}^{+\infty} D_i &= \{A, C\} \\ &\notin \mathcal{A} \\ &= \mathcal{F} \cup \mathcal{H}. \end{aligned}$$

As a consequence, $\mathcal{A} = \mathcal{F} \cup \mathcal{H}$ is not a $\sigma-$algebra.

2. Let: (2.0)

- $b_1, \dots, b_n$ be arbitrary positive numbers;
- $B_1, \dots, B_n$ be an arbitrary finite partition of the sample space Ω;
- P be a p.f. on $(\Omega, \mathcal{F})$.

Prove that the set function

$$Q(A) = \frac{\sum_{j=1}^n b_j \times P(A \cap B_j)}{\sum_{k=1}^n b_k \times P(B_k)}, \; A \in \mathcal{F},$$

is also a p.f. on $(\Omega, \mathcal{F})$.

- **Setting**
  Consider:
  (i) $b_1, \dots, b_n$ be arbitrary positive numbers;
  (ii) $B_1, \dots, B_n$ be an arbitrary finite partition of the sample space Ω;
  (iii) P be a p.f. on $(\Omega, \mathcal{F})$.

- **To prove**
  $Q(A) = \frac{\sum_{j=1}^n b_j \times P(A \cap B_j)}{\sum_{k=1}^n b_k \times P(B_k)}$, $A \in \mathcal{F}$, is also a p.f. on $(\Omega, \mathcal{F})$.

- **Proof**
  According to Def. 1.48, the p.f. P defined on $(\Omega, \mathcal{F})$ satisfies
  (iv) Axiom 1 — $P(A) \geq 0, \forall A \in \mathcal{F}$
  (v) Axiom 2 — $P(\Omega) = 1$
  (vi) Axiom 3 (countable additivity) — Whenever $A_1, A_2, \dots$ are (pairwise) disjoint events in $\mathcal{F}$, $P\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} P(A_i)$.
  As a result, the set function Q verifies:

$$\begin{aligned} Q(A) &= \frac{\sum_{j=1}^n b_j \times P(A \cap B_j)}{\sum_{k=1}^n b_k \times P(B_k)} \\ &\overset{(i),(iv)}{\geq} 0, \forall A \in \mathcal{F}; \end{aligned}$$

$$
\begin{aligned}
Q(\Omega) &= \frac{\sum_{j=1}^{n} b_j \times P(\Omega \cap B_j)}{\sum_{k=1}^{n} b_k \times P(B_k)} \\
&= \frac{\sum_{j=1}^{n} b_j \times P(B_j)}{\sum_{k=1}^{n} b_k \times P(B_k)} \\
&= 1;
\end{aligned}
$$

$$
\begin{aligned}
Q\left(\cup_{i=1}^{+\infty} A_i\right) &= \frac{\sum_{j=1}^{n} b_j \times P\left[\left(\cup_{i=1}^{+\infty} A_i\right) \cap B_j\right]}{\sum_{k=1}^{n} b_k \times P(B_k)} \\
&= \frac{\sum_{j=1}^{n} b_j \times P\left[\cup_{i=1}^{+\infty} (A_i \cap B_j)\right]}{\sum_{k=1}^{n} b_k \times P(B_k)} \\
&\overset{(ii),(vi)}{=} \frac{\sum_{j=1}^{n} \left[b_j \times \sum_{i=1}^{+\infty} P(A_i \cap B_j)\right]}{\sum_{k=1}^{n} b_k \times P(B_k)} \\
&= \sum_{i=1}^{+\infty} \frac{\sum_{j=1}^{n} b_j \times P(A_i \cap B_j)}{\sum_{k=1}^{n} b_k \times P(B_k)} \\
&= \sum_{i=1}^{+\infty} Q(A_i), \text{ for any } A_1, A_2, \ldots \text{(pairwise) disjoint events in } \mathcal{F}.
\end{aligned}
$$

Consequently, Q is indeed a p.f. on $(\Omega, \mathcal{F})$.

3. Let A_i, $i = 1, \ldots, n$, be events on a sample space Ω such that $P(A_i) = 1$, $i = 1, \ldots, n$.

   (a) Show that $P(\cup_{i=1}^{n} A_i) = 1$. (0.5)

   - **To prove**

     $A_i \in \Omega : P(A_i) = 1,\ i = 1, \ldots, n \quad \Rightarrow \quad P(\cup_{i=1}^{n} A_i) = 1$

   - **Proof**

     Invoking

     – the fact that the probability of any event belongs to the interval $[0, 1]$ and

     – Prop. 1.54, namely the monotonicity of any p.f.,

     yields:

$$
\begin{aligned}
A_j \subseteq \cup_{i=1}^{n} A_i,\ j = 1, \ldots, n \quad \Rightarrow \quad & P(A_j) \leq P\left(\cup_{i=1}^{n} A_i\right),\ j = 1, \ldots, n \\
& 1 \leq P\left(\cup_{i=1}^{n} A_i\right) \\
& P\left(\cup_{i=1}^{n} A_i\right) = 1.
\end{aligned}
$$

   (b) Prove by induction that $P(\cap_{i=1}^{n} A_i) = 1$. (1.0)

   - **To prove**

     $A_i \in \Omega : P(A_i) = 1,\ i = 1, \ldots, n \quad \Rightarrow \quad P(\cap_{i=1}^{n} A_i) = 1$

   - **Proof**

     The equality $P(\cap_{i=1}^{n} A_i) = 1$ obviously holds for $n = 1$. Let us suppose that equality $P(\cap_{i=1}^{n} A_i) = 1$ is true for n. By capitalising once again on the fact that the probability of any event belongs to the interval $[0, 1]$, on the monotonicity of any p.f. and also on the following results —

     (i) $P(\cup_{i=1}^{n+1} A_i) = 1$

     (ii) $P(C \cap D) = P(C) + P(D) - P(C \cup D)$ (addition law rewritten!)

     (iii) $\cap_{i=1}^{n} A_i \subseteq [(\cap_{i=1}^{n} A_i) \cup A_{n+1}] \quad \Rightarrow \quad 1 = P\left(\cap_{i=1}^{n} A_i\right) \leq P\left[(\cap_{i=1}^{n} A_i) \cup A_{n+1}\right] \quad \Rightarrow$ $P\left[(\cap_{i=1}^{n} A_i) \cup A_{n+1}\right] = 1$

     —, we get:

$$
\begin{aligned}
P(\cap_{i=1}^{n+1} A_i) &= P\left[(\cap_{i=1}^{n} A_i) \cap A_{n+1}\right] \\
&= P\left(\cup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left[(\cap_{i=1}^{n} A_i) \cup A_{n+1}\right] \\
&= 1 + 1 - 1 \\
&= 1.
\end{aligned}
$$

4. Demonstrate that if $P(A|B) > P(A)$ then $P(B|A) > P(B)$. (0.5)

   - **To prove**

     $P(A|B) > P(A) \quad \Rightarrow \quad P(B|A) > P(B)$

   - **Proof**

$$
\begin{aligned}
P(B|A) &\overset{Bayes\,Theo.}{=} \frac{P(A|B)}{P(A)} \times P(B) \\
&\overset{P(A|B)>P(A)}{>} P(B).
\end{aligned}
$$

Group III — Random variables and independence **8.0 points**

1. Let X and Y be two r.v. and A be an event. Prove that the function (2.0)

$$
Z(\omega) = \begin{cases} X(\omega), & \text{if } \omega \in A \\ Y(\omega), & \text{if } \omega \in A^c \end{cases}
$$

   is a r.v.

   - **To prove**

$$
X, Y \text{ r.v., } A \text{ event} \quad \Rightarrow \quad Z(\omega) = \begin{cases} X(\omega), & \text{if } \omega \in A \\ Y(\omega), & \text{if } \omega \in A^c \end{cases} \text{ is a r.v.}
$$

   - **Proof**

     Let:

     (i) X and Y be two real r.v.;

     (ii) $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable spaces.

     Then, by Def. 2.13 and for $W = X, Y$, we have $W : \Omega \to \mathbb{R}$ and

$$
W^{-1}(B) = \{\omega \in \Omega : W(\omega) \in B\} \in \mathcal{F},\ \forall B \in \mathcal{B}(\mathbb{R}).
$$

Adding to this the fact that $\mathcal{F}$ is a $\sigma-$algebra — thus, closed under countable union, countable intersection and complementation —, we can state that, $\forall x \in \mathbb{R}$,

$$\begin{aligned} Z^{-1}((-\infty, x]) &= \{\omega \in \Omega : Z(\omega) \leq x\} \\ &= \left\{\omega \in \Omega : \begin{cases} X(\omega) \leq x, & \text{if } \omega \in A \\ Y(\omega) \leq x, & \text{if } \omega \in A^c \end{cases}\right\} \\ &= \{\omega \in \Omega \cap A : X(\omega) \leq x\} \cup \{\omega \in \Omega \cap A^c : Y(\omega) \leq x\} \\ &= (\{\omega \in \Omega : X(\omega) \leq x\} \cap A) \cup (\{\omega \in \Omega : Y(\omega) \leq x\} \cap A^c) \in \mathcal{F}, \end{aligned}$$

i.e., Z is a r.v., according to Prop. 2.16.

2. The radial miss distances (in $10^2\,m$) of the landing point of two parachuting sky divers (say A and B) from the center of the target area are known to be r.v. X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{(1+x+y)^3}, & x,y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y two independent r.v.? (2.0)

- **Random vector**
  (X,Y)
  $X =$ radial miss distance of sky diver A (in $10^2\,m$)
  $Y =$ radial miss distance of sky diver B (in $10^2\,m$)

- **Joint p.d.f.**
  $$f_{X,Y}(x,y) = \begin{cases} \frac{2}{(1+x+y)^3}, & x,y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- **Checking whether X and Y are independent r.v.**
  The marginal p.d.f. of X is given by
  $$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \\ &= \begin{cases} \int_0^{+\infty} \frac{2}{(1+x+y)^3}\,dy = -\frac{1}{(1+x+y)^2}\Big|_0^{+\infty} = \frac{1}{(1+x)^2}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$
  We obtain the marginal p.d.f. of Y in a similar fashion:
  $$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx \\ &= \begin{cases} \frac{1}{(1+y)^2}, & y \geq 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$
  Since
  $$f_{X,Y}(x,y) \neq f_X(x) \times f_Y(y),$$
  for some $(x,y) \in \mathbb{R}^2$, Theorem 3.38 (independence criterion for absolutely continuous r.v.) leads to the conclusion that X and Y are NOT INDEPENDENT r.v.

(b) Calculate $P(X > Y)$. (2.0)

- **Requested probability**
  $$\begin{aligned} P(X > Y) &= \int\int_{\{(x,y)\in\mathbb{R}^2 : x>y\}} f_{X,Y}(x,y)\,dy\,dx \\ &= \int_0^{+\infty}\int_0^x \frac{2}{(1+x+y)^3}\,dy\,dx \\ &= \int_0^{+\infty} \left[-\frac{1}{(1+x+y)^2}\Big|_0^x\right] dx \\ &= \int_0^{+\infty} \left[\frac{1}{(1+x)^2} - \frac{1}{(1+2x)^2}\right] dx \\ &= -\frac{1}{(1+x)} + \frac{1}{2(1+2x)}\Big|_0^{+\infty} \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

(c) Derive the p.d.f. of the sum of the radial miss distances. (2.0)

- **New r.v.**
  $Z = X + Y$

- **Range**
  $\mathbb{R}_Z = \mathbb{R}_0^+$

- **P.d.f. of Z**
  According to Theo. 2.123,
  $$\begin{aligned} f_Z(z) &= f_{X+Y}(z) \\ &= \int_{-\infty}^{+\infty} f_{X,Y}(z-y,y)\,dy. \end{aligned}$$
  Moreover, for $z \geq 0$,
  $$\begin{aligned} f_Z(z) &\stackrel{y,z-y\geq 0}{=} \int_0^z f_{X,Y}(z-y,y)\,dy \\ &= \int_0^z \frac{2}{[1+(z-y)+y]^3}\,dy \\ &= \int_0^z \frac{2}{(1+z)^3}\,dy \\ &= \frac{2z}{(1+z)^3}. \end{aligned}$$

Group IV — Independence **4.0 points**

1. Show that if a sample space contains only three points then we are not able to identify two independent events none of which has probability zero or one. (1.5)

- **Events/probabilities**

  $P(A) = a$

  $P(B) = b$

  $P(C) = c$

  Note that these events form a partition of Ω, thus, are disjoint hence dependent.

- **Checking that we are not able to identify two independent events none of which has probability zero or one...**

  Let $D = A \cup B$ and $E = B \cup C$ be two events. D and E are independent if we consider a fixed $a \in (0,1)$ and

$$(b,c) \in (0,1)^2 \quad : \quad \begin{cases} P(A) + P(B) + P(C) = 1 \\ P(D \cap E) = P(D) \times P(E) \end{cases}$$

$$(a,b,c) \in (0,1)^3 \quad : \quad \begin{cases} a+b+c = 1 \\ P(B) = P(A \cup B) \times P(B \cup C) \end{cases}$$

$$\begin{cases} a+b+c = 1 \\ b = (a+b) \times (b+c) \end{cases}$$

$$\begin{cases} c = 1-a-b \\ b = (a+b) \times (b+1-a-b) \end{cases}$$

$$\begin{cases} - \\ b = (a+b) \times (1-a) \end{cases}$$

$$\begin{cases} - \\ b = a(1-a) + b - ab \end{cases}$$

$$\begin{cases} - \\ b = \frac{a(1-a)}{a} \end{cases}$$

$$\begin{cases} b = 1-a \\ c = 0 \end{cases}$$

  Thus: $C = \emptyset$; Ω has only 2 points, A and B; $D = A \cup B = \Omega$ and has probability equal to 1.

  We were indeed unable to identify two independent events none of which has probability (zero or) one when Ω has only 3 points.

2. The daily number of requests that arrive to a server is a r.v. $N \sim \text{Poisson}(\lambda)$. The associated processing times are i.i.d. r.v. X_i, which are independent of N and uniformly distributed in the interval $(0,1)$. (2.5)

   Determine the c.d.f. of the r.v. Y, the longest processing time in a randomly chosen day, having in mind that $Y = 0$ when $N = 0$.

- **R.v.**

  $N =$ daily number of requests

  $N \sim \text{Poisson}(\lambda)$

  $P(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}$, $n \in \mathbb{N}_0$

  $X_i =$ processing time of request i, $i \in \mathbb{N}$

  $X_i \stackrel{i.i.d.}{\sim} X \sim \text{Uniform}(0,1)$, $i \in \mathbb{N}$

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

- **Other r.v.**

  $Y =$ longest processing time in a randomly chosen day

  $Y|\{N = 0\} = 0$

  $Y|\{N = n\} =$ longest processing time given that $\{N = n\}$, $n \in \mathbb{N}$

- **C.d.f. of Y (trivial cases)**

  Since the range of Y is $\mathbb{R}_Y = 0 \cup (0,1)$, for:

  (i) $y < 0$,
  $$F_Y(y) = 0;$$
  (ii) $y = 0$,
  $$F_Y(y) = P(N = 0) = e^{-\lambda};$$
  (iii) $y \geq 1$,
  $$F_Y(y) = 1.$$

- **C.d.f. of Y $(0 < y < 1)$**

  Since $Y|\{N = n\} \stackrel{d}{=} \max_{i=1,\ldots,n} X_i$ because $X_i \perp\!\!\!\perp N$, $i \in \mathbb{N}$, we have

$$\begin{aligned} F_{Y|\{N=n\}}(y) &= P(Y \leq y|\{N = n\}) \\ &= y^n,\ 0 < y < 1, \end{aligned}$$

  and

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(Y = 0) + \sum_{n=1}^{+\infty} P(Y > y|\{N = n\}) \times P(N = n) \\ &= e^{-\lambda} + \sum_{n=1}^{+\infty} y^n \times e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} + e^{-\lambda} \sum_{n=1}^{+\infty} \frac{(\lambda y)^n}{n!} \\ &= e^{-\lambda} + e^{-\lambda}(e^{\lambda y} - 1) \\ &= e^{-\lambda(1-y)},\ 0 < y < 1. \end{aligned}$$