Department of Mathematics, IST — Section of Probability and Statistics

Probability Theory

2nd. Test — 1st. Semester — 2010/11

Duration: 1h30m — 2011/01/10 — 3PM, Room P8

- Please justify your answers.
- This test has two pages and four groups. The total of points: 20.0.

Group V — Independence and Bernoulli/Poisson processes **4.0 points**

1. *A Bernoulli process with parameter p has already been used in the investigation of earths magnetic field reversals,*[1] *with Bernoulli trials separated by 282 ky (i.e. 282 thousand years).* (2.0)

   Prove that, given that the number of geomagnetic reversals in the first 100 *Bernoulli trials is equal to* 4 *(that is,* $\{S_{100}=4\}$*), the joint distribution of* $(T_1,\ldots,T_4)$*, the vector of the number of 282 ky periods until the 1st., 2nd., 3rd. and 4th. geomagnetic reversals, is the same as the distribution of a random sample of* 4 *numbers chosen without replacement from* $\{1,2,\ldots,100\}$.

   - **Bernoulli process**

     $\{X_i,\, i\in\mathbb{N}\} \stackrel{i.i.d.}{\sim} \text{Bernoulli}(p)$

     $$X_i = \begin{cases} 1, & \text{if there is a geomagnetic reversal during the } i^{th} \text{ 282 ky period} \\ 0, & \text{otherwise} \end{cases}$$

   - **R.v.**

     According to Prop. 3.83:

     $$\begin{aligned} S_n &= \text{number of geomagnetic reversals in the last } n \text{ 282 ky periods} \\ &= \sum_{i=1}^n X_i \\ &\sim \text{Binomial}(n,p),\ n\in\mathbb{N}; \\ T_k &= \text{number of 282 ky periods until the } k^{th}\text{geomagnetic reversal} \\ &\sim \text{NegativeBinomial}(k,p),\ k\in\mathbb{N}. \end{aligned}$$

   - **Requested probability**

     For $1\le t_1<t_2<t_3<t_4\le 100$, $P(T_1=t_1,\ldots,T_4=t_4 \mid S_{100}=4)$ is equal to:

     $$P(T_1=t_1,\ldots,T_4=t_4 \mid S_{100}=4) = \frac{P(T_1=t_1,\ldots,T_4=t_4,\ S_{100}=4)}{P(S_{100}=4)},$$

     where

     $$\begin{aligned} P(T_1=t_1,\ldots,T_4=t_4,\ S_{100}=4) &= P\Big[\big(\cap_{i=1}^{t_1-1}\{X_i=0\}\big)\cap X_{t_1}=1, \\ &\quad \big(\cap_{i=t_1+1}^{t_2-1}\{X_i=0\}\big)\cap X_{t_2}=1, \\ &\quad \big(\cap_{i=t_2+1}^{t_3-1}\{X_i=0\}\big)\cap X_{t_3}=1, \\ &\quad \big(\cap_{i=t_3+1}^{t_4-1}\{X_i=0\}\big)\cap X_{t_4}=1, \\ &\quad \big(\cap_{i=t_4+1}^{100}\{X_i=0\}\big)\Big] \\ &= \big[(1-p)^{t_1-1}p\big]\times\big[(1-p)^{t_2-t_1-1}p\big] \\ &\quad \times\big[(1-p)^{t_3-t_2-1}p\big]\times\big[(1-p)^{t_4-t_3-1}p\big] \\ &\quad \times(1-p)^{100-t_4} \\ &= (1-p)^{100-4}\,p^4 \end{aligned}$$

     and

     $$P(S_{100}=4) = \binom{100}{4}\,p^4\,(1-p)^{100-4}.$$

     Thus,

     $$P(T_1=t_1,\ldots,T_4=t_4 \mid S_{100}=4) = \frac{1}{\binom{100}{4}},$$

     which is indeed the distribution of a random sample of 4 numbers chosen without replacement from $\{1,2,\ldots,100\}$.

2. *Suppose that customers arrive to do business at a bank according to a non-homogeneous Poisson process with rate function* $\lambda(t)=20+10\cos[2\pi(t-9.5)]$, $t\ge 9$. (2.0)

   What is the probability that twenty customers arrive between 9:30 and 10:30, and another twenty arrive in the following half hour?

   - **Stochastic process**

     $\{N(t),\, 9\le t\le 17\}\sim NHPP(\lambda(t))$

     $N(t)=$ number of arrivals at the bank until time t

     $\lambda(t)=$ intensity function $=20+10\cos[2\pi(t-9.5)]$, $9\le t\le 17$

   - **Distribution**

     In accordance to Def. 3.125, for $9\le t, t+s\le 17$,

     $$N(t+s)-N(t)\sim \text{Poisson}\left(\int_t^{t+s}\lambda(z)\,dz\right), \tag{1}$$

     and the process $\{N(t),\, 9\le t\le 17\}$ has Independent increments.

   - **Requested probability**

     Taking advantage of the independent increments and of result (1),

     $$\begin{aligned} P[N(10.5)-N(9.5)=20,\ N(11)-N(10.5)=20] &= P[N(10.5)-N(9.5)=20] \\ &\quad \times P[N(11)-N(10.5)=20] \end{aligned}$$

[1] *A geomagnetic reversal is a change in the orientation of Earth's magnetic field such that the positions of magnetic north and magnetic south become interchanged (http://en.wikipedia.org/wiki/Geomagnetic_reversal).*

$$P[N(10.5)-N(9.5)=20,\, N(11)-N(10.5)=20] = \frac{e^{\int_{9.5}^{10.5}\lambda(z)\,dz} \times \left[\int_{9.5}^{10.5}\lambda(z)\,dz\right]^{20}}{20} \times \frac{e^{\int_{10.5}^{11}\lambda(z)\,dz} \times \left[\int_{10.5}^{11}\lambda(z)\,dz\right]^{20}}{20},$$

where:

$$\begin{aligned}\int_{9.5}^{10.5}\lambda(z)\,dz &= \int_{9.5}^{10.5}\{20+10\cos[2\pi(z-9.5)]\}\,dz \\ &= 20z|_{9.5}^{10.5} + 10\times\frac{1}{2\pi}\sin[2\pi(z-9.5)]|_{9.5}^{10.5} \\ &= 20\times(10.5-9.5)+\frac{5}{\pi}[\sin(2\pi)-\sin(0)] \\ &= 20;\end{aligned}$$

$$\begin{aligned}\int_{10.5}^{11}\lambda(z)\,dz &= \int_{10.5}^{11}\{20+10\cos[2\pi(z-9.5)]\}\,dz \\ &= 20z|_{10.5}^{11} + 10\times\frac{1}{2\pi}\sin[2\pi(z-9.5)]|_{10.5}^{11} \\ &= 20\times(11-10.5)+\frac{5}{\pi}[\sin(3\pi)-\sin(2\pi)] \\ &= 10.\end{aligned}$$

Hence,

$$\begin{aligned}P[N(10.5)-N(9.5)=20,\, N(11)-N(10.5)=20] &= e^{-20}\frac{20^{20}}{20!}\times e^{-10}\frac{10^{20}}{20!} \\ &= \frac{e^{-30}\,2^{20}\,10^{40}}{(20!)^2} \\ &\simeq 1.657\times10^{-4}.\end{aligned}$$

Group VI — Independence and expectation **7.0 points**

1. *The lifetimes of n computer systems are assumed to be independent and exponentially distributed with expected value equal to θ. Let $L = X_{n:n}$ be the lifetime of the system that survives the longest.* (3.0)

   Show that $E(L)=\theta\sum_{i=1}^{n}\frac{1}{n-i+1}$.

   Hint: *Write L as a sum of independent r.v., capitalize on the lack of memory property of the exponential distribution and its closure under the minimum operation.*

   - **R.v.**
     X_i = lifetime of computer system i, $i=1,\ldots,n$
     $X_i \overset{i.i.d.}{\sim} X$, $i=1,2$
     $X\sim$ Exponential(θ^{-1})
     $P(X>x)=e^{-\theta^{-1}x}$, $x\geq 0$

   - **New r.v.**
     $X_{n:n}=\max_{i=1,\ldots,n}X_i$ = lifetime of the system that survives the longest

   - **Rewriting $X_{n:n}$**
     $X_{n:n}$ can be rewritten in terms of the times between consecutive failures:

     $$X_{n:n} = X_{1:n}+(X_{2:n}-X_{1:n})+(X_{3:n}-X_{2:n})+\cdots+(X_{n-1:n}-X_{n-2:n})+(X_{n:n}-X_{n-1:n}).$$

     (Schematics!)
     Now, capitalizing on the lack of memory of the exponentially distributed lifetimes, we can add that

     $$X_{n:n} \overset{d}{=} X_{1:n}+X_{1:n-1}+X_{1:n-2}+\cdots+X_{1:2}+X_{1:1},$$

     where $X_{1:n-i+1}\sim$ Exponential$\left(\frac{n-i+1}{\theta}\right)$, $i=1,\ldots,n$, according to Prop. 3.60.[2]

   - **Requested expected value**

     $$\begin{aligned}E(X_{n:n}) &= \sum_{i=1}^{n}E(X_{1:n-i+1}) \\ &= \sum_{i=1}^{n}\frac{\theta}{n-i+1} \\ &= \theta\sum_{i=1}^{n}\frac{1}{n-i+1}.\end{aligned}$$

     QED

2. *Let X be a r.v. described as follows:*
   - *$X=0.3$, with probability 0.2;*
   - *$X=0.7$, with probability 0.3;*
   - *$X\sim$ Uniform$([0.2,0.5]\cup[0.6,0.8])$, with probability 0.5.*

   Find $E(X)$. (2.0)

   - **R.v.**
     $X=0.3$, with probability 0.2
     $X=0.7$, with probability 0.3
     $X\sim$ Uniform$([0.2,0.5]\cup[0.6,0.8])$, with probability 0.5

   - **Defining X as a mixed r.v.**
     Let us consider:

[2] Also note that these summands are independent r.v. This fact is not relevant for the calculation of $E(X_{n:n})$. It would be if we had to obtain $V(X_{n:n})$.

$$
\begin{aligned}
\alpha &= P(X=0.3)+P(X=0.7)\\
&= 0.5;\\
X_d &: P(X_d=x)=\begin{cases}\frac{0.2}{\alpha}, & x=0.3\\ \frac{0.3}{\alpha}, & x=0.7\\ 0, & \text{otherwise;}\end{cases}\\
1-\alpha &= (0.5-0.2)+(0.8-0.6)\\
&= 0.5;\\
X_a &\stackrel{d}{=} \text{Uniform}([0.2,0.5]\cup[0.6,0.8]).
\end{aligned}
$$

Then we can indeed write the d.f. of X as a convex linear combination of the d.f. of X_d and X_a:

$$F_X = \alpha \times F_{X_d} + (1-\alpha)\times F_{X_a}$$

(check it!).

- **Requested expected value**

  According to Cor. 4.75,

$$
\begin{aligned}
E(X) &= \alpha \times E(X_d) + (1-\alpha)\times E(X_a)\\
&= \alpha \times \sum_i x_i \times P(X_d = x_i) + (1-\alpha)\times \int_{-\infty}^{+\infty} x \times f_{X_a}(x)\,dx\\
&= \alpha \times \left(0.3\times\frac{0.2}{\alpha} + 0.7\times\frac{0.3}{\alpha}\right)\\
&\quad (1-\alpha)\times\left[\int_{0.2}^{0.5} x\times\frac{1}{1-\alpha}\,dx + \int_{0.6}^{0.8} x\times\frac{1}{1-\alpha}\,dx\right]\\
&= 0.27 + \left(\left.\frac{x^2}{2}\right|_{0.2}^{0.5} + \left.\frac{x^2}{2}\right|_{0.6}^{0.8}\right)\\
&= 0.27 + 0.245\\
&= 0.515.
\end{aligned}
$$

Group VII — Expectation **5.0 points**

1. *Suppose that the number of insurance claims made in a year (X) is a Poisson r.v. with expected value $\lambda > 0$.*

   (a) *Show that $P(\{X \geq 2\lambda\}) \leq \frac{1}{1+\lambda}$.* (1.0)

   - **R.v.**

     X = number of insurance claims made in a year

     $X \sim \text{Poisson}(\lambda)$, $0 < \lambda < +\infty$

   - **Upper bound to $P(X \geq 2\lambda)$**

     Since $V(X) = \lambda < \infty$, $X \in L^2$ and, thus, we can apply the one-sided Chebyshev inequality to obtain:

$$
\begin{aligned}
P(X\geq 2\lambda) &= P[X-\lambda \geq \sqrt{\lambda}\,\sqrt{\lambda}]\\
&= P[X - E(X) \geq a\,\sqrt{V(X)}]\\
&\leq \frac{1}{1+a^2}\\
&= \frac{1}{1+\left(\sqrt{\lambda}\right)^2}\\
&= \frac{1}{1+\lambda}.
\end{aligned}
$$

   (b) *The minimum upper bound for $P(\{X \geq k\})$ — using Chernoff's inequality — is reached when $t = \ln\left(\frac{k}{\lambda}\right)$, $k > \lambda$. Compare it with the upper bound in part (a), for a few (say 3) values of λ.* (2.0)

   Hint: Use the fact that $E\left(e^{tX}\right) = e^{\lambda(e^t-1)}$.

   - **Minimum Chernoff's upper bound to $P(X \geq 2\lambda)$**

     Using Chernoff's bound in Prop. 4.99, we can state that

$$
\begin{aligned}
P(X\geq k) &\leq \frac{E\left(e^{tX}\right)}{e^{tk}}\\
&= \frac{e^{\lambda(e^t-1)}}{e^{tk}}.
\end{aligned}
$$

     Furthermore, the minimum the upper bound is reached when $t = \ln(k/\lambda)$, $k > \lambda$:

$$
\begin{aligned}
P(X\geq k) &\leq \frac{e^{\lambda[e^{\ln(k/\lambda)}-1]}}{e^{\ln(k/\lambda)\times k}}\\
&= \frac{e^{\lambda\left(\frac{k}{\lambda}-1\right)}}{\left(\frac{k}{\lambda}\right)^k}\\
&= \frac{e^{k-\lambda}}{\left(\frac{k}{\lambda}\right)^k}\\
&\stackrel{k=2\lambda>\lambda}{=} \frac{e^{2\lambda-\lambda}}{\left(\frac{2\lambda}{\lambda}\right)^{2\lambda}}\\
&= \left(\frac{e}{4}\right)^{\lambda}.
\end{aligned}
$$

   - **Comparison of the two upper bounds to $P(X \geq 2\lambda)$**

λ	$\frac{1}{1+\lambda}$	$\frac{e^\lambda}{4^\lambda}$	Which is better?
0.1	0.909091	0.962107	$\frac{1}{1+\lambda}$
1	0.5	0.679570	$\frac{1}{1+\lambda}$
4	0.2	0.213274	$\frac{1}{1+\lambda}$
5	0.166667	0.144935	$\frac{e^\lambda}{4^\lambda}$

2. *Let X_i be the light field being emitted from a laser at time t_i, $i = 1, 2$.* (2.0)

Laser light is said to be temporally coherent if X_1 and X_2 $(0 < t_1 < t_2)$ are dependent r.v. when $t_2 - t_1$ is not too large.

Admit that the joint p.d.f. of X_1 and X_2 is given by

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\,\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{1-\rho^2}\right), \; x_1, x_2 \in I\!R.$$

Obtain the expected value of the light field at time t_2, given that the light field at time t_1 exceeds 0.5 and $\rho = 0.75$.

- **Random vector**
  $\underline{X} = (X_1, X_2)$
  $X_i =$ light field at time t_i, $i = 1, 2$
- **Distribution**
  Since

  $$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\,\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{1-\rho^2}\right), \; x_1, x_2 \in \mathbb{R},$$

  Exercise 4.184 suggests that (X_1, X_2) has a bivariate normal distribution with $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and correlation coefficient $\rho \in (-1, 1)$.
- **Requested expected value**
  According to Def. 4.202 (inverse Mill's ratio),

  $$\begin{aligned} E(X_2 \mid X_1 > x_1) &= \rho\,\frac{\phi(x_1)}{\Phi(-x_1)} \\ &\stackrel{x_1=0.5,\rho=0.75}{=} 0.75 \times \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{0.5^2}{2}}}{1 - \Phi(0.5)} \\ &\stackrel{table}{\simeq} 0.75 \times \frac{0.352065}{1 - 0.6915} \\ &\simeq 0.855912. \end{aligned}$$

Group VIII — Convergence of sequences of r.v. **4.0 points**

1. *Let $X_2, X_3, \dots$ be a sequence of independent r.v. such that*

$$P(\{X_i = x\}) = \begin{cases} \frac{1}{2i\ln(i)}, & x = \pm i \\ 1 - \frac{1}{i\ln(i)}, & x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $\bar{X}_n = \frac{1}{n-1}\sum_{i=2}^n X_i \stackrel{q.m.}{\to} 0$. (1.5)

Hint: *Use the fact that $\sum_{i=2}^n \frac{i}{\ln(i)} \le \frac{n^2}{\ln(n)}$.*

- **Sequence of independent r.v.**
  $\{X_2, X_3, \dots\}$
- **P.f.**

  $$P(X_i = x) = \begin{cases} \frac{1}{2i\ln(i)}, & x = \pm i \\ 1 - \frac{1}{i\ln(i)}, & x = 0 \\ 0, & \text{otherwise.} \end{cases}$$
- **New r.v.**
  $\bar{X}_n = \frac{1}{n-1}\sum_{i=2}^n X_i$, $n = 2, 3, \dots$
- **Limiting distribution**
  $X \stackrel{d}{=} 0$
- **To prove**
  $\bar{X}_n \stackrel{q.m.}{\to} 0$
- **Proof**
  Firstly, we have to check if $X_2, X_3, \dots \in L^2$. Did is indeed true because, for $i = 2, 3, \dots$,

  $$\begin{aligned} E(X_i^2) &= (-i)^2 \times \frac{1}{2i\ln(i)} + 0^2 \times \left[1 - \frac{1}{i\ln(i)}\right] + i^2 \times \frac{1}{2i\ln(i)} \\ &= i^2 \times \frac{1}{i\ln(i)} \\ &= \frac{i}{\ln(i)} \\ &< +\infty. \end{aligned}$$

  Secondly, we have to check if $\bar{X}_2, \bar{X}_3, \dots \in L^2$. This is also true because, for $n = 2, 3, \dots$,

  $$\begin{aligned} E(\bar{X}_n^2) &= E\left[\frac{1}{(n-1)^2}\left(\sum_{i=2}^n X_i\right)^2\right] \\ &= \frac{1}{(n-1)^2} E\left(\sum_{i=2}^n X_i^2 + \sum_{i=2}^n \sum_{j=2, j\neq i}^{2} X_i X_j\right) \\ &\stackrel{X_i \perp\!\!\!\perp X_j, i\neq j}{=} \frac{1}{(n-1)^2}\left[\sum_{i=2}^n E(X_i^2) + \sum_{i=2}^n \sum_{j=2, j\neq i}^{2} E(X_i)\,E(X_j)\right] \\ &\stackrel{E(X_i)=E(X_j)=0}{=} \frac{1}{(n-1)^2} \sum_{i=2}^n \frac{i}{\ln(i)} \\ &\stackrel{Hint}{\le} \frac{1}{(n-1)^2}\,\frac{n^2}{\ln(n)} \\ &< +\infty. \end{aligned}$$

Finally, recall Def. 5.15 and we get

$$\begin{aligned}\lim_{n\to+\infty} E[(\bar{X}_n - X)^2] &\overset{X\overset{d}{=}0}{=} \lim_{n\to+\infty} E[(\bar{X}_n)^2] \\ &= \lim_{n\to+\infty} \frac{1}{(n-1)^2}\sum_{i=2}^{n}\frac{i}{\ln(i)} \\ &\overset{Hint}{\leq} \lim_{n\to+\infty}\frac{1}{(n-1)^2}\frac{n^2}{\ln(n)} \\ &= 0,\end{aligned}$$

i.e., $\bar{X}_n \overset{q.m.}{\to} 0$. QED

2. *Let the interval $[0,1]$ be partitioned into n disjoint sub-intervals with lengths $p_1,\ldots,p_m$ ($p_i > 0$, $\sum_{i=1}^m p_i = 1$). Then the (Shannon) entropy*[3] *of this partition is defined to be $h = -\sum_{i=1}^m p_i \times \ln(p_i)$.* (2.5)

   Consider $\{X_1, X_2, \ldots\}$ a sequence of independent r.v. having the uniform distribution on $[0,1]$, and let $Z_n(i)$ be the number of the $X_1, X_2, \ldots, X_n$ which lie in the ith interval in the partition above.

   Show that $W_n = -\frac{1}{n}\sum_{i=1}^m Z_n(i)\times\ln(p_i) \overset{P}{\to} h$.[4]

   Hint: *Identify the joint distribution of $(Z_n(1),\ldots,Z_n(m))$, and obtain $E(W_n)$ and $V(W_n)$.*

   - **Sequence of i.i.d. r.v.**

     $\{X_1, X_2, \ldots\}$

     $X_i \overset{i.i.d.}{\sim}$ Uniform($[0,1]$)

   - **Related r.v.**

     $Z_n(i)$ = number of the $X_1, X_2, \ldots, X_n$ which lie in the i^{th}interval in the partition of $[0,1]$ with n disjoint sub-intervals with lengths $p_1,\ldots,p_m$ ($p_i > 0$, $\sum_{i=1}^m p_i = 1$), for $n \in \mathbb{N}$ and $i = 1,\ldots,m$

   - **Marginal and joint distributions**

     $Z_n(i) \sim$ Binomial(n, p_i), $i = 1,\ldots,m$

     $(Z_n(1),\ldots,Z_n(m)) \sim$ Multinomial$_{m-1}(n,(p_1,\ldots,p_m))$

   - **Related sequence of r.v.**

     $\{W_1, W_2, \ldots\}$

     $W_n^* = -\frac{1}{n}\sum_{i=1}^m Z_n(i)\times\ln(p_i)$, a weighted mean of the components of the random vector $(Z_n(1),\ldots,Z_n(m))$

[3] *The (Shannon) entropy is a measure of the average information content one is missing when one does not know the value of the r.v. (http://en.wikipedia.org/wiki/Entropy_(information_theory)).*

[4] *Almost sure convergence also holds. Moreover, $W_n^* = -\frac{1}{n}\sum_{i=1}^m Z_n(i)\times\ln(Z_n(i)/n)$ is a consistent estimator of the entropy.*

   - **Expected value and variance of W_n**

$$\begin{aligned}E(W_n) &= -\frac{1}{n}\sum_{i=1}^m E[Z_n(i)]\times\ln(p_i) \\ &= -\frac{1}{n}\sum_{i=1}^m np_i \times \ln(p_i) \\ &\equiv h \\ &= \text{Shannon entropy}\end{aligned}$$

$$\begin{aligned}V(W_n) &\overset{(4.113)}{=} \frac{1}{n^2}\left\{\sum_{i=1}^m [\ln(p_i)]^2\times V[Z_n(i)] + 2\sum_{i=1}^m\sum_{j>i}^m \ln(p_i)\,\ln(p_j)\,cov(Z_n(i),Z_n(j))\right\} \\ &= \frac{1}{n^2}\left\{\sum_{i=1}^m [\ln(p_i)]^2\times np_i(1-p_i) + 2\sum_{i=1}^m\sum_{j>i}^m \ln(p_i)\,\ln(p_j)\,(-n\,p_i\,p_j)\right\} \\ &= \frac{1}{n}\sum_{i=1}^m [\ln(p_i)]^2\times p_i(1-p_i) - \frac{2}{n}\sum_{i=1}^m\sum_{j>i}^m \ln(p_i)\,\ln(p_j)\,p_i\,p_j \\ &= \frac{a}{n} - \frac{2b}{n} \\ &\to 0\end{aligned}$$

     (The convergence follows from the fact that a and b are both finite.)

   - **To prove**

     $W_n \overset{P}{\to} h$

   - **Proof**

     The use of the Chebyshev-Bienaymé inequality leads to

$$\begin{aligned}\lim_{n\to+\infty} P(|W_n - h| > \epsilon) &= \lim_{n\to+\infty} P\left[|W_n - E(W_n)| > \frac{\epsilon}{\sqrt{V(W_n)}}\sqrt{V(W_n)}\right] \\ &\leq \lim_{n\to+\infty}\frac{1}{\left[\frac{\epsilon}{\sqrt{V(W_n)}}\right]^2} \\ &= \frac{1}{\epsilon^2}\lim_{n\to+\infty} V(W_n) \\ &= 0,\end{aligned}$$

     i.e., $W_n \overset{P}{\to} h$. QED