Department of Mathematics, IST — Probability and Statistics Unit

# Probability Theory

1st. Test — 1st. Semester — 2010/11

Duration: 1h30m — 2010/11/13 — 9AM, Room P8

- Please justify your answers.
- This test has two pages and four groups. The total of points is 20.0.

## Group I — Warm up and probability spaces **3.0 points**

*During World War II, random walk was used to model the distance that an escaped prisoner of war would travel in a given time.*

*A military expert assumes that the position of a prisoner at time $n$ ($n = 1, 2, \dots$), $X_n$, is governed by an* **a***symmetric random walk — starting at "0" (the prison camp) and with probability of an northward (resp. southward) "step" (of one km in one time unit) equal to $p$ (resp. $1-p$), where $p \in \left(0, \frac{1}{2}\right)$.*

*Let $A_k$ be the event that the prisoner ever reaches point $k$ ($k = 0, 1, 2, \dots$) north of the prison camp.*

(a) *Argue that $A_k \supseteq A_{k+1}$, $k = 0, 1, 2, \dots$, i.e., $A_k \downarrow$.* (1.0)

- **Process**
  Asymmetric random walk
- **R.v.**
  $Y_n =$ size of the $n^{th}$ step
  $Y_n \stackrel{i.i.d.}{\sim} Y$, $n \in \mathbb{N}$
  $$P(Y = y) = \begin{cases} p, & y = 1 \text{ (step of 1 km north of the prison camp)} \\ 1-p, & y = -1 \text{ (step of 1 km south of the prison camp)} \\ 0, & \text{otherwise} \end{cases}$$
  $X_n = \sum_{i=1}^n Y_i =$ position of the prisoner after $n$ steps ($n \in \mathbb{N}$)
- **Initial condition**
  $X_0 = 0$ (prison camp)
- **Event**
  $A_k =$ prisoner ever reaches point $k$ north of the prison camp, $k \in \mathbb{N}_0$
- **To argue**
  $A_k \downarrow$, i.e., $\{A_k,\, k \in \mathbb{N}_0\}$ is a non-increasing sequence of events
- **Arguing that $A_k \downarrow$**
  Note that:
  (i) $A_0 = \Omega$ because the prisoner starts at the prison camp — $X_0 = 0$ — and therefore $A_0 \supseteq A_1$ is obviously true;



  (ii) exploiting the "continuity" of the RW lead us to conclude that to reach position $k+1$ the prisoner has to pass through position $k$, thus the occurrence of $A_{k+1}$ implies the one of $A_k$, i.e., $A_k \supseteq A_{k+1}$ is true for $k \in \mathbb{N}$.

(b) *Derive $P(A_r)$, for $r = 1, 2, \dots$, by using the fact that $A_k \downarrow$, $P(A_0) = 1$ and $P(A_{k+1}|A_k) = \frac{p}{1-p}$, $k = 0, 1, 2, \dots$.* (2.0)

- **Requested probability**

$$\begin{aligned} P(A_r) &\stackrel{A_k\downarrow}{=} P\left(\cap_{k=0}^r A_k\right) \\ &\stackrel{mult. rule}{=} P(A_0) \times P(A_1 \mid A_0) \times P(A_2 \mid A_0 \cap A_1) \times \dots \\ &\qquad \times P(A_r \mid A_0 \cap A_1 \cap \dots \cap A_{r-1}) \\ &\stackrel{A_k\downarrow}{=} P(A_0) \times P(A_1 \mid A_0) \times P(A_2 \mid A_1) \times \dots \times P(A_r \mid A_{r-1}) \\ &= 1 \times \prod_{k=0}^{r-1} P(A_{k+1}|A_k) \\ &\stackrel{P(A_{k+1}|A_k)=\frac{p}{1-p}}{=} \prod_{k=0}^{r-1} \left(\frac{p}{1-p}\right) \\ &= \left(\frac{p}{1-p}\right)^r, \; r \in \mathbb{N}. \end{aligned}$$

## Group II — Probability spaces and random variables **5.5 points**

1. *Consider the sample space $\Omega = \{A, B, C\}$, and let $\mathcal{F} = \{\emptyset, \{A\}, \{B, C\}, \Omega\}$ and $\mathcal{H} = \{\emptyset, \{A, B\}, \{C\}, \Omega\}$ be two $\sigma-$algebras on $\Omega$.*

   *Show that $\mathcal{F} \cup \mathcal{H}$, the collection of subsets of $\Omega$ lying in either $\mathcal{F}$ or $\mathcal{H}$, is not a $\sigma-$algebra.* (1.5)

   - **Sample space**
     $\Omega = \{A, B, C\}$
   - **$\sigma-$algebras on $\Omega$**
     $\mathcal{F} = \{\emptyset, \{A\}, \{B, C\}, \Omega\}$
     $\mathcal{H} = \{\emptyset, \{A, B\}, \{C\}, \Omega\}$
   - **To prove**
     $\mathcal{F} \cup \mathcal{H}$ is not a $\sigma-$algebra
   - **Proof**
     We ought to mention that, according to Def. 1.38, a minimal set of postulates for a non-empty class of subsets $\mathcal{A}$ of $\Omega$ to be a $\sigma-$algebra on $\Omega$ is:
     (i) $\Omega \in \mathcal{A}$;
     (ii) $A \in \mathcal{A} \;\Rightarrow\; A^c \in \mathcal{A}$;
     (iii) $A_1, A_2, \dots \in \mathcal{A} \;\Rightarrow\; \cup_{i=1}^{+\infty} A_i \in \mathcal{A}$.

Hence, we have to prove that at least one of these 3 postulates are not true for $\mathcal{A} = \mathcal{F} \cup \mathcal{H}$.

Firstly, note that

$$\mathcal{A} = \mathcal{F} \cup \mathcal{H} = \{\emptyset, \{A\}, \{C\}, \{A, B\}, \{B, C\}, \Omega\}.$$

Thus, $\Omega \in \mathcal{A} = \mathcal{F} \cup \mathcal{H}$.

Secondly,

| $D$ | $D^c = \Omega \backslash D$ | $D^c \in \mathcal{A} = \mathcal{F} \cup \mathcal{H}$ ? |
|---|---|---|
| $\emptyset$ | $\Omega$ | TRUE |
| $\{A\}$ | $\{B, C\}$ | TRUE |
| $\{A, B\}$ | $\{C\}$ | TRUE |
| $\{B, C\}$ | $\{A\}$ | TRUE |
| $\{C\}$ | $\{A, B\}$ | TRUE |
| $\Omega$ | $\emptyset$ | TRUE |

Thirdly, the following disjoint events

$$\begin{aligned} D_1 &= \{A\} \\ D_2 &= \{C\} \\ D_3 &= D_4 = \cdots = \emptyset \end{aligned}$$

yield

$$\begin{aligned} \cup_{i=1}^{+\infty} D_i &= \{A, C\} \\ &\notin \mathcal{A} = \mathcal{F} \cup \mathcal{H}. \end{aligned}$$

As a consequence, $\mathcal{A} = \mathcal{F} \cup \mathcal{H}$ is not a $\sigma -$ algebra.

2. *Suppose that $P_1$ and $P_2$ are probability functions on $(\Omega, \mathcal{F})$ and that $\alpha \in [0, 1]$. Prove that the set function* (2.0)

$$P(A) = \alpha \times P_1(A) + (1 - \alpha) \times P_2(A),\ A \in \mathcal{F},$$

*is also a probability function on $(\Omega, \mathcal{F})$.*

- **Setting**

  Consider two p.f. on $(\Omega, \mathcal{F})$:

  $$P_j : \Omega \to [0, 1],\ j = 1, 2.$$

- **To prove**

  $P(A) = \alpha \times P_1(A) + (1 - \alpha) \times P_2(A)$, $A \in \mathcal{F}$, where $\alpha \in [0, 1]$, is also a p.f. on $(\Omega, \mathcal{F})$.

- **Proof**

  According to Def. 1.48, the two p.f. defined of $(\Omega, \mathcal{F})$, $P_j$, $j = 1, 2$, satisfy

  (i) Axiom 1 — $P_j(A) \geq 0,\ \forall A \in \mathcal{F}$

  (ii) Axiom 2 — $P_j(\Omega) = 1$

  (iii) Axiom 3 (countable additivity) — Whenever $A_1, A_2, \ldots$ are (pairwise) disjoint events in $\mathcal{F}$, $P_j\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} P_j(A_i)$,

  for $j = 1, 2$.

  As a result, the set function $P$ verifies:

$$\begin{aligned} P(A) &= \alpha \times P_1(A) + (1 - \alpha) \times P_2(A) \\ &\geq 0, \text{ for } A \in \mathcal{F} \quad \text{(by (i))}; \\ P(\Omega) &= \alpha \times P_1(\Omega) + (1 - \alpha) \times P_2(\Omega) \\ &= \alpha \times 1 + (1 - \alpha) \times 1 \quad \text{(by (ii))} \\ &= 1; \\ P\left(\bigcup_{i=1}^{+\infty} A_i\right) &= \alpha \times P_1\left(\bigcup_{i=1}^{+\infty} A_i\right) + (1 - \alpha) \times P_2\left(\bigcup_{i=1}^{+\infty} A_i\right) \\ &= \alpha \times \sum_{i=1}^{+\infty} P_1(A_i) + (1 - \alpha) \times \sum_{i=1}^{+\infty} P_2(A_i) \quad \text{(by (iii))} \\ &= \sum_{i=1}^{+\infty} [\alpha \times P_1(A_i) + (1 - \alpha) \times P_2(A_i)] \\ &= \sum_{i=1}^{+\infty} P(A_i), \text{ for any } A_1, A_2, \ldots \text{(pairwise) disjoint events in } \mathcal{F}. \end{aligned}$$

  That is, $P$ is indeed a p.f. on $(\Omega, \mathcal{F})$.

3. *A parent particle can split into 0, 1 or 2 particles with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. It disappears after splitting. Beginning with one particle (the progenitor) let us denote by $X_i$ the number of particles in generation $i$ $(i = 0, 1, 2, \ldots)$ and remind the reader that $P(\{X_0 = 1\}) = 1$.* (2.0)

   *Find $P(\{X_2 > 0\})$.*

- **R.v.**

  $X_0 = 1$ (we began with one particle, the progenitor)

  $X_1$ = number of particles in which the progenitor splits into

  $\vdots$

  $X_i$ = number of particles in generation $i$, $i \in \mathbb{N}$

- **P.f. of $X_1$**

$$P(X_1 = x) = \begin{cases} \frac{1}{4}, & x = 0, 2 \\ \frac{1}{2}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Other r.v.**

  $Y_1^{(0)} = X_1$

  $Y_j^{(i)}$ = number of particles in which particle $j$ splits into in generation $i$, $i, j \in \mathbb{N}$

  $\{Y_j^{(i)}, j \in \mathbb{N}\} \stackrel{i.i.d.}{\sim} Y_1^{(0)}$, for all $i \in \mathbb{N}$

- **Obs.**

  $X_1 = Y_1^{(0)}$

  $X_2 = \sum_{j=1}^{X_1} Y_j^{(1)}$

  $\vdots$

  $X_{i+1} = \sum_{j=1}^{X_i} Y_j^{(i)}$, $i \in \mathbb{N}_0$

- **Requested probability**

$$
\begin{aligned}
P(X_2 > 0) &= 1 - P(X_2 = 0) \\
&\stackrel{total\,prob.\,law}{=} 1 - \sum_{i=0}^{2} P(X_2 = 0 \mid X_1 = i) \times P(X_1 = i) \\
&= 1 - [P(X_2 = 0 \mid X_1 = 0) \times P(X_1 = 0) \\
&\quad + P(X_2 = 0 \mid X_1 = 1) \times P(X_1 = 1) \\
&\quad + P(X_2 = 0 \mid X_1 = 2) \times P(X_1 = 2)] \\
&= 1 - \left[1 \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \left(\frac{1}{4}\right)^2 \times \frac{1}{4}\right] \\
&= 1 - \frac{16 + 8 + 1}{64} \\
&= \frac{39}{64}.
\end{aligned}
$$

## Group III — Random variables **5.0 points**

*Consider a cubical fuel tank with capacity equal to 1000 liters. For safety reasons, the tank is kept three-quarters full (750 liters). Moreover, the only tank wall subjected to corrosion is the front one; and if a hole appears in that wall at height $X$ (in meters), all the fuel above the hole leaks out of the tank.*

*Let $Y$ be the amount of fuel (in liters) that will remain in the tank after a hole appears and assume that $X \sim Uniform(0, 1)$.*

(a) *Define $Y$ in terms of $X$ and show that $Y$ is also a r.v.* (2.0)

- **R.v.**

  $X$ = height (in meters) at which the hole appears in the front wall of the fuel tank

- **Distribution**

  $X \sim$ Uniform$(0, 1)$



- **Transform**

$$
\begin{aligned}
Y &= \text{amount of fuel (in liters) that will remain in the tank after leakage} \\
&= \begin{cases} 1000X, & \text{if } 0 \leq X \leq 0.75 \\ 750, & \text{if } 0.75 < X \leq 1 \end{cases} \\
&= \min\{1000X, 750\} \\
&= g(X)
\end{aligned}
$$

- **Important**

  Let:

  - $X$ be a real r.v.;
  - $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable spaces.

  Then, by Def. 2.13, $X : \Omega \to \mathbb{R}$ and is such that

$$
X^{-1}(B) = \{w \in \Omega : X(\omega) \in B\} \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R}), \tag{1}
$$

  namely,

$$
X^{-1}((-\infty, x]) = \{w \in \Omega : X(\omega) \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R},
$$

  according to Prop. 2.16.

- **To prove**

  $Y = g(X) = \min\{1000X, 750\}$ is a Borel measurable function, thus, also a r.v., by Corollary 2.40.

- **Proof**

  Firstly, let us remind the reader that a function $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff

$$
g^{-1}(B) \in \mathcal{B}(I\!R), \forall B \in \mathcal{B}(I\!R),
$$

  according to Def. 2.36, with $n = m = 1$. Furthermore, in order that $g : \mathbb{R} \to \mathbb{R}$ be Borel measurable it suffices that $g^{-1}((-\infty, y]) \in \mathcal{B}(\mathbb{R})$, $\forall y \in I\!R$, according to Remark 2.47 (with $n = 1$).

  Secondly, note that the range of $X$ is

$$
\mathbb{R}_X = [0, 1],
$$

  therefore the range of $Y$ is given by

$$
\begin{aligned}
\mathbb{R}_Y &= g(\mathbb{R}_X) \\
&= [0, 750].
\end{aligned}
$$

  Thirdly, $g^{-1}((-\infty, y])$ equals:

  - for $y < 0$,

$$
\begin{aligned}
\{x \in \mathbb{R} : g(x) \leq y\} &= \emptyset \\
&\in \mathcal{B}(\mathbb{R});
\end{aligned}
$$

- for $0 \leq y \leq 750$,

$$\begin{aligned} \{x \in \mathbb{R} : g(x) \leq y\} &= \{x \in \mathbb{R} : 1000x \leq y\} \\ &= (-\infty, y/1000] \\ &\in \mathcal{B}(\mathbb{R}); \end{aligned}$$

- for $y > 750$,

$$\begin{aligned} \{x \in \mathbb{R} : g(x) \leq y\} &= \Omega \\ &\in \mathcal{B}(\mathbb{R}). \end{aligned}$$

Consequently, we can conclude that $g$ is indeed a Borel measurable function and therefore Y is a r.v., by Corollary 2.40. QED

(b) *Derive and draw the graph of the d.f. of $Y$ and calculate $P(\{Y = 750\})$.* (3.0)

- **D.f. of $X$**

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 0 \end{cases}$$

- **D.f. of $Y$**

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\min\{1000X, 750\} \leq y) \\ &= \begin{cases} P(\emptyset) = 0, & y < 0 \\ P(1000X \leq y) = F_X\left(\frac{y}{1000}\right) = \frac{y}{1000}, & 0 \leq y < 750 \\ P(\Omega) = 1, & y \geq 750 \end{cases} \end{aligned}$$

- **Graph of $F_Y(y)$**

(It has a discontinuity at $y = 750$, etc.)

- **Requested probability**

Since $F_Y(y)$ has a discontinuity point at $y = 750$,

$$\begin{aligned} P(Y = 750) &= F_Y(750) - F_Y(750^-) \\ &= 1 - 0.75 \\ &= 0.25. \end{aligned}$$

This probability coincides with $P(0.75 < X \leq 1) = P(\text{hole above the mark of } 0.75\text{m})$.



## Group IV — Independence and random variables **6.5 points**

1. *If $A, B, C$ are independent events, show directly that $A \cup B$ is independent of $C$.* (1.5)

- **Setting**

$A, B, C$ are independent events, i.e.,

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ P(A \cap C) &= P(A) \times P(C) \\ P(B \cap C) &= P(B) \times P(C) \\ P(A \cap B \cap C) &= P(A) \times P(B) \times P(C) \end{aligned}$$

- **To prove**

$A \cup B \perp\!\!\!\perp C$, that is, $P[(A \cup B) \cap C] = P(A \cup B) \times P(C)$.

- **Proof**

Invoking the independence, namely the *factorisations above*, we get:

$$\begin{aligned} P[(A \cup B) \cap C] &= P[(A \cap C) \cup (B \cap C)] \\ &= P[A \cap C) + P(B \cap C) - P[(A \cap C) \cap (B \cap C)] \\ &= P[A \cap C) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A) \times P(C) + P(B) \times P(C) - P(A) \times P(B) \times P(C) \\ &= [P(A) + P(B) - P(A) \times P(B)] \times P(C) \\ &= P(A \cup B) \times P(C). \end{aligned}$$

QED

2. *The maximum flying time, $V$ (in hours), of a small aircraft is defined by $V = 30XY$, where: the r.v. $X$ represents the amount of fuel (in tons) in the beginning of the flight; and the r.v. $Y$ represents the reciprocal of the fuel consumption rate (in hours per ton of fuel).*

*Admit that $X$ and $Y$ are independent r.v. such that $X \sim$ Uniform$(0,1)$ and $Y \sim$ Beta$(2,1)$.*

(a) *Obtain the survival function of $V$.* (3.0)

- **R.v.**

$X =$ amount of fuel (in tons) in the beginning of the flight
$Y =$ reciprocal of the fuel consumption rate (in hours per ton of fuel)

- **Distributions**

$X \sim \text{Uniform}(0,1)$
$\perp\!\!\!\perp$
$Y \sim \text{Beta}(2,1)$

- **Transformation**

$V =$ maximum flying time (in hours)

- **P.d.f. de $X$**

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- **P.d.f. de $Y$**

$$f_Y(y) = \begin{cases} \frac{\Gamma(2+1)}{\Gamma(2)\,\Gamma(1)}\, y^{2-1}\,(1-y)^{1-1} = 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Survival function of $Y$**

$$\begin{aligned} \bar{F}_Y(y) &= 1 - F_Y(y) \\ &= \begin{cases} 1, & y \le 0 \\ 1 - \int_0^y 2t\,dt = 1 - y^2, & 0 < y < 1 \\ 0, & y \ge 1. \end{cases} \end{aligned}$$

- **Range of $V$**

$I\!R_V = [0, 30]$

- **Survival function of $V$**

For $0 \le v \le 30$,

$$\begin{aligned} \bar{F}_V(v) &= P(V > v) \\ &= P(30\,X\,Y > v) \\ &= P\left(Y > \frac{v}{30X}\right) \\ &\overset{X \perp\!\!\!\perp Y}{=} \int\int_{\{(x,y)\in[0,1]^2:\, y > \frac{v}{30x}\}} f_X(x) \times f_Y(y)\,dy\,dx \quad \text{(schematics!)} \\ &= \int_{\frac{v}{30}}^1 f_X(x) \times \left[\int_{\frac{v}{30x}}^1 f_Y(y)\,dy\right] dx \\ &= \int_{\frac{v}{30}}^1 f_X(x) \times \bar{F}_Y\left(\frac{v}{30x}\right) dx \\ &= \int_{\frac{v}{30}}^1 1 \times \left[1 - \left(\frac{v}{30x}\right)^2\right] dx \\ &= \int_{\frac{v}{30}}^1 dx - \left(\frac{v}{30}\right)^2 \times \int_{\frac{v}{30}}^1 x^{-2}\,dx \\ &= \left(1 - \frac{v}{30}\right) + \left(\frac{v}{30}\right)^2 \times \left.\frac{1}{x}\right|_{\frac{v}{30}}^1 \\ &= \left(1 - \frac{v}{30}\right)^2. \end{aligned}$$

(b) *Describe a method to generate (pseudo-)random numbers from the distribution of $V$.* (2.0)

- **Quantile function**

Let $u \in (0, 1)$ then the quantile function is given by:

$$\begin{aligned} F_V(v) &= u \\ 1 - \bar{F}_V(v) &= u \\ 1 - \left(1 - \frac{v}{30}\right)^2 &= u \\ F_W^{-1}(u) &= v \\ &= 30(1 - \sqrt{1-u}). \end{aligned}$$

- **Quantile transformation**

By Prop. 2.140, if

$$U \sim \text{Uniform}(0,1)$$

then $F_V^{-1}(U) \overset{d}{=} V$, i.e.,

$$30(1 - \sqrt{1-U}) \overset{d}{=} V.$$

Consequently, if we want to generate (pseudo-)random numbers from the distribution of $V$ then we have to:

(1) generate $u$ from a Uniform$(0,1)$ distribution;

(2) assign $v = 30(1 - \sqrt{1-u})$.