Department of Mathematics, IST — Section of Probability and Statistics

Probability Theory

1st. Exam/ 2nd. Test — 1st. Semester — 2009/10

Duration: 3h/ 1h30m — 2010/01/19 — 5PM, Room P8

- This **exam** has three pages and **eight groups**. The corresponding number of points equals half the ones that are marked, thus, the total number of points of an exam is 20.0.
- If you plan to do the **2nd. test** you should only **solve groups V, VI, VII, VIII**. The total number of points is 20.0.
- Please justify your answers.

Group I — Warm up **2.0 points**

Random walk models are often found in physics, from particle motion to a simple description of a polymer. (2.0)

A physicist assumes that the position of a particle at time n, X_n, is governed by an **a***symmetric random walk — starting at 0 and with probability of an upward (resp. downward) step equal to p (resp. $1-p$), where $p \in (0,1)\backslash\{\frac{1}{2}\}$.*

Derive $P(X_{2n} = 0 \mid X_{2n-2} = 0)$, for $n = 2, 3, \ldots$

- **Process**

  Asymmetric random walk

- **R.v.**

  Y_n = size of the n^{th} step

  $Y_n \stackrel{i.i.d.}{\sim} Y$, $n \in \mathbb{N}$

  $$P(Y = y) = \begin{cases} p, & y = 1 \\ 1-p, & y = -1 \\ 0, & \text{otherwise} \end{cases}$$

  $X_n = \sum_{i=1}^{n} Y_i$ = position of the particle at time n ($n \in \mathbb{N}$)

- **Initial condition**

  $X_0 = 0$

- **Requested probability**

  $$\begin{aligned} P(X_{2n} = 0 \mid X_{2n-2} = 0) &= \frac{P(X_{2n} = 0, X_{2n-2} = 0)}{P(X_{2n-2} = 0)} \\ &= \frac{P(X_{2n} = 0, X_{2n-1} = -1, X_{2n-2} = 0)}{P(X_{2n-2} = 0)} \\ &\quad + \frac{P(X_{2n} = 0, X_{2n-1} = 1, X_{2n-2} = 0)}{P(X_{2n-2} = 0)} \end{aligned}$$

  where, by the multiplication rule and the independence between X_{2n-2} and Y_{2n-1} and between X_{2n-1} and Y_{2n},

  $$\begin{aligned} P(X_{2n} = 0, X_{2n-1} = -1, X_{2n-2} = 0) &= P(X_{2n-1} = -1, X_{2n-2} = 0) \\ &\quad \times P(X_{2n} = 0 \mid X_{2n-1} = -1, X_{2n-2} = 0) \\ &= P(X_{2n-1} = -1, X_{2n-2} = 0) \\ &\quad \times P(X_{2n} = 0 \mid X_{2n-1} = -1) \\ &= P(X_{2n-1} = -1, X_{2n-2} = 0) \\ &\quad \times \frac{P(X_{2n} = 0, X_{2n-1} = -1)}{P(X_{2n-1} = -1)} \\ &= P(Y_{2n-1} = -1, X_{2n-2} = 0) \\ &\quad \times \frac{P(Y_{2n} = 1, X_{2n-1} = -1)}{P(X_{2n-1} = -1)} \\ &= P(Y_{2n-1} = -1) \times P(X_{2n-2} = 0) \\ &\quad \times \frac{P(Y_{2n} = 1) \times P(X_{2n-1} = -1)}{P(X_{2n-1} = -1)} \\ &= P(X_{2n-2} = 0) \times (1-p) \times p \end{aligned}$$

  $$\begin{aligned} \frac{P(X_{2n} = 0, X_{2n-1} = -1, X_{2n-2} = 0)}{P(X_{2n-2} = 0)} &= \frac{P(X_{2n-2} = 0) \times (1-p) \times p}{P(X_{2n-2} = 0)} \\ &= (1-p) \times p \end{aligned}$$

  and, similarly,

  $$\frac{P(X_{2n} = 0, X_{2n-1} = 1, X_{2n-2} = 0)}{P(X_{2n-2} = 0)} = p \times (1-p).$$

  Thus,

  $$P(X_{2n} = 0 \mid X_{2n-2} = 0) = 2 \times p \times (1-p).$$

- **Obs.**

  We could have argued that the event $\{X_{2n-2} = 0 | X_{2n} = 0\}$ is equivalent to the following event: "the particle moves up and down" (with probability $p \times (1-p)$) "or down and up" (with probability $(1-p) \times p$).

Group II — Probability spaces **7.0 points**

1. *Let $B_1, B_2, \ldots$ be events such that $\sum_{k=1}^{n} P(B_k) > n - 1$.* (2.0)

   Prove that $P\left(\cap_{k=1}^{n} B_k\right) > 0$.

   - **To prove**

     $B_1, B_2, \cdots : \sum_{k=1}^{n} P(B_k) > n - 1 \quad \Rightarrow \quad P\left(\cap_{k=1}^{n} B_k\right) > 0$

- **Proof**

  By applying De Morgans law, we obtain

$$\begin{aligned} P\left(\cap_{k=1}^n B_k\right) &= P\left[\left(\cup_{k=1}^n B_k^c\right)^c\right] \\ &= 1 - P\left(\cup_{k=1}^n B_k^c\right), \end{aligned}$$

  where, according to the Bonferroni inequality in (1.48),

$$\begin{aligned} P\left(\cup_{k=1}^n B_k^c\right) &\leq \sum_{k=1}^n P\left(B_k^c\right) \\ &= \sum_{k=1}^n [1 - P(B_k)] \\ &= n - \sum_{k=1}^n P(B_k) \\ &\overset{\sum_{k=1}^n P(B_k) > n-1}{<} n - (n-1) \\ &< 1. \end{aligned}$$

  Consequently,

$$\begin{aligned} P\left(\cap_{k=1}^n B_k\right) &= 1 - P\left(\cup_{k=1}^n B_k^c\right) \\ &> 0. \end{aligned}$$

QED

2. *Let:* (3.0)

   - $(\Omega, \mathcal{F})$ *be a measurable space;*
   - P *be a nonnegative set function on* $\mathcal{F}$ *with* $P(\Omega) = 1$.

   Prove that if P *is* $\sigma-$*additive then whenever* $A_n \uparrow A$ *in* $\mathcal{F}$, $P(A_n) \uparrow P(A)$.

   - **Setting**

     $(\Omega, \mathcal{F})$ is a measurable space

     $P : \mathcal{F} \to [0, 1], \quad P(\Omega) = 1$

   - **To prove**

     P is $\sigma-$additive, $A_n \uparrow A$ in $\mathcal{F}$ $\Rightarrow$ $P(A_n) \uparrow P(A)$ (monotone continuity!).

   - **Proof**

     On one hand, recall that $A_n \uparrow A$ means that:

     (i) A_n is an increasing sequence of events (of $\mathcal{F}$), i.e., $\emptyset = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$

     (ii) $A = \lim A_n = \limsup A_n = \liminf A_n$.

     One the other hand, (i) implies that

$$P(A_n) \uparrow$$

     by the monotonicity of the p.f. stated in Prop. 1.54, equation (1.41). Moreover, as a result of (i) and (ii)

$$A = \cup_{i=1}^{+\infty} A_i,$$

     by Prop. 1.27. Furthermore,

$$\begin{aligned} P(A) &= P\left(\cup_{i=1}^{+\infty} A_i\right) \\ &= P\left[\cup_{i=1}^{+\infty} (A_i \backslash A_{i-1})\right] \qquad \text{(by the "disjointification" technique)} \\ &= \sum_{i=1}^{+\infty} P(A_i \backslash A_{i-1}) \qquad \text{(because of the } \sigma-\text{additivity of } P\text{)} \\ &= \sum_{i=1}^{+\infty} [P(A_i) - P(A_i \cap A_{i-1})] \\ &= \sum_{i=1}^{+\infty} [P(A_i) - P(A_{i-1})] \qquad \text{(because } A_{i-1} \subseteq A_i\text{)} \\ &= \lim_{n \to +\infty} \sum_{i=1}^{n} [P(A_i) - P(A_{i-1})] \\ &= \lim_{n \to +\infty} P(A_n) \qquad \text{(this is a telescopic sum with } P(A_0) = P(\emptyset) = 0\text{).} \end{aligned}$$

     Thus, $P(A_n) \uparrow$ and $\lim_{n \to +\infty} P(A_n) = P(A)$, that is,

$$P(A_n) \uparrow P(A).$$

QED

3. *Five missiles are fired against an aircraft carrier in the ocean. It takes at least two direct hits to sink the carrier. All five missiles are on the correct trajectory but must get through the "point defense" guns of the carrier. Moreover, it is known that the "point defense" guns can destroy a missile with probability 0.9.* (2.0)

   What is the probability that the carrier will still be afloat after the encounter?

   - **R.v.**

     X = number of direct hits, out of 5 missiles sent

   - **Distribution**

     $X \sim \text{Binomial}(n, p)$

   - **Parameters**

$$\begin{aligned} n &= 5 \text{ missiles sent} \\ p &= P(\text{direct hit}) \\ &= P(\text{missile gets through "point defense" guns of the carrier}) \\ &= 1 - P(\text{missile destroyed by "point defense" guns of the carrier}) \\ &= 1 - 0.9 \\ &= 0.1 \end{aligned}$$

- **Requested probability**

$$\begin{aligned} P(\text{carrier still afloat after the encounter}) &= 1 - P(\text{at least 2 direct hits, out of 5 missiles sent}) \\ &= 1 - P(X \geq 2) \\ &= P(X \leq 1) \\ &= F_{Binomial(5,0.1)}(1) \\ &\stackrel{table}{=} 0.9185. \end{aligned}$$

Group III — Random variables **7.0 points**

It is well known from electric-curcuit theory that the current I flowing through a resistor $r > 0$ dissipates an amount of power W given by $W = rI^2$.

(a) *Let I be a r.v.* (1.5)

Show that W is also a r.v.

- **R.v.**
  $I =$ current flowing through a resistor
- **Important**
  Let:
  - I be a real r.v.;
  - $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two measurable spaces.

  Then, by Def. 2.13, $I : \Omega \to \mathbb{R}$ and is such that

  $$I^{-1}(B) = \{\omega \in \Omega : I(\omega) \in B\} \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R}),$$

  in particular

  $$I^{-1}((-\infty, x]) = \{\omega \in \Omega : I(\omega) \leq x\} \in \mathcal{F}, \forall x \in \mathbb{R},$$

  according to Prop. 2.16.
- **To prove**
  $W = g(I) = rI^2$ is a Borel measurable function, therefore also a r.v., by Corollary 2.40.
- **Proof**
  Firsty, let us remind the reader that a function $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff

  $$g^{-1}(B) \in \mathcal{B}(I\!R), \forall B \in \mathcal{B}(I\!R),$$

  according to Def. 2.36, with $n = m = 1$. Furthermore, in order that $g : \mathbb{R} \to \mathbb{R}$ be Borel measurable it suffices that $g^{-1}((-\infty, w]) \in \mathcal{B}(\mathbb{R}), \forall w \in I\!R$, according to Remark 2.47 (with $n = 1$).
  Secondly, $g^{-1}((-\infty, w])$ equals:
  - for $w \leq 0$,

  $$\begin{aligned} \{i \in \mathbb{R} : g(i) \leq w\} &= \emptyset \\ &\in \mathcal{B}(\mathbb{R}); \end{aligned}$$

  - for $w > 0$,

  $$\begin{aligned} \{i \in \mathbb{R} : g(i) \leq w\} &= \{i \in \mathbb{R} : r\, i^2 \leq w\} \\ &= \left[-\sqrt{w/r}, \sqrt{w/r}\right] \\ &\in \mathcal{B}(\mathbb{R}) \end{aligned}$$

  As a result, g is indeed a Borel measurable function and therefore W is a r.v., by Corollary 2.40. QED

(b) *Derive the d.f. of W when $I \sim Uniform(-\pi, \pi)$.* (3.0)

- **D.f. of I**

  $$\begin{aligned} F_I(x) &= P(I \leq x) \\ &= \begin{cases} 0, & x < -\pi \\ \frac{x+\pi}{2\pi}, & -\pi \leq x \leq \pi \\ 1, & x > \pi \end{cases} \end{aligned}$$

- **Transformation**
  $W = rI^2$
- **Range of W**
  $\mathbb{R}_W = g(\mathbb{R}_I) = [0, r\pi^2]$
- **D.f. of W**
  For

  $$\begin{aligned} F_W(w) &= P(W \leq w) \\ &= P\left(-\sqrt{w/r} \leq I \leq \sqrt{w/r}\right) \\ &= F_I\left(\sqrt{w/r}\right) - F_I\left(-\sqrt{w/r}\right) \\ &= \begin{cases} 0, & w < 0 \\ \frac{\sqrt{w/r}+\pi}{2\pi} - \frac{\sqrt{w/r}+\pi}{2\pi} = \frac{\sqrt{w/r}}{\pi}, & 0 \leq w \leq r\pi^2 \\ 1, & w > r\pi^2. \end{cases} \end{aligned}$$

(c) *Describe a method to generate (pseudo-)random numbers from the distribution of W.* (2.5)

- **Quantile function**
  Let $u \in (0, 1)$ then the quantile function is given by:

  $$\begin{aligned} F_W(w) &= u \\ \frac{\sqrt{w/r}}{\pi} &= u \\ F_W^{-1}(u) &= w \\ &= r(u\pi)^2. \end{aligned}$$

- **Quantile transformation**

By Prop. 2.140, if

$$U \sim \text{Uniform}(0,1)$$

then $F_W^{-1}(U) \stackrel{d}{=} W$, i.e.,

$$r(U\pi)^2 \stackrel{d}{=} W.$$

Consequently, if we want to generate (pseudo-)random numbers from the distribution of W then we have to:

(1) generate u from a Uniform$(0,1)$ distribution;

(2) assign $w = r(u\pi)^2$.

Group IV — Random variables and expectation **4.0 points**

A linear amplifier is an electronic circuit whose output Y is proportional to its input X. Consider a linear amplifier with cutoff, whose output is given by

$$Y = g(X) = \begin{cases} 0, & |X| \geq 1 \\ 2X, & -1 < X < 1, \end{cases}$$

where $X \sim$ Normal$(0.5, 1)$.

(a) *Derive the d.f. of Y. What sort of r.v. is Y?* (2.5)

- **R.v.**
  $X =$ input of the amplifier
  $X \sim \text{Normal}(0.5,1)$
- **Transformation**
  $$\begin{aligned} Y &= \text{output using the linear amplifier with cutoff} \\ &= g(X) \\ &= \begin{cases} 0, & |X| \geq 1 \\ 2X, & -1 < X < 1, \end{cases} \end{aligned}$$
- **Range of Y**
  $\mathbb{R}_Y = g(\mathbb{R}_X) = (-2,2)$ (schematics!)
- **D.f. of Y**
  Let $F_Y(y) = P(Y \leq y)$, $y \in \mathbb{R}$, be the d.f. of Y. Then:
  - for $y \leq -2$,
    $$F_Y(y) = 0;$$
  - for $-2 < y < 0$,
    $$\begin{aligned} F_Y(y) &= P(-2 < 2X \leq y) \\ &= P\left(-1 < X \leq \frac{y}{2}\right) \\ &= \Phi\left(\frac{\frac{y}{2} - 0.5}{\sqrt{1}}\right) - \Phi\left(\frac{-1-0.5}{\sqrt{1}}\right) \\ &= \Phi\left(\frac{y-1}{2}\right) - \Phi(-1.5); \end{aligned}$$
  - for $y = 0$,
    $$\begin{aligned} F_Y(y) &= P(-2 < 2X \leq 0) + P(Y = 0) \\ &= P(-1 < X \leq 0) + P(|X| \geq 1) \\ &= [\Phi(-0.5) - \Phi(-1.5)] + [1 - P(-1 < X < 1)] \\ &= [\Phi(-0.5) - \Phi(-1.5)] + \{1 - [\Phi(0.5) - \Phi(-1.5)]\} \\ &= 2 \times [1 - \Phi(0.5)]; \end{aligned}$$
  - for $0 < y < 2$,
    $$\begin{aligned} F_Y(y) &= P(Y \leq 0) + P(0 < Y \leq y) \\ &= 2 \times [1 - \Phi(0.5)] + P\left(0 < X \leq \frac{y}{2}\right) \\ &= 2 \times [1 - \Phi(0.5)] + \left[\Phi\left(\frac{y-1}{2}\right) - \Phi(-0.5)\right] \\ &= 2 \times [1 - \Phi(0.5)] + \left\{\Phi\left(\frac{y-1}{2}\right) - [1 - \Phi(0.5)]\right\} \\ &= 1 - \Phi(0.5) + \Phi\left(\frac{y-1}{2}\right); \end{aligned}$$
  - for $y \geq 2$,
    $$F_Y(y) = 1.$$
  - **What sort of r.v. is Y**
    $F_Y(y)$ is a continuous except at point $y = 0$. In fact,
    $$\begin{aligned} F_Y(0^-) &= \Phi(-0.5) - \Phi(-1.5) \\ &\neq 2 \times [1 - \Phi(0.5)] \\ &= F_Y(0). \end{aligned}$$
    Hence, Y is NOT an ABSOLUTELY CONTINUOUS R.V. Moreover, note that
    $$\begin{aligned} P(Y = 0) &= F_Y(0) - F_Y(0^-) \\ &= 2 \times [1 - \Phi(0.5)] - [\Phi(-0.5) - \Phi(-1.5)] \\ &= 1 - \Phi(0.5) + \Phi(-1.5) \\ &< 1. \end{aligned}$$
    Thus, there is no countable set C such that $P(Y \in C) = 1$ and, as a result, Y is NOT A DISCRETE R.V.
    Consequently, Y is a MIXED R.V.[1]

[1] We are indeed able to identify a discrete d.f., F_{Y_d}, an absolutely continuous d.f., F_{Y_a}, and $\alpha \in (0,1)$ such that $F_Y = \alpha \times F_{Y_d} + (1-\alpha) \times F_{Y_a}$ (see (4.39) and (c)).

(b) *Obtain a simplified expression for* $E(Y)$. (1.5)

- **Defining Y as a mixed r.v.**

Since the d.f. of Y has just one discontinuity point at $y = 0$, let us consider:

$$\begin{aligned}
\alpha &= P(Y=0) \\
&= 1-\Phi(0.5)+\Phi(-1.5); \\
Y_d &\stackrel{d}{=} 0; \\
F_{Y_d}(y) &= \begin{cases} 0, & y<0 \\ 1, & y \geq 0; \end{cases} \\
1-\alpha &= \Phi(0.5)-\Phi(-1.5); \\
Y_a &\stackrel{d}{=} (2X \mid -1 < X < 1); \\
F_{Y_a}(y) &= P(2X \leq y \mid -1 < X < 1) \\
&= \begin{cases} 0, & y \leq -2 \\ \frac{P\left(-1<X\leq \frac{y}{2}\right)}{P(-1<X<1)} = \frac{\Phi\left(\frac{y-1}{2}\right)-\Phi(-1.5)}{1-\alpha}, & -2<y<2 \\ 1, & y \geq 2. \end{cases}
\end{aligned}$$

Then

$$F_Y = \alpha \times F_d + (1-\alpha) \times F_a$$

(check it!).

- **Expected value of Y (simplified expression)**

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{+\infty} y\, dF_Y(y) \\
&\stackrel{Cor. 4.75}{=} \alpha \times \sum_i y_i \times P(Y_d = y_i) + (1-\alpha) \times \int_{-\infty}^{+\infty} y \times f_{Y_a}(y)\, dy \\
&= \alpha \times 0 + (1-\alpha) \int_{-\infty}^{+\infty} y \times \frac{d\, F_{Y_a}(y)}{dy}\, dy \\
&= \int_{-2}^{+2} y \times \frac{1}{2}\phi\left(\frac{y-1}{2}\right) dy \\
&= \int_{-2}^{+2} \frac{y}{2} \times \frac{1}{\sqrt{2\pi}} \exp\left[-\left(\frac{y-1}{2}\right)^2\right] dy.
\end{aligned}$$

Group V — Independence **4.0 points**

Men and women enter a supermarket according to two independent Poisson processes having respective rates two and four per minute.

(a) *What is the probability that the number of arrivals (men and women) exceeds ten in the first 20 minutes?* (1.5)

- **Two INDEPENDENT stochastic processes**

$\{N_1(t),\, t \geq 0\} \sim PP(\lambda_1 = 2)$

$N_1(t)$ = number of men who entered the supermarket by time t

$\{N_2(t),\, t \geq 0\} \sim PP(\lambda_2 = 4)$

$N_2(t)$ = number of women who entered the supermarket by time t

- **Merged process**

According to Prop. 3.118,

$$\{N(t) = N_1(t) + N_2(t),\, t \geq 0\} \sim PP(\lambda = \lambda_1 + \lambda_2 = 6).$$

Moreover,

$$\begin{aligned}
N(t) &= \text{number of men and women who entered the supermarket by time } t \\
&\sim \text{Poisson}(6t).
\end{aligned}$$

- **Requested probability**

Using the normal approximation to the Poisson distribution, we obtain:

$$\begin{aligned}
P[N(20) > 10] &= 1 - F_{Poisson(6\times 20)}(10) \\
&\simeq 1 - \Phi\left(\frac{10 - 6\times 20}{\sqrt{6 \times 20}}\right) \\
&\simeq 1 - \Phi(-10.04) \\
&\simeq 1.
\end{aligned}$$

(b) *Compute the probability that the first male customer arrives after the first two female customers.* (2.5)

- **Relevant r.v.**

$S_1^{(1)}$ = time of arrival of the 1st. man to the supermarket

$S_2^{(2)}$ = time of arrival of the 2nd. woman to the supermarket

According to Prop. 3.107,

$$\begin{aligned}
S_1^{(1)} &\sim \text{Exponential}(\lambda_1) \\
S_2^{(2)} &\sim \text{Gamma}(2, \lambda_2).
\end{aligned}$$

- **Requested probability**

Capitalizing on the total probability law and then on the fact that $S_1^{(1)}$ and $S_2^{(2)}$ are independent r.v., we get:

$$\begin{aligned}
P\left[S_1^{(1)} > S_2^{(2)}\right] &= \int_0^{+\infty} P\left[S_1^{(1)} > S_2^{(2)} \mid S_2^{(2)} = x\right] f_{S_2^{(2)}}(x)\, dx \\
&= \int_0^{+\infty} [1 - F_{Exponential(\lambda_1)}(x)]\, f_{Gamma(2,\lambda_2)}(x)\, dx
\end{aligned}$$

$$
\begin{aligned}
P\left[S_1^{(1)} > S_2^{(2)}\right] &= \int_0^{+\infty} e^{-\lambda_1 x} \times \frac{\lambda_2^2}{\Gamma(2)}\, x^{2-1} e^{-\lambda_2 x}\, dx \\
&= \frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2} \int_0^{+\infty} \frac{(\lambda_1+\lambda_2)^2}{\Gamma(2)}\, x^{2-1} e^{-(\lambda_1+\lambda_2)x}\, dx \\
&= \frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2} \int_0^{+\infty} f_{Gamma(2,\lambda_1+\lambda_2)}(x)\, dx \\
&= \frac{4^2}{(2+4)^2} \\
&= \frac{4}{9}.
\end{aligned}
$$

Group VI — Independence and expectation **7.0 points**

1. *A machine is only capable of supplying electrical power if 2 out of 3 components are functioning. Admit that the time to failure (in months) of these components are i.i.d. r.v. with common survival function $S(x) = e^{-x^{1.5}}$, $x \geq 0$.*[2]

   Derive a simplified expression for the expected value of the time to failure of such a machine. (3.5)

   - **R.v.**

     $X_i =$ time to failure of component i, $i = 1, 2, 3$

     $X_i \overset{i.i.d.}{\sim} X$, $i = 1, 2, 3$

     $S(x) = P(X > x) = e^{-x^{1.5}}$, $x \geq 0$

   - **New r.v.**

     According to Example 3.71,

$$
\begin{aligned}
T &= \text{time to failure of a } 2-\text{out-of}-3 \text{ system} \\
&= X_{(n-k+1)} \\
&= X_{(3-2+1)} \\
&= X_{(2)}.
\end{aligned}
$$

   - **Survival function of T**

     Capitalizing once again on Example 3.71, we get

$$
\begin{aligned}
S_T(x) &= P(X_{(n-k+1)} > x) \\
&= F_{Binomial(n,1-S(x))}(n-k) \\
&= F_{Binomial(3,1-e^{-x^{1.5}})}(3-2) \\
&= \left(e^{-x^{1.5}}\right)^3 + 3\left(1-e^{-x^{1.5}}\right)\left(e^{-x^{1.5}}\right)^2 \\
&= 3e^{-2x^{1.5}} - 2e^{-3x^{1.5}}.
\end{aligned}
$$

[2] *Note that $X \sim$ Weibull(α, β), where α and β are the scale and shape parameters (respectively), iff* $S_X(x) = \exp\left[-(x/\alpha)^\beta\right]$, $x \geq 0$.

   - **Expected value of T**

     Since T is a non-negative r.v., we can apply Cor. 4.69; moreover, taking into account the information in the footnote, we get

$$
\begin{aligned}
E(T) &= \int_0^{+\infty} S_T(x)\, dx \\
&= \int_0^{+\infty} \left(3e^{-2x^{1.5}} - 2e^{-3x^{1.5}}\right) dx \\
&= 3\int_0^{+\infty} S_{Weibull(\alpha=2^{-1/1.5},\,\beta=1.5)}(x)\, dx \\
&\quad -2\int_0^{+\infty} S_{Weibull(\alpha=3^{-1/1.5},\,\beta=1.5)}(x)\, dx \\
&= 3 \times E[\text{Weibull}(\alpha = 2^{-1/1.5},\, \beta = 1.5)] \\
&\quad -2 \times E[\text{Weibull}(\alpha = 3^{-1/1.5},\, \beta = 1.5)] \\
&= \left(3 \times 2^{-1/1.5} - 2 \times 3^{-1/1.5}\right) \times \Gamma\left(1 + \frac{1}{1.5}\right) \\
[&\simeq 0.838092].
\end{aligned}
$$

2. *Suppose that the number of telephone calls made in a day is a Poisson r.v. with mean 1000.*

   (a) *Find a bound for the probability that more than 1142 calls are made in a day using Markov's inequality.* (1.5)

   - **R.v.**

     $X =$ number of calls in a day

     $X \sim \text{Poisson}(\lambda = 1000)$

   - **Bound to $P(X > 1142)$**

     Since $X \geq 0$ and $E(|X|) = E(X) = V(X) = \lambda$, we can add that $X \in L^1$. Consequently, we can apply Markov's inequality and conclude that

$$
\begin{aligned}
P(X > 1142) &= P(|X| \geq 1143) \\
&\leq \frac{E(|X|)}{1143} \\
&= \frac{1000}{1143} \\
&\simeq 0.874891.
\end{aligned}
$$

   (b) *Try to improve this bound using a more convenient inequality.* (2.0)

   - **Another bound to $P(X > 1142)$**

     Since $V(X) = 1000 < \infty$, $X \in L^2$ and therefore we can apply the one-sided Chebyshev inequality to get:

$$
\begin{aligned}
P(X > 1142) &= P[X - E(X) \geq 1143 - E(X)] \\
&= P\left[X - E(X) \geq \frac{143}{\sqrt{V(X)}}\sqrt{V(X)}\right]
\end{aligned}
$$

$$\begin{aligned} P(X > 1142) &\leq \frac{1}{1 + \left(\frac{143}{\sqrt{1000}}\right)^2} \\ &= \frac{1000}{1000 + 143^2} \\ &\simeq 0.046622. \end{aligned}$$

- **Comment**
  This last upper bound is much smaller than the previous one, therefore we have considerably improved the upper bound to $P(X > 1142)$.[3]

Group VII — Expectation **7.0 points**

1. *Let $\{Y_k,\, k \in \mathbb{N}\}$ be a collection of non negative r.v. such that $\sum_{k=1}^{+\infty} Y_k(\omega) < +\infty,\, \forall \omega$.* (3.0)

   Prove that

$$E\left(\sum_{k=1}^{+\infty} Y_k\right) = \sum_{k=1}^{+\infty} E(Y_k).$$

- **Setting**
  $Y_k,\, k \in \mathbb{N}$
  $Y_k \geq 0$
  $\sum_{k=1}^{+\infty} Y_k(\omega) < \infty,\, \forall \omega$ (i.e., the series is convergent!)
- **To prove**
  $E\left(\sum_{k=1}^{+\infty} Y_k\right) = \sum_{k=1}^{+\infty} E(Y_k)$
- **Proof**
  Let $X_n = \sum_{k=1}^{n} Y_k$. Then $\{X_n,\, n \in \mathbb{N}\}$ is an increasing sequence of non-negative r.v. Furthermore, since $\sum_{k=1}^{+\infty} Y_k(\omega) < \infty,\, \forall \omega$, we can conclude that

$$X_n \uparrow X = \sum_{k=1}^{+\infty} Y_k$$

  and, by the monotone convergence theorem (Th. 4.29), that

$$\begin{aligned} E(X_n) &\uparrow E(X) \\ E\left(\sum_{k=1}^{n} Y_k\right) = \sum_{k=1}^{n} E(Y_k) &\uparrow E\left(\sum_{k=1}^{+\infty} Y_k\right) \\ \lim_{n\to+\infty} E\left(\sum_{k=1}^{n} Y_k\right) = \lim_{n\to+\infty} \sum_{k=1}^{n} E(Y_k) &= E\left(\sum_{k=1}^{+\infty} Y_k\right) \\ \sum_{k=1}^{+\infty} E(Y_k) &= E\left(\sum_{k=1}^{+\infty} Y_k\right). \end{aligned}$$

  QED

[3] Please note that, according to *Mathematica* $P(X > 1142) \simeq 5.16208 \times 10^{-6}$, which is indeed smaller than and much closer to 0.046622 than the previous upper bound 0.874891.

- **Obs.**
  This result, a consequence of the monotone convergence theorem, allows us to interchange expectation and limit signs.

2. *Let X_i be the light field being emitted from a laser at time t_i, $i = 1, 2$.*

   Laser light is said to be temporally coherent if X_1 and X_2 $(0 < t_1 < t_2)$ are dependent r.v. when $t_2 - t_1$ is not too large.

   Admit that the joint p.d.f. of X_1 and X_2 is given by

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\,\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{1-\rho^2}\right),\; x_1, x_2 \in I\!R.$$

   (a) *Prove that, given $\{X_1 = x_1\}$, the light field at time t_2 is more likely to take on positive values if $\rho x_1 > 0$.* (1.5)

- **Random vector**
  (X_1, X_2)
  $X_i =$ light field at time t_i, $i = 1, 2$
- **Distribution**
  Since

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\,\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{1-\rho^2}\right),\; x_1, x_2 \in \mathbb{R},$$

  Exercise 4.184 suggests that (X_1, X_2) has a bivariate normal distribution with $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and correlation coefficient $\rho \in (-1, 1)$.
- **New r.v.**
  $(X_2 \mid X_1 = x_1)$
- **Distribution**
  Adapting result (4.147) of Th. 4.60 leads to

$$(X_2 \mid X_1 = x_1) \sim \text{Normal}\left(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x_1 - \mu_1) = \rho\, x_1,\; \sigma_2^2(1-\rho^2) = 1 - \rho^2\right).$$

- **Proof**
  Given $\{X_1 = x_1\}$, the light field at time t_2 is more likely to take on positive values if

$$\begin{aligned} P(X_2 > 0 \mid X_1 = x_1) &> \frac{1}{2} \\ 1 - \Phi\left(\frac{0 - \rho\, x_1}{\sqrt{1-\rho^2}}\right) &> \frac{1}{2} \\ \Phi\left(\frac{\rho\, x_1}{\sqrt{1-\rho^2}}\right) &> \frac{1}{2} \\ \frac{\rho\, x_1}{\sqrt{1-\rho^2}} &> \Phi^{-1}\left(\frac{1}{2}\right) \\ \rho\, x_1 &> 0. \end{aligned}$$

  QED

(b) *Obtain a lower bound for $E(Y^2)$, where $Y \stackrel{d}{=} X_2|\{X_1 = x_1\}$.* (2.5)
Compare it with the true value of $E(Y^2)$. Comment.

- **R.v.**
  $Y \stackrel{d}{=} (X_2 \mid X_1 = x_1) \sim \text{Normal}\,(\rho\, x_1,\, 1-\rho^2)$
- **Transformation**
  $g(Y) = Y^2$
- **True value of** $E[g(y)] = E(Y^2)$

$$\begin{aligned} E[g(Y)] &= E(Y^2) \\ &= V(Y) + E^2(Y) \\ &= (1-\rho^2) + (\rho x_1)^2 \end{aligned}$$

- **Lower bound to** $E[g(y)] = E(Y^2)$
  Since g is a convex function of Y and $Y, g(Y) \in L^1$, we can make use of the Jensen's inequality (Th. 4.116) and add that

$$\begin{aligned} E[g(Y)] &\geq g[E(Y)] \\ &= g(\rho x_1) \\ &= (\rho x_1)^2 \\ &= \underline{E(Y^2)}. \end{aligned}$$

- **Comment**
  If we admit that $\rho \in (-1, 1)$ then

$$\begin{aligned} E(Y^2) - \underline{E(Y^2)} &= (1-\rho^2) + (\rho x_1)^2 - (\rho x_1)^2 \\ &= 1-\rho^2 \\ &\in (0,1). \end{aligned}$$

  Unsurprisingly, $\underline{E(Y^2)}$ underestimates $E(Y^2)$. In addition, the larger $|\rho|$ is, the closer the lower bound is to the true value of $E(Y^2)$; this result makes sense because the larger the absolute value of the correlation, the more informative is the observation of X_1 when it comes to the prediction of X_2.

Group VIII — Sequences of random variables **2.0 points**

Let $\{X_2, X_3, \dots\}$ be a sequence of independent r.v. such that (2.0)

$$F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2} - \frac{1}{n}, & 0 \leq x < 1 \\ 1, & x \geq 1, \end{cases}$$

i.e. $X_n \sim$ Bernoulli $\left(\frac{1}{2} + \frac{1}{n}\right)$, $n = 2, 3, \dots$

Prove that $X_n \stackrel{d}{\to} X$, where $X \sim$ Bernoulli $\left(\frac{1}{2}\right)$,[4] but $X_n \stackrel{P}{\not\to} X$. Comment.

[4] *Assume that X_n and X are independent r.v. for every n.*

- **Sequence of independent r.v.**
  $\{X_2, X_3, \dots\}$

$$F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2} - \frac{1}{n}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

  $X_n \stackrel{indep}{\sim}$ Bernoulli $\left(\frac{1}{2} + \frac{1}{n}\right)$, $n = 2, 3, \dots$
- **Limiting distribution**
  $X \sim \text{Bernoulli}(\frac{1}{2})$
- **Checking if** $X_n \stackrel{d}{\to} X$
  Note that

$$\begin{aligned} \lim_{n\to+\infty} F_{X_n}(x) &= \begin{cases} 0, & x < 0 \\ \lim_{n\to+\infty}\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \\ &\equiv F_{Bernoulli(\frac{1}{2})}(x) \\ &= F_X(x). \end{aligned}$$

  Thus, we can state that $\lim_{n\to+\infty} F_{X_n}(x) = F_X(x)$, for all x at which F_X is continuous, i.e.,

$$X_n \stackrel{d}{\to} X,$$

  in accordance to Def. 5.23. QED
- **Proving that** $X_n \stackrel{P}{\not\to} X$
  We draw the attention of the reader to the fact that X_n and X are independent r.v. and that $|X_n - X|$ takes the following values: 0, in case $X_n = X = 0$ and $X_n = X = 1$; 1, if $X_n = 0$, $X = 1$ and $X_n = 1$, $X = 0$. Therefore, for $\epsilon > 0$,

$$\begin{aligned} P(|X_n - X| > \epsilon) &= \begin{cases} P(|X_n - X| = 1) & \\ = P(X_n = 0, X = 1) + P(X_n = 1, X = 0) & \\ = \left(\frac{1}{2} - \frac{1}{n}\right) \times \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{n}\right) \times \frac{1}{2} & \\ = \frac{1}{2}, & 0 < \epsilon < 1 \\ 0, & \epsilon \geq 1 \end{cases} \\ &\not\to 0 \end{aligned}$$

  (recall that $|X_n - X|$ takes values 0 and 1, thus, it cannot be larger than a fixed quantity $\epsilon \geq 1$). That is, $X_n \stackrel{P}{\not\to} X$. QED
- **Comment**
  We have just shown that, although

$$X_n \stackrel{P}{\to} X \Rightarrow X_n \stackrel{d}{\to} X,$$

  for all sequences of r.v. (Prop. 5.59),

$$X_n \stackrel{d}{\to} X \not\Rightarrow X_n \stackrel{P}{\to} X,$$

  for our particular sequence of r.v.