# Chapter 10

# Time and frequency responses

We already know that we can use the Laplace transform (and its inverse) to find out the output of any transfer function for any particular input. In this chapter we study several usual particular cases. This allows us to find approximate responses in many cases, and to characterise with simplicity more complex responses. It also paves the way to the important concept of frequency responses.

## 10.1 Time responses: steps and impulses as inputs

The following inputs are routinely used to test systems:

- The **impulse**: *Impulse*

$$u(t) = \delta(t) \tag{10.1}$$
$$\mathscr{L}[u(t)] = 1 \tag{10.2}$$

- The **step**, with amplitude $d$: *Step*

$$u(t) = d\,H(t) \tag{10.3}$$
$$\mathscr{L}[u(t)] = \frac{d}{s} \tag{10.4}$$

- In particular, the **unit step**, with amplitude 1: *Unit step*

$$u(t) = H(t) \tag{10.5}$$
$$\mathscr{L}[u(t)] = \frac{1}{s} \tag{10.6}$$

- The **ramp**, with slope $d$: *Ramp*

$$u(t) = d\,t \tag{10.7}$$
$$\mathscr{L}[u(t)] = \frac{d}{s^2} \tag{10.8}$$

- In particular, the **unit ramp**, with slope 1:

*Unit ramp*

$$u(t) = t \tag{10.9}$$

$$\mathscr{L}\left[u(t)\right] = \frac{1}{s^2} \tag{10.10}$$

*Parabola*

- The **parabola**, with second derivative $2d$:

$$u(t) = d\,t^2 \tag{10.11}$$

$$\mathscr{L}\left[u(t)\right] = \frac{2d}{s^3} \tag{10.12}$$

*Parabola*

- In particular, the **unit parabola**, with second derivative 2:

$$u(t) = t^2 \tag{10.13}$$

$$\mathscr{L}\left[u(t)\right] = \frac{2}{s^3} \tag{10.14}$$

You can either find the Laplace transforms above in Table 2.1, or calculate them yourself.

**Remark 10.1.** Notice that:

- the unit step is the integral of the impulse: $\int_0^t \delta(t)\,\mathrm{d}t = H(t)$;
- the unit ramp is the integral of the unit step: $\int_0^t H(t)\,\mathrm{d}t = t$;
- the unit parabola is not the integral of the unit ramp: $\int_0^t t\,\mathrm{d}t = \frac{1}{2}t^2 \neq t^2$. □

*Properties of $\delta(t)$*

*$\delta(t)$ is not a function*

**Remark 10.2.** Remember that while the Heaviside function $H(t)$ is a function, and so are $t$ and $t^2$, the Dirac delta $\delta(t)$ is not. It is a generalised function, and the limit of the following family of functions:

$$f(t,\epsilon) = \begin{cases} \frac{1}{\epsilon}, & \text{if } 0 \leq t \leq \epsilon \\ 0, & \text{if } t < 0 \vee t > \epsilon \end{cases} \tag{10.15}$$

$$\delta(t) = \lim_{\epsilon \to 0^+} f(t,\epsilon) \tag{10.16}$$

Since

$$\int_{-\infty}^{+\infty} f(t,\epsilon)\,\mathrm{d}t = \int_0^{\epsilon} f(t,\epsilon)\,\mathrm{d}t = \int_0^{\epsilon} \frac{1}{\epsilon}\,\mathrm{d}t = 1,\ \forall \epsilon \in \mathbb{R}^+ \tag{10.17}$$

*Its integral in $\mathbb{R}$ is 1*

we have also

$$\int_{-\infty}^{+\infty} \delta(t)\,\mathrm{d}t = 1 \tag{10.18}$$

Furthermore, for a continuous function $g(t)$,

$$
\begin{aligned}
f(t,\epsilon)\min_{0\le t\le\epsilon} g(t) \le f(t,\epsilon)g(t) \le f(t,\epsilon)\max_{0\le t\le\epsilon} g(t) \\
\Rightarrow \int_0^\epsilon f(t,\epsilon)\min_{0\le t\le\epsilon} g(t)\,\mathrm{d}t \le \int_0^\epsilon f(t,\epsilon)g(t)\,\mathrm{d}t \le \int_0^\epsilon f(t,\epsilon)\max_{0\le t\le\epsilon} g(t)\,\mathrm{d}t \\
\Leftrightarrow \min_{0\le t\le\epsilon} g(t)\int_0^\epsilon f(t,\epsilon)\,\mathrm{d}t \le \int_0^\epsilon f(t,\epsilon)g(t)\,\mathrm{d}t \le \max_{0\le t\le\epsilon} g(t)\int_0^\epsilon f(t,\epsilon)\,\mathrm{d}t \\
\Rightarrow \min_{0\le t\le\epsilon} g(t) \le \int_0^\epsilon f(t,\epsilon)g(t)\,\mathrm{d}t \le \max_{0\le t\le\epsilon} g(t) \qquad (10.19)
\end{aligned}
$$

where we used (10.17). Making $\epsilon \to 0^+$, we get

$$g(0) \le \int_0^\epsilon f(t,\epsilon)g(t)\,\mathrm{d}t \le g(0) \Leftrightarrow \int_0^\epsilon f(t,\epsilon)g(t)\,\mathrm{d}t = g(0) \qquad (10.20)$$

A consequence of this is that

$$\mathscr{L}[\delta(t)] = \int_0^{+\infty} \delta(t)e^{-st}\,\mathrm{d}t = e^{-s0} = 1 \quad \square \qquad (10.21)$$

The reasons why (10.1)–(10.13) are routinely used as inputs to test systems are:

- They are simple to create.
- Calculations are simple, given their Laplace transforms.
- They can be used to model many real inputs exactly, and even more as approximations.

**Example 10.1.** The following situations can be modelled as steps:

- A metal workpiece is taken from an oven and quenched in oil at a lower temperature.
- A sluice gate is suddenly opened, letting water into an irrigation canal.
- A switch is closed and a tension is thereby applied to the motor that rotates the joint of a welding robot.
- A finished part is dropped onto a conveyer belt.
- A car advancing at constant speed descends a sidewalk onto the street pavement. □

**Example 10.2.** The following situations can be modelled as ramps:

- A deep space probe moves out of the solar system at constant speed along a straight line in an inertial system of coordinates, due to inertia, far from the gravitational influence of any close celestial body.
- A high-speed train moves from one station to another at cruiser speed.
- A welding robot creates a welding joint at constant speed, to ensure a uniform thickness.

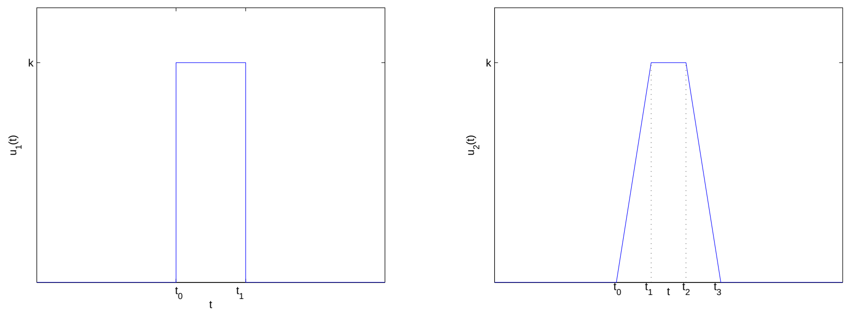



Figure 10.1: Two functions that can be approximated by an impulse if $t_0 \approx t_1$ (left) or $t_0 \approx t_3$ (right).

Notice that, save for the first example, the ramp is limited in time: sooner or later, the train and the welding robot will have to stop. In fact, unlimited ramps are seldom found. □

**Remark 10.3.** The impulse is in fact impossible to create: there are no physical quantities applied during no time at all, with an infinite intensity. However, the impulse is a good approximation of inputs that have a very short duration. Figure 10.1 shows two inputs in that situation: a sequence of two steps

$$\begin{aligned} u_1(t) &= k\,H(t-t_0) - k\,H(t-t_1) \\ &= \begin{cases} k, & \text{if } t_0 \le t \le t_1 \\ 0, & \text{if } t < t_0 \vee t > t_1 \end{cases} \approx k(t_1 - t_0)\delta(t) \end{aligned} \tag{10.22}$$

and, even more realistically, a sequence of two ramps, approximated by

$$\begin{aligned} \delta(t) \int_0^{+\infty} u_2(t)\,\mathrm{d}t &= \delta(t) \left[ \frac{1}{2} k(t_1 - t_0) + k(t_2 - t_1) + \frac{1}{2} k(t_3 - t_2) \right] \\ &= \delta(t) \frac{k}{2} (t_3 + t_2 - t_1 - t_0) \end{aligned} \tag{10.23}$$

Of course, any input with a form such as that of Figure 10.1 can be approximated by an impulse (multiplied by the integral over time of the input). □

**Remark 10.4.** Unit steps are almost exclusively used because amplitude 1 makes calculations easier. Since we are assuming linearity, if the amplitude of the step is $d$ instead of 1, the output will be that for the unit step, multiplied by $d$. The same can be said for unit ramps and unit parabolas. When steps (or ramps, or parabolas) are applied experimentally, amplitude 1 may be too big or too small, and a different one will have to be used instead. □

**Example 10.3.** Suppose you want to test a car's suspension, when the wheel climbs or descends a step. Obviously nobody with a sound mind would apply a 1 m step for this purpose (see Figure 10.3). A 10 cm step would for instance be far more reasonable. Of course, if our model is linear, we can apply a unit step, knowing well that the result will be nonsense, and then simply scale down the result. □

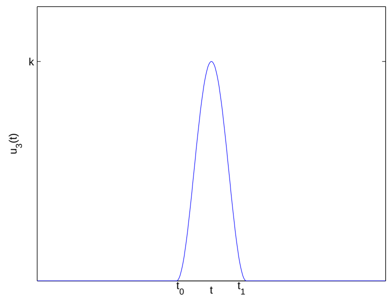



Figure 10.2: General form of a function that can be approximated by an impulse if $t_0 \approx t_1$.

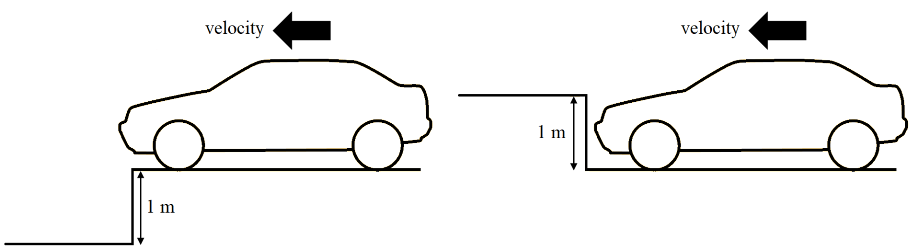



Figure 10.3: Would you test a car's suspension like this? (Source: Wikimedia, modified)

Figure 10.4: The Rasteirinho mobile robot, without the laptop computer with which it is controlled.

**Example 10.4.** The Rasteirinho (see Figure 10.4) is a mobile robot, of which about a dozen units are used at IST in laboratory classes of different courses. It is controlled by a laptop computer, fixed with velcro. Its maximum speed depends on the particular unit; in most, it is around 80 cm/s. Consequently, it is useless to try to make its position follow a unit ramp, which would correspond to a 1 m/s velocity. Once more, we could simulate its behaviour with a linear model for a unit ramp and then scale the output down. □

**Example 10.5.** In the WECs of Figures 3.2 and 3.3, the air inside the device is compressed by the waves. A change of air pressure of 1 Pa is ludicrously small; it is useless even to try to measure it. But if our model of the WEC is linear we can simulate how much energy it produces when a unit step is applied in the air pressure and then scale the result up to a more reasonable value of the pressure variation. □

In what follows we will concentrate on the impulse and unit step responses, and mention responses to unit ramps and steps with amplitudes which are not 1 whenever appropriate.

*Impulse response of a system*

**Theorem 10.1.** The impulse response of a transfer function has a Laplace transform which is the transfer function itself.

*Proof.* Since $G(s) = \frac{Y(s)}{U(s)}$, where $G(s)$ is a transfer function, $Y(s)$ is the Laplace transform of the output, and $U(s)$ is the Laplace transform of the input, and since the Laplace transform of an impulse is 1, the result is immediate. □

**Remark 10.5.** This allows defining a system's transfer function as the Laplace transform of its output when the input is an impulse. This definition is an alternative to Definition 4.1 found in many textbooks. □

**Corollary 10.1.** The output of a transfer function $G(s)$ for any input $u(t)$ is equal to the convolution of the input with the transfer function's impulse response $g(t)$:

$$y(t) = g(t) * u(t) = \int_0^t g(t-\tau)u(\tau)\,\mathrm{d}\tau \tag{10.24}$$

*Proof.* This is an immediate result of Theorem 10.1 and of (2.78). □

**Remark 10.6.** It is usually easier to calculate the Laplace transform of the input $U(s)$ to find the Laplace of the output as $Y(s) = G(s)U(s)$ and then finally the output as $y(t) = \mathscr{L}^{-1}[G(s)U(s)]$, than to calculate the output directly as $y(t) = g(t) * u(t)$. □

The following MATLAB functions are useful to find time responses:

- `step` plots a system's response to a unit step (and can return the values plotted in vectors);
- `impulse` does the same for an impulse input;
- `lsim`, already studied in Section 4.2, can be used for any input.

Just like `lsim`, both `step` and `impulse` use numerical methods to find the responses, rather than analytical computations.

**Example 10.6.** The impulse, unit step and unit ramp responses of a plant are shown in Figure 10.5 and obtained as follows:

*MATLAB's command `impulse`*

*MATLAB's command `step`*

```
>> s = tf('s'); G = 1/(s+1);
>> figure, impulse(G), figure, step(G)
>> t = 0 : 0.01 : 6; figure, plot(t, lsim(G, t, t))
>> xlabel('t [s]'), ylabel('output')
>> title('response to a unit ramp')
```

The time range is chosen automatically by `step` and `impulse`. □

**Example 10.7.** The response of the transfer function from Example 10.6 to a step with amplitude 10 during 20 s can be found in two different manners, both providing, of course, the same result:

```
>> [stepresp, timevector] = step(G, 20);
>> t = 0 : 0.01 : 20;
>> figure, plot(t, lsim(G, 10*ones(size(t)), t), timevector, 10*stepresp)
>> xlabel('t [s]'), ylabel('output'), title('Step response')
```

There is, in fact, a slight difference in the two plots shown in Figure 10.6, because function `step` chooses the sampling time automatically, and it is different from the one explicitly fed to `lsim`. □

## 10.2 Steady-state response and transient response

The impulse, unit step, and unit ramp responses of

$$G(s) = \frac{1}{s+1} \tag{10.25}$$

from Example 10.6, shown in Figure 10.5 as they are numerically calculated by Matlab, can be found analytically as follows:

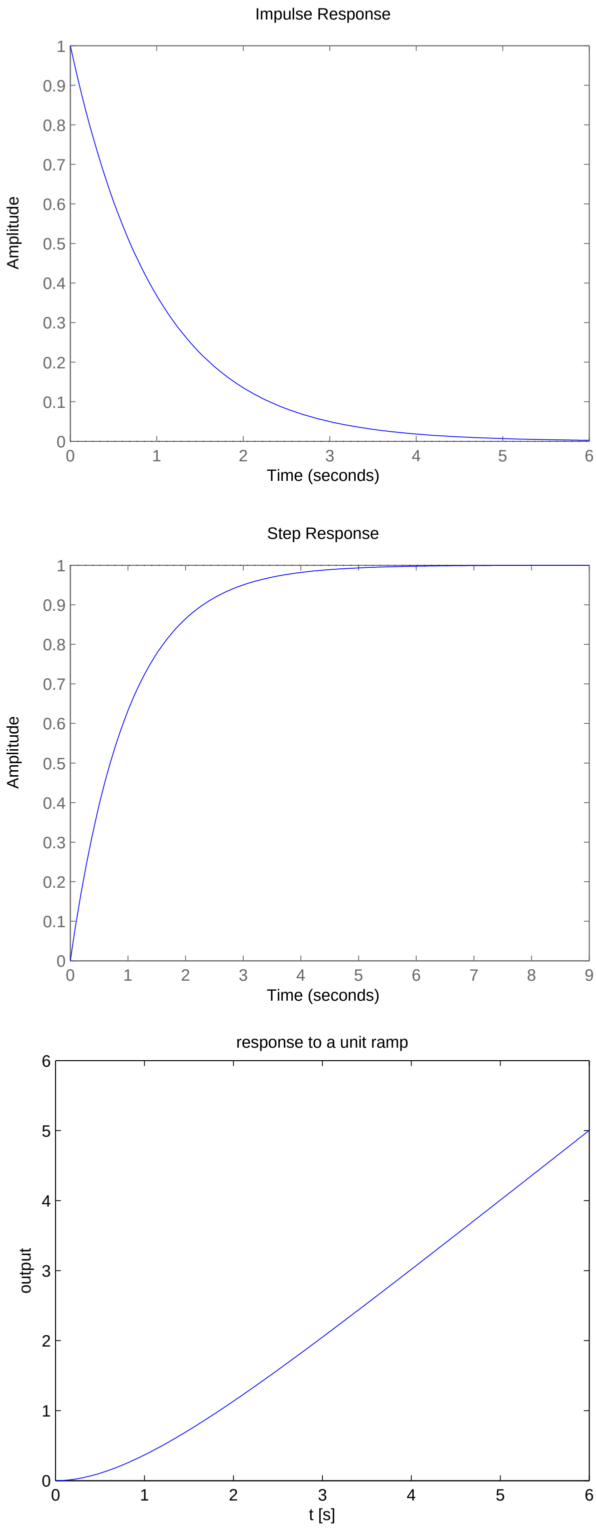



Figure 10.5: Impulse, unit step and unit ramp responses of $G(s) = \frac{1}{s+1}$, from Example 10.6.

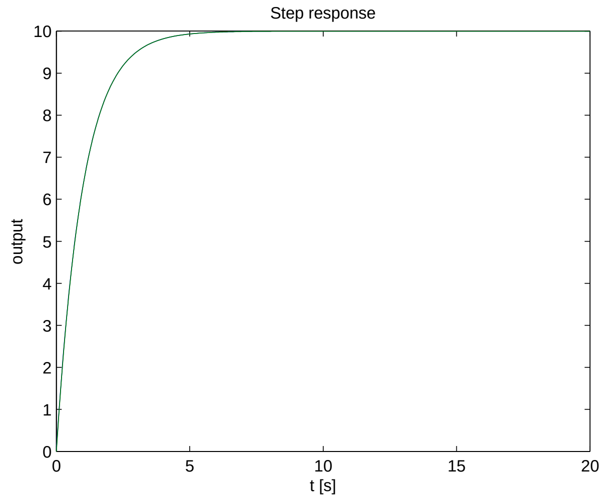



Figure 10.6: Response of $G(s) = \frac{1}{s+1}$ for a step with amplitude 10, from Example 10.7.

- Impulse response:

$$y_i(t) = \mathscr{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \tag{10.26}$$

- Unit step response:

$$y_s(t) = \mathscr{L}^{-1}\left[\frac{1}{s+1}\,\frac{1}{s}\right] = 1 - e^{-t} \tag{10.27}$$

- Unit ramp response:

$$y_r(t) = \mathscr{L}^{-1}\left[\frac{1}{s+1}\,\frac{1}{s^2}\right] = \mathscr{L}^{-1}\left[-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right] = t - 1 + e^{-t} \tag{10.28}$$

In each of them we can separate the terms that tend to zero as the time increases from those that do not. The first make up what we call the **transient response**. The latter make up what we call the **steady-state response**. *Transient* *Steady-state*

$$y_i(t) = \underbrace{0}_{\text{steady-state}} + \underbrace{e^{-t}}_{\text{transient}} \tag{10.29}$$

$$y_s(t) = \underbrace{1}_{\text{steady-state}} - \underbrace{e^{-t}}_{\text{transient}} \tag{10.30}$$

$$y_r(t) = \underbrace{t-1}_{\text{steady-state}} + \underbrace{e^{-t}}_{\text{transient}} \tag{10.31}$$

In other words, a time response $y(t)$ can be separated into two parts, the transient response $y_t(t)$ and the steady-state response $y_{ss}(t)$, such that

$$y(t) = y_t(t) + y_{ss}(t) \tag{10.32}$$

$$\lim_{t\to+\infty} y_t(t) = 0 \tag{10.33}$$

$$\lim_{t\to+\infty} y_{ss}(t) \neq 0 \tag{10.34}$$

We also call transient to the period of time in which the response is dominated by the transient response, and steady-state to the period of time in which the transient response is neglectable and the response can be assumed equal to the steady-state response. Whether a transient response can or cannot be neglected depends on how precise our knowledge of the response has to be. Below in Sections 10.5 and 10.6 we will see usual criteria for this.

The steady-state response can be:

- zero, as the impulse response of (10.25), shown in Figure 10.5;
- a non-null constant, as the unit step response of (10.25), shown in Figure 10.5;
- an oscillation with constant amplitude, as the step response of $\frac{1}{(s^2+1)(s+1)}$, shown in Figure 10.7;
- infinity, with the output increasing or decreasing monotonously, as the unit step response of (10.25), shown in Figure 10.5;
- infinity, with the output oscillating with increasing amplitude, as the impulse response of $\frac{s}{(s^2+1)^2}$, shown in Figure 10.7.

What the steady-state response is depends on what the system is and on what its input is.

**Remark 10.7.** Most systems never reach infinity. The probe of Example 10.2 can move away to outer space, but temperatures do no rise to infinite values (before that the heat source is exhausted, or something will burn), robots reach the end of their workspace, high electrical currents will activate a circuit breaker, etc.; in other words, for big values of the variables involved, the linear model of the system usually ceases in one way or another to be valid. □

Over the next sections we will learn several ways to calculate steady-state responses without having to find an explicit expression for the output, and then calculating its limit. When the steady-state response is constant or infinity, it can be found from the final value theorem (Theorem 2.4), i.e. applying (2.74).

*Final value theorem*

**Example 10.8.** The steady-states of the impulse, step and ramp responses (10.26)–(10.28) are as follows:

$$\lim_{t\to+\infty} y_i(t) = \lim_{t\to+\infty} e^{-t} = 0 \tag{10.35}$$

$$\lim_{t\to+\infty} y_s(t) = \lim_{t\to+\infty} 1 - e^{-t} = 1 \tag{10.36}$$

$$\lim_{t\to+\infty} y_r(t) = \lim_{t\to+\infty} t - 1 + e^{-t} = +\infty \tag{10.37}$$

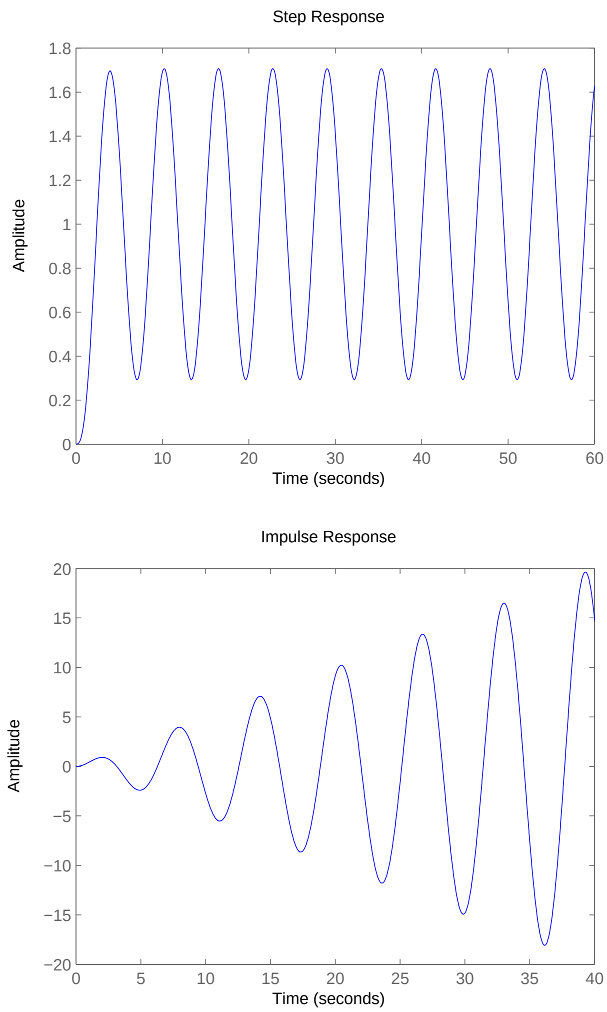



Figure 10.7: Time responses with oscillations: unit step response of $\frac{1}{(s^2+1)(s+1)}$ (top) and impulse response of $\frac{s}{(s^2+1)^2}$ (bottom).

They can be found without the inverse Laplace transform using (2.74):

$$\lim_{t\to+\infty} y_i(t) = \lim_{s\to 0} s\,\frac{1}{s+1} = 0 \tag{10.38}$$

$$\lim_{t\to+\infty} y_s(t) = \lim_{s\to 0} s\,\frac{1}{s+1}\,\frac{1}{s} = 1 \tag{10.39}$$

$$\lim_{t\to+\infty} y_r(t) = \lim_{s\to 0} s\,\frac{1}{s+1}\,\frac{1}{s^2} = +\infty \quad \square \tag{10.40}$$

**Example 10.9.** Remember that (2.74) applies when the limit in time exists. Figure 10.7 shows two cases where this limit clearly does not exist because of oscillations with an amplitude that does not decrease. But the two corresponding limits are

$$\lim_{t\to+\infty} y(t) = \lim_{s\to 0} s\,\frac{1}{(s^2+1)(s+1)}\,\frac{1}{s} = 1 \tag{10.41}$$

$$\lim_{t\to+\infty} y(t) = \lim_{s\to 0} s\,\frac{s}{(s^2+1)^2} = \infty \tag{10.42}$$

In the first case we got the average value of the steady-state response; in the second, infinity. Neither case is a valid application of the final value theorem. We need to know first if the time limit exists. □

The former example illustrates the importance of the concept of stability.

*Bounded signal*

**Definition 10.1.** A signal $x(t)$ is **bounded** if $\exists K \in \mathbb{R}^+ : \ \forall t, \ |x(t)| < K$. □

*BIBO stability*

**Definition 10.2.** A system is:

- **stable** if, for every input which is bounded, its output is bounded too;
- **not stable** if there is at least a bounded input for which its output is not bounded.

This definition of **stability** is known as bounded input, bounded output stability (BIBO stability). □

*All poles of stable transfer functions are on the left complex half-plane*

**Theorem 10.2.** A transfer function is stable if and only if all its poles are on the left complex half-plane.

*Proof.* We will prove this in two steps:

- A transfer function $G(s)$ is stable if and only if its impulse response $g(t)$ is absolutely integrable, i.e. iff $\exists M \in \mathbb{R}^+$

$$\int_0^{+\infty} |g(t)|\, \mathrm{d}t < M \tag{10.43}$$

- A transfer function's impulse response is absolutely integrable if and only if all its poles are on the left complex half-plane.

□

**Lemma 10.1.** A transfer function is stable if and only if its impulse response is absolutely integrable.

*Proof.* Let us suppose that the impulse response $g(t)$ is absolutely integrable, and that

$$\int_0^{+\infty} |g(\tau)|\,\mathrm{d}\tau = K \tag{10.44}$$

Let us also suppose that the input $u(t)$ is bounded, as required by the definition of BIBO stability:

$$|u(t)| \leq U,\ \forall t \tag{10.45}$$

From (10.24) we get

$$\begin{aligned}
|y(t)| = |g(t) * u(t)| &= \left|\int_0^t g(\tau)u(t-\tau)\,\mathrm{d}\tau\right| \\
&\leq \int_0^t |g(\tau)u(t-\tau)|\,\mathrm{d}\tau \\
&\leq \int_0^t |g(\tau)|\,|u(t-\tau)|\,\mathrm{d}\tau \\
&\leq U\int_0^t |g(\tau)|\,\mathrm{d}\tau \leq UK
\end{aligned} \tag{10.46}$$

So the output is bounded, proving that the condition (impulse response absolutely integrable) is sufficient.

Reductio ad absurdum proves that it is also necessary. Suppose that the impulse response $g(t)$ is not absolutely integrable; thus, there is a time instant $T \in \mathbb{R}^+$ such that

$$\int_0^T |g(\tau)|\,\mathrm{d}\tau = +\infty \tag{10.47}$$

Now let the input $u(t)$ be given by

$$u(T-t) = \operatorname{sign}(g(t)) \tag{10.48}$$

This is a bounded input, $-1 \leq u(t) \leq 1,\ \forall t$, and so, if the transfer function were stable, the output would have to be bounded. But in time instant $T$

$$y(T) = \int_0^T g(\tau)u(t-\tau)\,\mathrm{d}\tau = \int_0^T |g(\tau)|\,\mathrm{d}\tau = +\infty \tag{10.49}$$

and thus $y(t)$ is not bounded. This shows that the condition is not only sufficient but also necessary. □

**Lemma 10.2.** A transfer function's impulse response is absolutely integrable if and only if all its poles are on the left complex half-plane.

*Proof.* A transfer function $G(s)$ has an impulse response given by $\mathscr{L}^{-1}[G(s)]$. Transfer function $G(s)$ can be expanded into a partial fraction expansion, where the fractions have the poles of $G(s)$ in the denominator. Poles can be divided into four cases.

- The pole is real, $p \in \mathbb{R}$, and simple. In this case the fraction $\frac{k}{s-p}$ (where $k \in \mathbb{R}$ is some real numerator) has the inverse Laplace transform $k\,e^{pt}$.

– If $p = 0$, then $\lim_{t \to +\infty} k\,e^{pt} = k$. In this case the impulse response is not absolutely integrable, since

$$\int_0^{+\infty} |k| \;\mathrm{d}t = \lim_{t \to +\infty} |k|t = +\infty \tag{10.50}$$

– If $p > 0$, the exponential tends to infinity: $\lim_{t \to +\infty} k\,e^{pt} = \pm\infty$ (depending on the sign of $k$). If in the last case the response was not absolutely integrable, even more so in this one.

– If $p < 0$, the exponential tends to zero: $\lim_{t \to +\infty} k\,e^{pt} = 0$. The impulse response is absolutely integrable, since

$$\int_0^{+\infty} \left| k\,e^{pt} \right| \;\mathrm{d}t = k \int_0^{+\infty} e^{pt} \;\mathrm{d}t = k \left[ \frac{1}{p} e^{pt} \right]_0^{+\infty} = \frac{k}{p}(0-1) = -\frac{k}{p} \in \mathbb{R}^+ \tag{10.51}$$

- The pole is real and its multiplicity $n$ is 2 or higher. In this case there will be, in the expansion, fractions of the form $\frac{k_n}{(s-p)^n}, \frac{k_{n-1}}{(s-p)^{n-1}}, \frac{k_{n-2}}{(s-p)^{n-2}} \cdots \frac{k_1}{s-p}$. (Here the $k_i \in \mathbb{R}$, $i = 1 \ldots n$ are the numerators in the expansion.) The corresponding inverse Laplace transforms are of the form $\frac{k_i}{(i-1)!} t^{i-1} e^{pt}$, $i = 1 \ldots n$.

  – If $p = 0$, then the exponential tends to 1, but the power does diverge to infinity: $\lim_{t \to +\infty} \frac{k_i}{(i-1)!} t^{i-1} e^{pt} = \pm\infty$ (depending on the sign of $k$), $\forall i \geq 2$. So in this case the impulse response is not absolutely integrable, as seen above.

  – If $p > 0$, then $\lim_{t \to +\infty} \frac{k_i}{(i-1)!} t^{i-1} e^{pt} = \pm\infty$, $\forall i$. Again, the impulse response is not absolutely integrable.

  – If $p < 0$, then $\lim_{t \to +\infty} \frac{k_i}{(i-1)!} t^{i-1} e^{pt} = 0$, $\forall i$, since the effect of the exponential prevails. For the same reason, the impulse response is absolute integrable, just as in (10.51).

- The pole is complex, $p = a + bj \in \mathbb{C} \backslash \mathbb{R}$, $a, b \in \mathbb{R}$, and simple. Remember once more that complex poles must appear in pairs of complex conjugates, since all polynomial coefficients are real (otherwise real inputs would case complex outputs). In this case the fraction $\frac{k}{s-p} = \frac{k}{s-(a+bj)}$ (where $k \in \mathbb{C}$ is some complex numerator) has the inverse Laplace transform

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{k}{s-p}\right] &= \mathscr{L}^{-1}\left[\frac{k}{s-(a+bj)}\right] = k\,e^{pt} = k\,e^{at}e^{bjt} = k\,e^{at}e^{bjt} \\ &= k\,e^{at}\left(\cos bt + j \sin bt\right) \end{aligned} \tag{10.52}$$

and the fraction $\frac{\bar{k}}{s-\bar{p}} = \frac{\bar{k}}{s-(a-bj)}$ (where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$) has the inverse Laplace transform

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{\bar{k}}{s-\bar{p}}\right] &= \mathscr{L}^{-1}\left[\frac{\bar{k}}{s-(a-bj)}\right] = \bar{k}\,e^{\bar{p}t} = \bar{k}\,e^{at}e^{-bjt} = \bar{k}\,e^{at}e^{-bjt} \\ &= \bar{k}\,e^{at}\left(\cos(-bt) + j\sin(-bt)\right) = \bar{k}\,e^{at}\left(\cos bt - j\sin bt\right) \end{aligned} \tag{10.53}$$

Their effect on the impulse response is their sum:

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{k}{s-p}\right]+\mathscr{L}^{-1}\left[\frac{\bar{k}}{s-\bar{p}}\right] &= k\,e^{at}\left(\cos bt+j\sin bt\right)+\bar{k}\,e^{at}\left(\cos bt-j\sin bt\right)\\ &= (k+\bar{k})\,e^{at}\cos bt = 2\Re(k)\,e^{at}\cos bt\end{aligned} \quad (10.54)$$

Notice that the imaginary parts cancel out, and we are left with oscillations having:

- period $\frac{2\pi}{b}$, where $b$ is the positive imaginary part of the poles;
- amplitude $2\Re(k)e^{at}$, where $a$ is the real part of the poles. The exponential is the important term, since it is the exponential that may cause this term to vanish or diverge.

So:

- If $a=0$, then the amplitude of the oscillations remains constant; they do not go to zero neither do they diverge to an infinite amplitude. This means that the impulse response is not absolutely integrable, since

$$\begin{aligned}\int_0^{+\infty}\left|2\Re(k)\,\cos bt\right|\,\mathrm{d}t &= 2|\Re(k)|\int_0^{+\infty}\left|\cos bt\right|\,\mathrm{d}t\\ &= 2|\Re(k)|\lim_{n\to+\infty} n\int_0^{\frac{2\pi}{b}}\left|\cos bt\right|\,\mathrm{d}t\\ &= 4|\Re(k)|\lim_{n\to+\infty} n\int_0^{\frac{\pi}{b}}\sin bt\,\mathrm{d}t\\ &= \frac{4|\Re(k)|}{b}\lim_{n\to+\infty} n\left[-\cos bt\right]_0^{\frac{\pi}{b}}\\ &= \frac{8|\Re(k)|}{b}\lim_{n\to+\infty} n = +\infty\end{aligned} \quad (10.55)$$

- If $a>0$, the amplitude of the oscillations tends to infinity. Consequently the impulse response will not be absolutely integrable.
- If $a<0$, the exponential tends to zero, and so will the oscillations. In this case the impulse response is absolutely integrable, since

$$\int_0^{+\infty}\left|2\Re(k)\,e^{at}\cos bt\right|\,\mathrm{d}t \le 2|\Re(k)|\int_0^{+\infty} e^{at}\,\mathrm{d}t \quad (10.56)$$

  and we end up with a case similar to (10.51).

- The pole is complex and its multiplicity $n$ is 2 or higher. This case is a mixture of the last two. There will be terms of the form $\frac{k_i}{(s-(a+bj))^i}+\frac{\overline{k_i}}{(s-(a-bj))^i}$, $i=1\dots n$. The corresponding inverse Laplace transform is $\frac{2\Re(k_i)}{(i-1)!}\,t^{i-1}e^{at}\cos bt$. So:
  - If $a=0$, then $e^{at}=1$ but the amplitude of the oscillations still grows to infinity, because of the power function, if $i\ge 2$. So in this case the impulse response will not be absolutely integrable.

– If $a > 0$, the amplitude of the oscillations tends to infinity. The same conclusion follows.

– If $a < 0$, the exponential tends to zero, and for large times its effect prevails; so the the impulse response will be absolutely integrable.

It is clear that one single term not tending exponentially to zero suffices to prevent the impulse response from being absolutely integrable. Consequently, the only way for the impulse response to tend to zero is that all poles should have negative real parts; in other words, that all poles should lie on the left complex half-plane. □

While some authors call unstable to all systems that are not stable, the following distinction is current.

*Unstable and marginally stable systems*

*Unstable systems*

*Marginal stability*

**Definition 10.3.** A system is:

- **unstable** if, for every input which is bounded, its output is not bounded;
- **marginally stable** if there are at least a bounded input for which its output is bounded and a bounded input for which its output is not bounded. □

*Marginally stable systems have simple poles on the imaginary axis*

**Theorem 10.3.** Marginally stable systems have no poles on the right complex half-plane, and one or more simple poles on the imaginary axis.

*Proof.* It is clear from the proof of Lemma 10.2 that simple poles on the imaginary axis correspond to:

- impulse responses which are bounded:
  - a pole at the origin has a constant impulse response;
  - a pair of complex conjugate imaginary poles has constant amplitude sinusoidal oscillations as impulse response;
- responses to bounded inputs which are not bounded, since systems with such poles are not stable.

A single pole $p$ on the right complex half-plane makes a system unstable, since, whatever the input may be, in the partial fraction expansion of the output there will be a fraction of the form $\frac{k}{s-p}$, and the proof of Lemma 10.2 shows that such terms always diverge exponentially to infinity. The same happens with multiple poles on the imaginary axis. □

The effect of each pole on the stability of a system justifies the following nomenclature.

*Stable, marginally stable, and unstable poles*

**Definition 10.4.** Poles are:

- **stable**, when located on the left complex half-plane;
- **marginally stable**, when simple and located on the imaginary axis;
- **unstable**, when multiple and located on the imaginary axis, or when located on the right complex half-plane. □

*lity depends on pole ion*

A system is:

- **stable**, when all its poles are stable;
- **marginally stable**, when it has no unstable poles, and one or more of its poles are marginally stable;
- **unstable**, when it has one or more unstable poles.

**Example 10.10.** From the location of the poles, we can conclude the following about the stability of these transfer functions:

- $\frac{s+4}{(s+1)(s+2)(s+3)}$; poles: $-1$, $-2$, $-3$; stable transfer function
- $\frac{s-5}{s^2+6}$; poles: $\pm\sqrt{6}j$; marginally stable transfer function
- $\frac{s+7}{(s^2+8)^2}$; poles: $\pm\sqrt{8}j$ (double); unstable transfer function
- $\frac{(s-12)(s+13)}{(s+9)^2(s^2+20s+221)}$; poles: $\pm-3$ (double), $-10\pm11j$; stable transfer function
- $\frac{14}{s-15}$; poles: 15; unstable transfer function
- $\frac{16}{s}$; poles: 0; marginally stable transfer function
- $\frac{-17}{s^2}$; poles: 0 (double); unstable transfer function
- $\frac{18}{s(s^2+18)}$; poles: 0, $\pm\sqrt{18}j$; marginally stable transfer function
- $\frac{19}{(s+20)(s+21)(s+22)(s-23)}$; poles: $-20$, $-21$, $-22$, 23; unstable transfer function □

**Remark 10.8.** Never forget that zeros have nothing to do with stability. □

*Poles, not zeros, determine stability*

## 10.3 Time responses: periodic inputs

Consider the weaving loom in Figure 10.8. The shuttle that carries the yarn that will become the weft thread moves without cease from the left to the right and then back. Meanwhile, half the warp threads are pulled up by a harness, which will then lower then while the other half goes up, and this too without cease. The corresponding references are similar to those in Figure 10.9. They are called **square wave** and **triangle wave**, and are examples of **periodic signals**.

*Square wave*
*Triangle wave*

**Definition 10.5.** A **periodic signal** is one for which $\exists\mathfrak{T}\in\mathbb{R}^+$

*Periodic signals*

$$f(t+\mathfrak{T}) = f(t),\ \forall t \tag{10.57}$$

$T = \min\mathfrak{T}$ is the **period** of signal $f(t)$. □

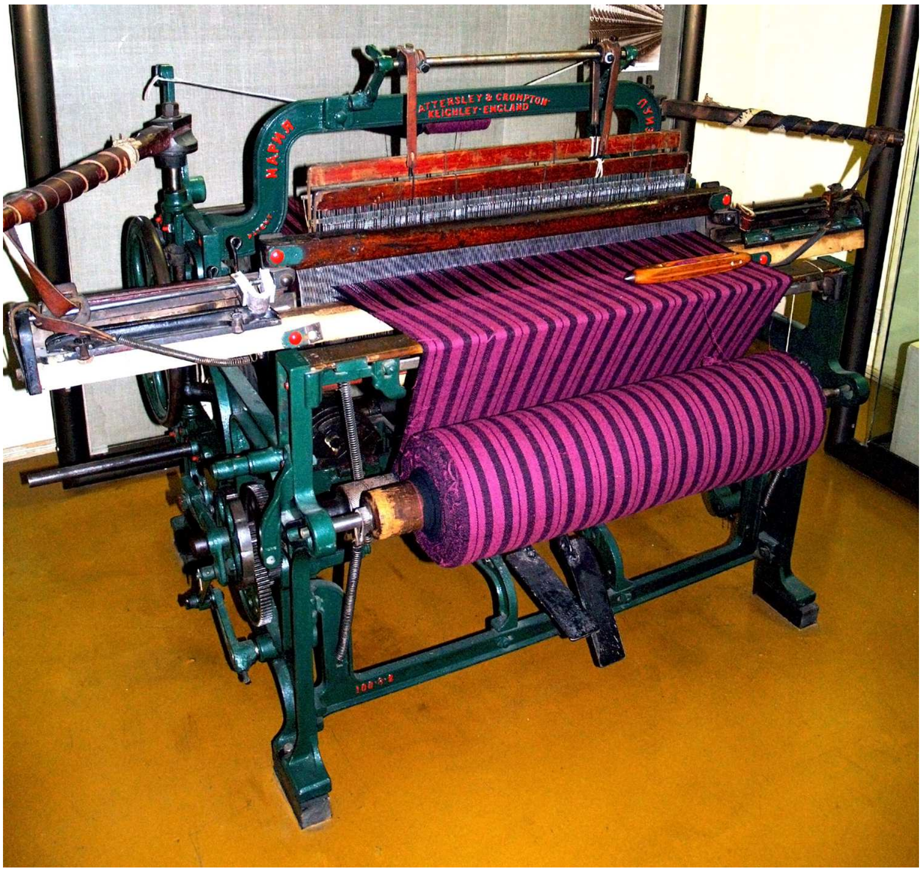



Figure 10.8: A weaving loom (source:Wikimedia).

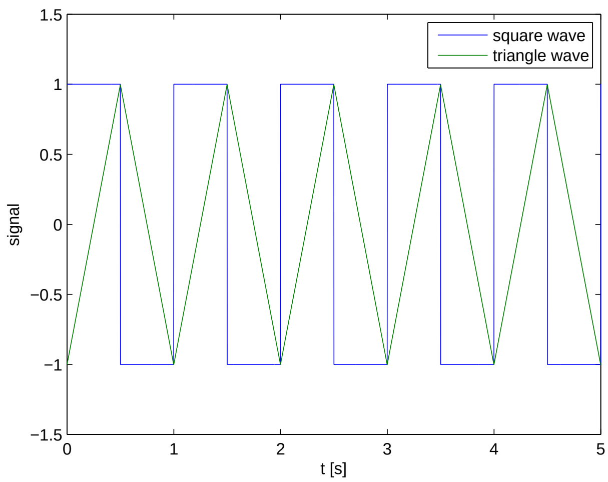



Figure 10.9: A square wave and a triangle wave (both with period 1 and amplitude 1).

**Remark 10.9.** Notice that the different values of $\mathfrak{T}$ are in fact the integer multiples of $T$, i.e.

$$f(t+T)=f(t),\ \forall t \Rightarrow f(t+nT)=f(t),\ \forall t,n\in\mathbb{N} \quad \square \tag{10.58}$$

Triangle waves are also a useful alternative to ramps, since they avoid the inconvenience of an infinitely large signal. Square waves are useful in experimental settings for another reason: they allow seeing successive step responses, and consequently allow measuring parameters several times in a row. For this purpose, the period must be large enough for the transient regime to disappear.

*MATLAB's command `square`*

**Example 10.11.** We can find the output of $G(s)=\frac{15}{s+20}$ to a square wave with period 1 s and amplitude 1 using MATLAB as follows:

```
>> t = 0 : 0.001 : 3;
>> u = square(t*2*pi);
>> figure, plot(t,u, t,lsim(15/(s+20),u,t))
>> axis([0 3 -1.5 1.5])
>> xlabel('t [s]'), ylabel('input and output'), legend({'input','output'})
```

Notice that the amplitude of the first step is 1 and the amplitude of the following steps is the peak to peak amplitude, twice as big, viz. 2. Also notice that there is a step every half period, i.e. every 0.5 s.

The period was appropriately chosen since (as we shall see in Section 10.5) the transient response is practically gone after 0.5 s. A period four times smaller would not allow seeing a complete step response. Both cases are shown in Figure 10.10. $\square$

Another useful periodic signal is the sinusoid, which appears naturally e.g. when working with tides and with any phenomena that are the projection onto a plane of a circular movement on a perpendicular plane (and this includes such different things as motor vibrations or daily thermal variations).

*Sinusoidal inputs cause sinusoidal outputs in steady state*

**Theorem 10.4.** The stationary response $y(t)$ of a stable linear plant $G(s)$ subject to a sinusoidal input $u(t)=\sin(\omega t)$ is

$$y(t)=|G(j\omega)|\sin(\omega t+\angle G(j\omega)) \tag{10.59}$$

where $\angle z$ is the phase, or argument, of $z\in\mathbb{C}$ (also notated often as $\arg z$), so that $z=|z|e^{j\angle z}$.

*Proof.* The output is

$$y(t)=\mathscr{L}^{-1}\left[Y(s)\right] \tag{10.60}$$

and

$$Y(s)=G(s)U(s)=G(s)\mathscr{L}\left[\sin(\omega t)\right]=G(s)\frac{\omega}{s^2+\omega^2}=G(s)\frac{\omega}{(s+j\omega)(s-j\omega)} \tag{10.61}$$

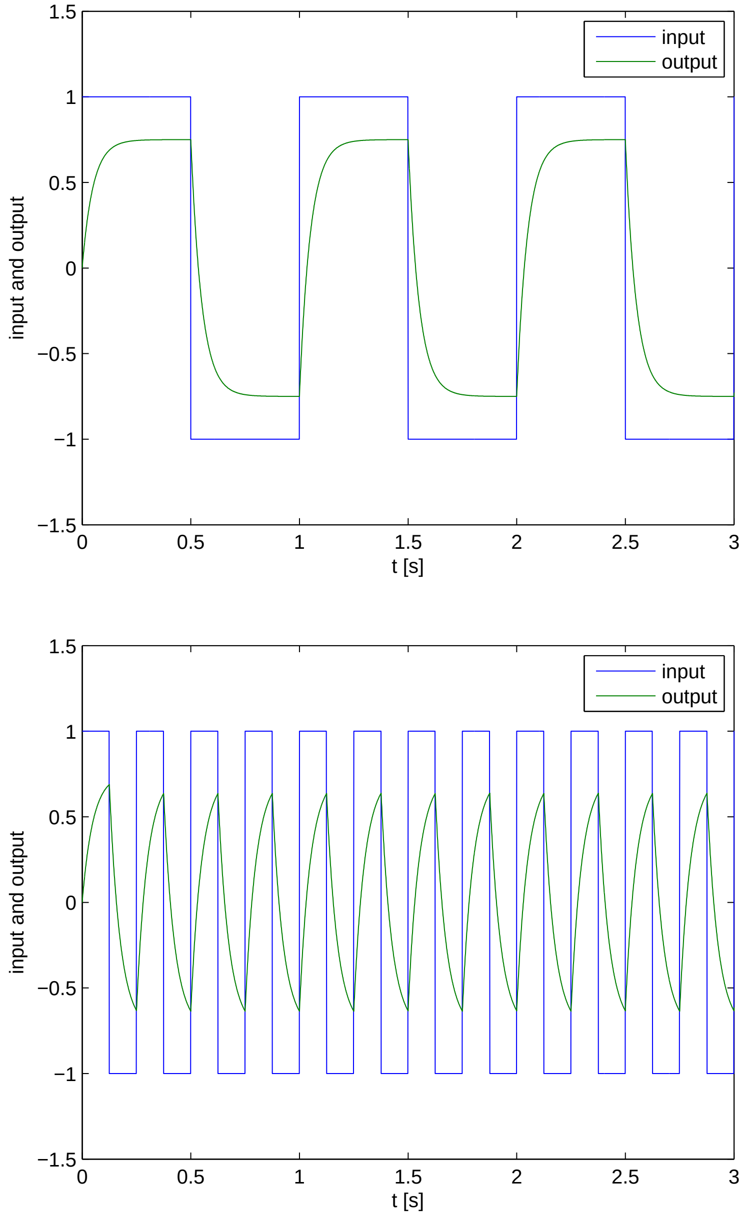



Figure 10.10: Response of $G(s) = \dfrac{15}{s+20}$ to two square waves with different periods.

If all poles $p_k,\ k = 1, \ldots, n$ of $G(s)$ are simple, we can perform a partial fraction expansion of $Y(s)$ as follows:

$$\begin{aligned} Y(s) &= \frac{b_0}{s + j\omega} + \frac{\overline{b_0}}{s - j\omega} + \sum_{k=1}^{n} \frac{b_k}{s - p_k} \\ \Rightarrow y(t) &= \underbrace{b_0 e^{-j\omega t} + \overline{b_0} e^{j\omega t}}_{\text{steady-state response } y_{ss}(t)} + \underbrace{\sum_{k=1}^{n} b_k e^{p_k t}}_{\substack{\text{transient} \\ \text{response } y_t(t)}} \end{aligned} \tag{10.62}$$

We know that all terms in the transient response $y_t(t)$ belong there because the exponentials are vanishing, since the poles are on the left complex half-plane. If there are multiple poles, the only difference is that there will be terms of the form $\frac{b_k}{(i-1)!} t^{i-1} e^{p_k t},\ i \in \mathbb{N}$ in the transient response $y_t(t)$, which will still, of course, be vanishing with time. In either case, the steady-state response is the same.

From (10.61) we know that $Y(s) = G(s)\frac{\omega}{(s+j\omega)(s-j\omega)}$, and from (10.62) we know that $Y(s) = \frac{b_0}{s+j\omega} + \frac{\overline{b_0}}{s-j\omega} + \mathscr{L}[y_t(t)]$. We can multiply both by $s + j\omega$ and obtain

$$G(s)\frac{\omega}{s - j\omega} = b_0 + \left(\frac{\overline{b_0}}{s - j\omega} + \mathscr{L}[y_t(t)]\right)(s + j\omega) \tag{10.63}$$

Now we evaluate this equality at $s = -j\omega$:

$$G(-j\omega)\frac{\omega}{-2j\omega} = b_0 \tag{10.64}$$

Replacing $b_0 = G(-j\omega)\frac{1}{-2j}$ and $\overline{b_0} = G(j\omega)\frac{1}{2j}$ in $y_{ss}(t) = b_0 e^{-j\omega t} + \overline{b_0} e^{j\omega t}$, we obtain

$$\begin{aligned} y_{ss}(t) &= G(-j\omega)\frac{1}{-2j}e^{-j\omega t} + G(j\omega)\frac{1}{2j}e^{j\omega t} \\ &= |G(-j\omega)|e^{j\angle G(-j\omega)}\frac{1}{-2j}e^{-j\omega t} + |G(j\omega)|e^{j\angle G(j\omega)}\frac{1}{2j}e^{j\omega t} \\ &= -|G(j\omega)|e^{j(\angle G(-j\omega)-\omega t)}\frac{1}{2j} + |G(j\omega)|e^{j(\angle G(j\omega)+\omega t)}\frac{1}{2j} \\ &= \frac{1}{2j}|G(j\omega)|\left(e^{j(\angle G(j\omega)+\omega t)} - e^{j(\angle G(-j\omega)-\omega t)}\right) \\ &= \frac{1}{2j}|G(j\omega)|\Big(\cos(\angle G(j\omega)+\omega t) + j\sin(\angle G(j\omega)+\omega t) \\ &\qquad - \cos\Big(-(\angle G(j\omega)+\omega t)\Big) - j\sin\Big(-(\angle G(j\omega)+\omega t)\Big)\Big) \\ &= \frac{1}{2j}|G(j\omega)|\Big(\cos(\angle G(j\omega)+\omega t) + j\sin(\angle G(j\omega)+\omega t) - \cos(\angle G(j\omega)+\omega t) + j\sin(\angle G(j\omega)+\omega t)\Big) \\ &= \frac{1}{2j}|G(j\omega)|2j\sin(\angle G(j\omega)+\omega t) \\ &= |G(j\omega)|\sin(\omega t + \angle G(j\omega))\square \end{aligned} \tag{10.65}$$

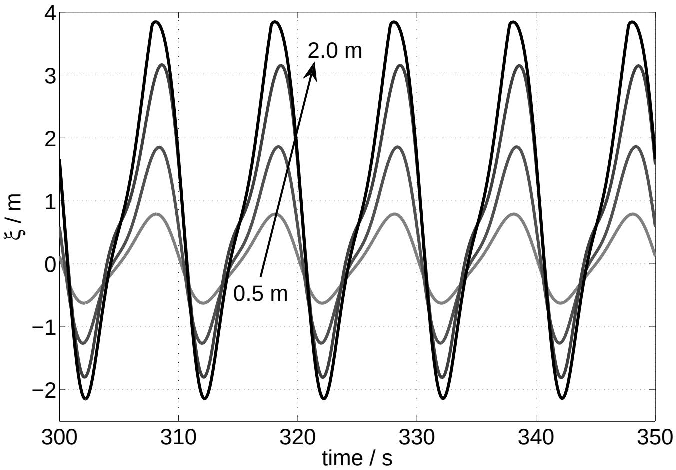



Figure 10.11: Vertical position of the AWS from Figure 3.2, simulated assuming sinusoidal sea waves.

**Corollary 10.2.** Since $G(s)$ is not only stable but also linear, if the input is $u(t) = A\sin(\omega t)$ instead, the output is

$$y(t) = A|G(j\omega)|\sin(\omega t + \angle G(j\omega)) \tag{10.66}$$

**Example 10.12.** Figure 10.11 shows the simulated vertical position of the Wave Energy Converter of Figure 3.2, the Archimedes Wave Swing, when subject to sinusoidal waves of different amplitudes. The device is in steady-state, as is clear both from the regularity of its movements and from the time already passed since the beginning of the simulation. As the input is sinusoidal, if the model were linear, the output should be sinusoidal too. But the shape of the output is not sinusoidal; it is not even symmetrical around its mean value; its amplitude does not increase linearly with the amplitude of the input. The model used to obtain these simulation results is obviously non-linear. □

## 10.4 Frequency responses and the Bode diagram

(10.66) shows that, if a stable system $G(s)$ has a sinusoidal input, the steady-state output is related to the input through $G(j\omega)$, which is the Fourier transform (2.87) of the differential equation describing the system's dynamics:

*Frequency, amplitude, and phase of output for sinusoidal inputs*

- if the input is sinusoidal, the steady-state output is sinusoidal too;
- if the input has frequency $\omega$, the steady-state output has frequency $\omega$ too;
- if the input has amplitude $A$ (or peak-to-peak amplitude $2A$), the steady-state output has amplitude $A|G(j\omega)|$ (or peak-to-peak amplitude $2A|G(j\omega)|$);
- if the input has phase $\theta$ at $t = 0$, the steady-state output has phase $\theta + \angle G(j\omega)$ at $t = 0$.

Remember that:

- the steady-state output is sinusoidal, but the transient is not: you must wait for the transient to go away to have a sinusoidal output; *The transient is not sinusoidal*
- unstable systems have transient responses that do not go away, so you will never have a sinusoidal output;
- $\omega$ is the frequency in radians per second. *$\omega$ is in rad/s*

**Definition 10.6.** Given a system $G(s)$:

- its **frequency response** is $G(j\omega)$, a function of $\omega$; *Frequency response*
- its **gain** at frequency $\omega$ is $|G(j\omega)|$; *Gain*
- its **gain in decibel** (denoted by symbol dB) is $20\log_{10}|G(j\omega)|$ (gain $|G(j\omega)|$ is often called gain in absolute value, to avoid confusion with the gain in decibel); *Gain in dB* *Gain in absolute value*
- its **phase** at frequency $\omega$ is $\angle G(j\omega)$. □ *Phase*

**Remark 10.10.** These definitions are used even if $G(s)$ is not stable. If the system is stable:

- the gain is the ratio between the amplitude of the steady-state output and the amplitude of the input;
- the phase is the difference in phase between the steady-state output sinusoid and the input sinusoid. □

**Example 10.13.** Figure 10.12 shows the output of $G(s) = \frac{300(s+1)}{(s+10)(s+100)}$ for a sinusoidal input of frequency 1 rad/s, found as follows:

```
>> s = tf('s');
>> G = 300*(s+1)/((s+10)*(s+100));
>> t = 0 : 0.001 : 30;
>> figure, plot(t,sin(t), t,lsim(G,sin(t),t))
>> xlabel('time [s]'), ylabel('output'), grid
```

The amplitude of the input is 1, by construction; the amplitude of the output is 0.4219. So the gain at 1 rad/s is $\frac{0.4219}{1} = 0.4219$ in absolute value, or $20\log_{10} 0.4219 = -7.50$ dB. This maximum value is taking place at 26 s, while the corresponding maximum of the input takes place later, at $4\times 2\pi + \frac{\pi}{2} = 26.7$ s. As the period is $2\pi = 6.28$ s, the phase is $\frac{26.7-26}{6.28} \times 360° = 40°$.

Figure 10.12 also shows the output of $G(s)$ when the frequency is 200 rad/s:

```
>> t = 0 : 0.0001 : 0.2;
>> figure, plot(t,sin(200*t), t,lsim(G,sin(200*t),t))
>> xlabel('time [s]'), ylabel('output'), grid
```

In that case, the amplitude of the input is still 1 and the amplitude of the output is 1.313. So the gain at 200 rad/s is $\frac{1.313}{1} = 1.313$ in absolute value, or $20\log_{10} 1.313 = 2.37$ dB. This maximum value is taking place at 0.1703 s, while the corresponding maximum of the input takes place earlier, at $5 \times \frac{2\pi}{200} + \frac{\frac{\pi}{2}}{200} =$

0.1649 s. As the period is $\frac{2\pi}{200} = 0.0314$ s, the phase is $\frac{0.1649-0.1703}{0.0314} \times 360° = -62°$.

In both cases, it is visible that the first oscillations are not sinusoidal, because of both their shape and their varying amplitudes. In other words, the transient has not yet disappeared by then. □

In the example above, the amplitude of the output was larger than that of the input in one case, and smaller in the other. Also in one case the extremes of the output sinusoid took place earlier than those of the input sinusoid, while in the other case it was the other way round.

**Definition 10.7.** Given

- a stable system $G(s)$,
- with sinusoidal input of frequency $\omega$ and amplitude $A_u$,
- with steady-state sinusoidal output also of frequency $\omega$ and amplitude $A_y = A_u|G(j\omega)|$,

then:

*Amplification*

- If the amplitude of the output is larger than the amplitude of the input, $A_y > A_u$, the system is **amplifying** its input:

$$A_y > A_u \Rightarrow |G(j\omega)| = \frac{A_y}{A_u} > 1 \Rightarrow 20\log_{10}|G(j\omega)| > 0 \text{ dB} \quad (10.67)$$

  That is to say:

  – the gain in absolute value is larger than 1;

  – the gain in decibel is larger than 0 dB.

*Attenuation*

- If the amplitude of the output is smaller than the amplitude of the input, $A_y > A_u$, the system is **attenuating** its input:

$$A_y < A_u \Rightarrow |G(j\omega)| = \frac{A_y}{A_u} < 1 \Rightarrow 20\log_{10}|G(j\omega)| < 0 \text{ dB} \quad (10.68)$$

  That is to say:

  – the gain in absolute value is smaller than 1;

  – the gain in decibel is smaller than 0 dB.

- If the amplitude of the output and the amplitude of the input are the same, $A_y = A_u$, the system is neither amplifying nor attenuating its input:

$$A_y = A_u \Rightarrow |G(j\omega)| = \frac{A_y}{A_u} = 1 \Rightarrow 20\log_{10}|G(j\omega)| = 0 \text{ dB} \quad (10.69)$$

  That is to say:

  – the gain in absolute value is 1;

  – the gain in decibel is 0 dB.

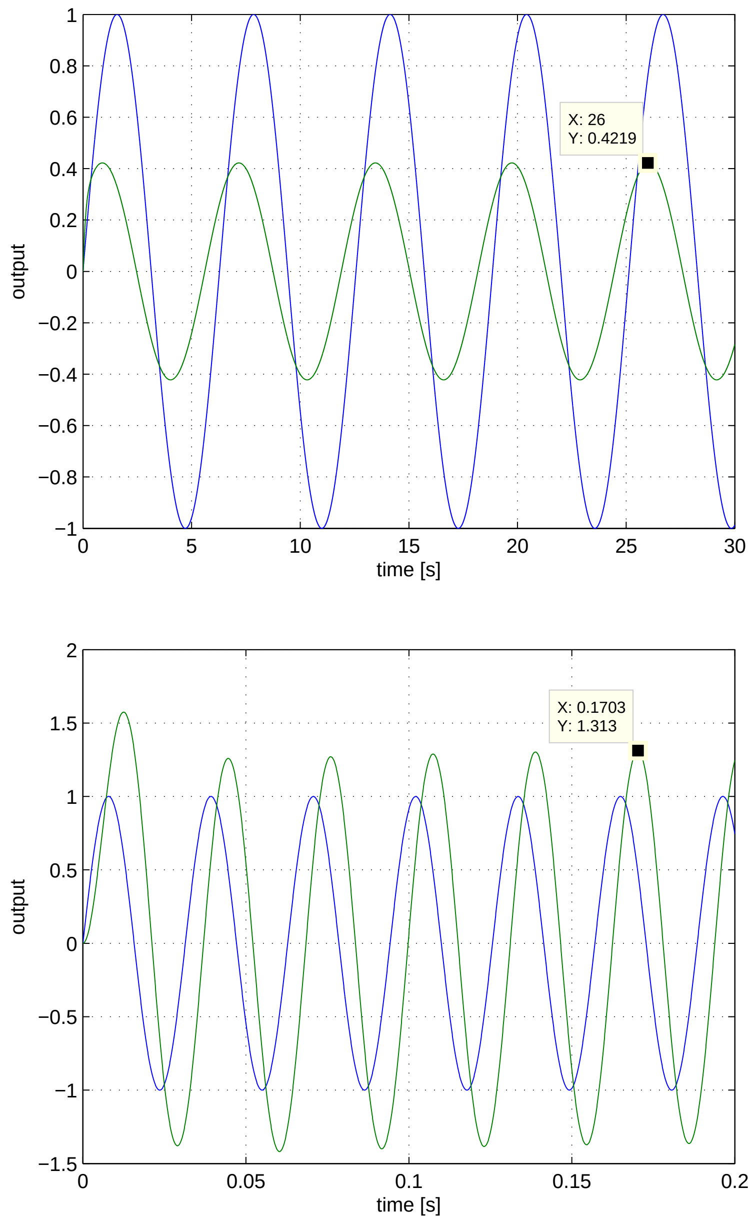



Figure 10.12: Response of $G(s) = \dfrac{300(s+1)}{(s+10)(s+100)}$ to two sinusoids with different periods.

Table 10.1: Gain values; $A_u$ is the amplitude of the input sinusoid and $A_y$ is the amplitude of the steady-state output sinusoid

| | Gain in absolute value | Gain in decibel | Amplitudes |
|---|---|---|---|
| Minimum value | $\lvert G(j\omega)\rvert = 0$ | $20\log 10\lvert G(j\omega)\rvert = -\infty$ dB | $A_y = 0$ |
| Attenuation | $0 < \lvert G(j\omega)\rvert < 1$ | $20\log 10\lvert G(j\omega)\rvert < 0$ dB | $A_y < A_u$ |
| Input and output with same amplitude | $\lvert G(j\omega)\rvert = 1$ | $20\log 10\lvert G(j\omega)\rvert = 0$ dB | $A_y = A_u$ |
| Amplification | $\lvert G(j\omega)\rvert > 1$ | $20\log 10\lvert G(j\omega)\rvert > 0$ dB | $A_y > A_u$ |

Furthermore:

*Phase lead*

- If the extremes of the output take place earlier than the corresponding extremes of the input, the output **leads** in relation to the input; this means that

$$\angle G(j\omega) > 0 \tag{10.70}$$

*Phase lag*

- If the extremes of the output take place later than the corresponding extremes of the input, the output **lags** in relation to the input; this means that

$$\angle G(j\omega) < 0 \tag{10.71}$$

- If the extremes of the output and the corresponding extremes of the input take place at the same time, the output and the input are in phase; this means that

$$\angle G(j\omega) = 0 \tag{10.72}$$

*Phase opposition*

- If the maxima of the output and the minima of the input take place at the same time, and vice versa, the output and the input are in **phase opposition**; this means that

$$\angle G(j\omega) = \pm 180^\circ = \pm\pi \text{ rad} \quad \square \tag{10.73}$$

**Remark 10.11.** Notice that, since sinusoids are periodic, the phase is defined up to 360° shifts: a 90° phase is undistinguishable from a −270° phase, or for that matter from a 3690° phase or any $90^\circ + k360^\circ$, $k \in \mathbb{Z}$ phase. While each of these values can be in principle arbitrarily chosen, it usual to make the phase vary continuously (as much as possible) with frequency, starting from values for low frequencies determined as we will see below in Section 10.7. □

Gain values can be summed up as shown in Table 10.1.

**Example 10.14.** Consider the responses to sinusoidal inputs of $G(s) = \frac{1}{s^2+0.5s+1}$ in Figure 10.13.

- For $\omega = 0.5$ rad/s:
  - The amplitude of the output is larger than that of the input, so we must have

$$|G(j0.5)| > 1 \Leftrightarrow 20\log_{10}|G(j0.5)| > 0 \text{ dB} \tag{10.74}$$

– In fact, the gain is

$$|G(j0.5)| = \left|\frac{1}{(j0.5)^2 + 0.5j0.5 + 1}\right| = \left|\frac{1}{1 - 0.25 + j0.25}\right| = \frac{1}{\sqrt{0.75^2 + 0.25^2}} = 1.26$$
$$\Rightarrow 20\log_{10} G(j0.5) = 20\log_{10} 1.26 = 2 \text{ dB} \quad (10.75)$$

– The output is delayed in relation to the input, so we must have $\angle G(j0.5) < 0$.

– In fact, the phase is

$$\angle G(j0.5) = \angle\left(\frac{1}{0.75 + j0.25}\right) = \angle 1 - \angle(0.75 + j0.25) = 0^\circ - \arctan\frac{0.25}{0.75} = -18^\circ \quad (10.76)$$

- For $\omega = 1$ rad/s:

  – The amplitude of the output is even larger now, so

$$|G(j)| > |G(j0.5)| = 1.26 \Leftrightarrow 20\log_{10}|G(j)| > 20\log_{10} G(j0.5) = 2 \text{ dB} \quad (10.77)$$

  – In fact, the gain is

$$|G(j)| = \left|\frac{1}{j^2 + 0.5j + 1}\right| = \left|\frac{1}{j0.5}\right| = \frac{1}{0.5} = 2$$
$$\Rightarrow 20\log_{10} G(j) = 20\log_{10} 2 = 6 \text{ dB} \quad (10.78)$$

  – The output is delayed in relation to the input. Furthermore, the output crosses zero as the input is already at a peak or at a through. So the phase is negative, and equal to $-90^\circ$.

  – In fact,

$$\angle G(j) = \angle\left(\frac{1}{j0.5}\right) = \angle 1 - \angle(j0.5) = 0^\circ - 90^\circ = -90^\circ \quad (10.79)$$

- For $\omega = 2$ rad/s:

  – The amplitude of the input is larger than that of the output, so we must have

$$|G(j2)| < 1 \Leftrightarrow 20\log_{10}|G(j2)| < 0 \text{ dB} \quad (10.80)$$

  – In fact, the gain is

$$|G(j2)| = \left|\frac{1}{(2j)^2 + 0.5j2 + 1}\right| = \left|\frac{1}{-3 + j}\right| = \frac{1}{\sqrt{9+1}} = 0.316$$
$$\Rightarrow 20\log_{10} G(j2) = 20\log_{10} 10^{-\frac{1}{2}} = -10 \text{ dB} \quad (10.81)$$

  – The output is delayed in relation to the input. Furthermore, input and output are almost in phase opposition, but not yet. So we must have $0^\circ < \angle G(j2) < -180^\circ$, but close to the latter value.

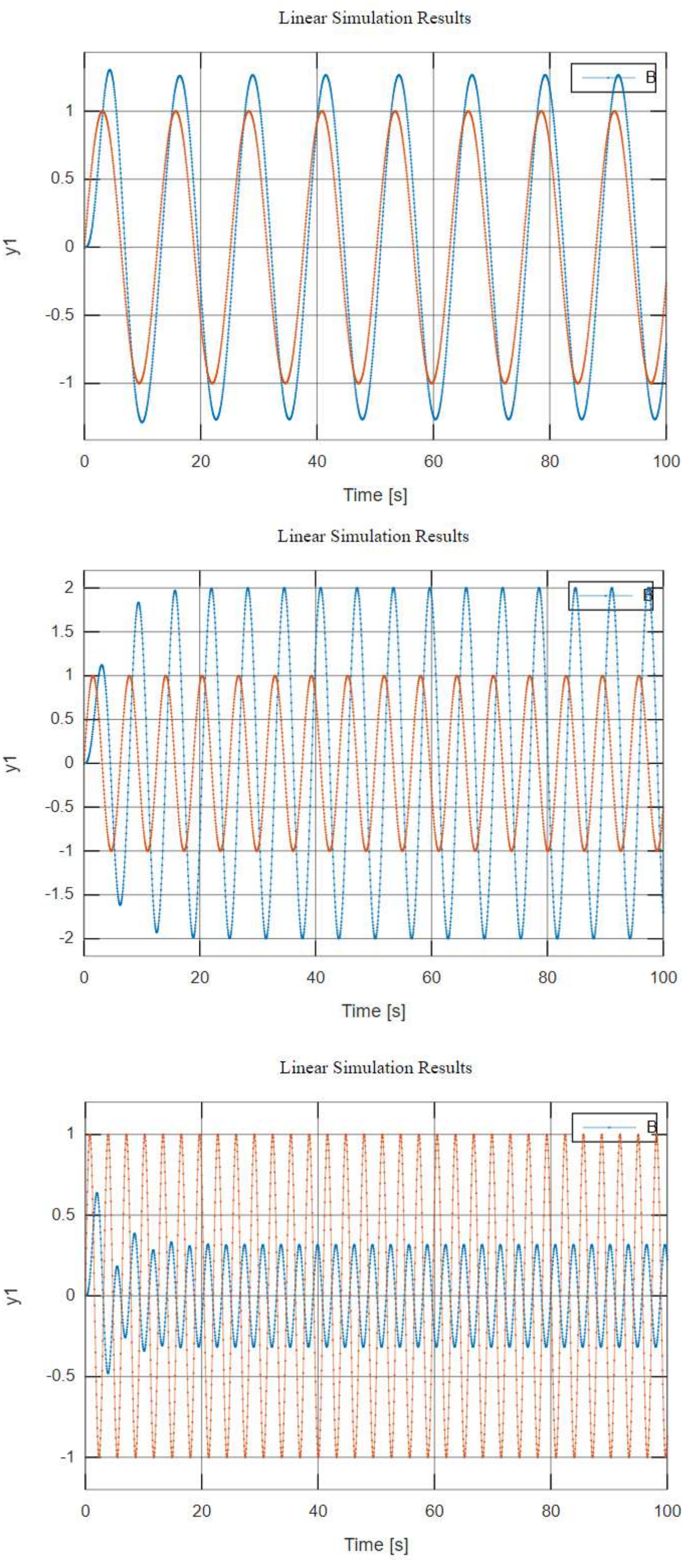



Figure 10.13: Responses of $G(s) = \frac{1}{s^2+0.5s+1}$ (blue) to input sinusoids (red) with 0.5 rad/s (top), 1 rad/s (centre) and 2 rad/s (bottom).

– In fact, the phase is

$$\angle G(j2) = \angle\left(\frac{1}{-3+j}\right) = \angle 1 - \angle(-3+j) = 0^\circ - \arctan\frac{1}{-3} = -162^\circ \quad \square \tag{10.82}$$

The **Bode diagram**, or Bode plot, is a graphical representation of the frequency response of a system, as a function of frequency. This diagram comprises two plots:

*Bode diagram*

- a top plot, showing the gain in dB ($y$–axis) as a function of frequency in a semi-logarithmic scale ($x$–axis);
- a bottom plot, showing the phase in degrees ($y$–axis) as a function of frequency in a semi-logarithmic scale ($x$–axis).

Frequency is usually given in rad/s, but sometimes in Hz.

In the following sections we will learn how to plot by hand the Bode diagram of any plant (or at least a reasonable approximation thereof); meanwhile, the following MATLAB commands can be used instead:

- `bode` plots the Bode diagram of a system;
- `freqresp` calculates the frequency response of a system.

**Example 10.15.** The Bode diagram in Figure 10.14 of $G(s) = \frac{300(s+1)}{(s+10)(s+100)}$ from Example 10.13 is found as follows:

*MATLAB's command `bode`*

```
>> s = tf('s');
>> G = 300*(s+1)/((s+10)*(s+100));
>> figure, bode(G), grid
```

The gains and phases at $\omega = 1$ rad/s and $\omega = 200$ rad/s found in Example 10.13 can be observed in the diagram.

This way we first find the frequency response and then use it to plot the Bode diagram:

*MATLAB's command `freqresp`*

```
>> [Gjw, w] = freqresp(G); % Gjw returned as a 3-dimensional tensor...
>> Gjw = squeeze(Gjw); % ...must now be squeezed to a vector
>> figure, subplot(2,1,1), semilogx(w, 20*log10(abs(Gjw)))
>> grid, xlabel('frequency [rad/s]'), ylabel('gain [dB]'), title('Bode diagram')
>> subplot(2,1,2), semilogx(w, rad2deg(unwrap(angle(Gjw))))
>> grid, ylabel('phase [degrees]') % unwrap avoids jumps of 360 degrees
```

To find the gains and phases to confirm those found found in Example 10.13:

```
>> Gjw = freqresp(G, [1 200])
Gjw(:,:,1) =
   0.3294 + 0.2640i
Gjw(:,:,2) =
   0.6525 - 1.1704i
>> gains = 20*log10(abs(Gjw))
gains(:,:,1) =
```

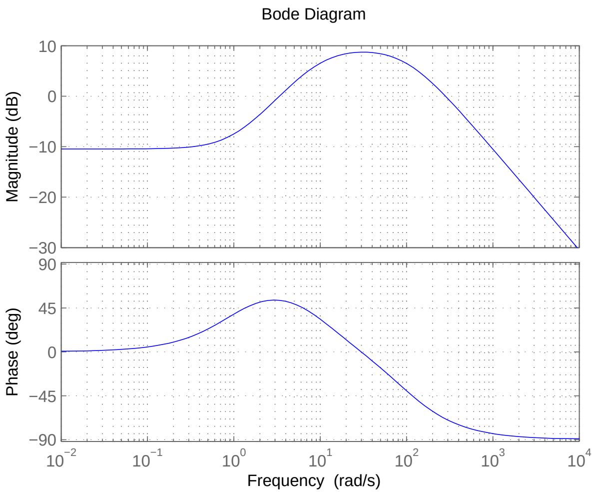



Figure 10.14: Bode diagram of $G(s) = \dfrac{300(s+1)}{(s+10)(s+100)}$.

```
   -7.4909
gains(:,:,2) =
    2.5420
>> phases = rad2deg(unwrap(angle(Gjw)))
phases(:,:,1) =
   38.7165
phases(:,:,2) =
  -60.8590
```

Here's another way of find the same values:

```
>> w = [1 200];
>> Gjw = 300*(1i*w+1)./((1i*w+10).*(1i*w+100));
Gjw =
   0.3271 + 0.2676i   0.6186 - 1.2212i
>> gains = 20*log10(abs(Gjw))
gains =
   -7.4909    2.5420
>> phases = rad2deg(unwrap(angle(Gjw)))
phases =
   38.7165  -60.8590
```

Notice the small differences due to numerical errors. □

**Example 10.16.** The Bode diagram in Figure 10.15 of $G(s) = \frac{1}{s^2+0.5s+1}$ from Example 10.14 shows the gains and phases found in that example, that can also be found as follows:

```
>> G = tf(1,[1 .5 1])

G =
```

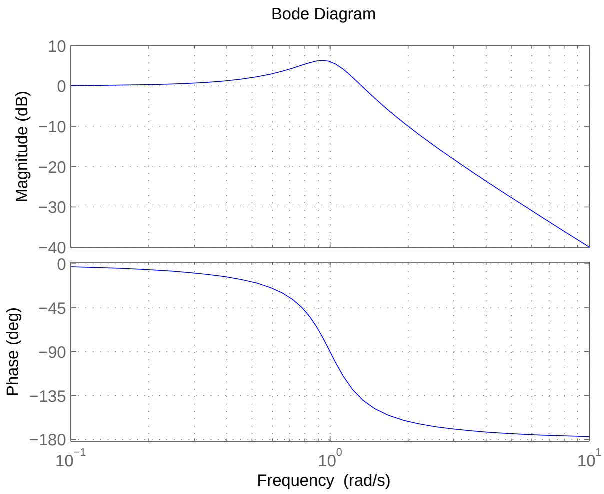



Figure 10.15: Bode diagram of $\frac{1}{s^2 + 0.5s + 1}$.

```
          1
  ---------------
  s^2 + 0.5 s + 1

Continuous-time transfer function.

>> figure,bode(G),grid on
>> Gjw = squeeze(freqresp(G, [.5 1 2]))
Gjw =
   1.2000 - 0.4000i
   0.0000 - 2.0000i
  -0.3000 - 0.1000i
>> gains = 20*log10(abs(Gjw))
gains =
    2.0412
    6.0206
  -10.0000
>> phases = rad2deg(unwrap(angle(Gjw)))
phases =
  -18.4349
  -90.0000
 -161.5651
```

□

**Example 10.17.** From the Bode diagram in Figure 10.16, even without knowing what transfer function it belongs to, we can conclude the following:

- At $\omega = 0.1$ rad/s, the gain is 20 dB (i.e. $10^{\frac{20}{20}} = 10$ in absolute value) and the phase is $0° = 0$ rad. So, if the input is

$$u(t) = 5\sin(0.1t + \frac{\pi}{6}) \tag{10.83}$$

the steady-state output will be

$$y(t) = 5 \times 10\sin(0.1t + \frac{\pi}{6}) = 50\sin(0.1t + \frac{\pi}{6}) \tag{10.84}$$

- At $\omega = 10$ rad/s, the gain is 17 dB (i.e. $10^{\frac{17}{20}} = 7.1$ in absolute value) and the phase is $-45° = -\frac{\pi}{4}$ rad. So, if the input is

$$u(t) = 5\sin(10t + \frac{\pi}{6}) \tag{10.85}$$

the steady-state output will be

$$y(t) = 5 \times 7.1\sin(10t + \frac{\pi}{6} - \frac{\pi}{4}) = 35.5\sin(10t - \frac{\pi}{12}) \tag{10.86}$$

- At $\omega = 100$ rad/s, the gain is 0 dB (i.e. $10^0 = 1$ in absolute value) and the phase is $-85° = -1.466$ rad. So, if the input is

$$u(t) = 5\sin(100t + \frac{\pi}{6}) \tag{10.87}$$

the steady-state output will be

$$y(t) = 5 \times 1\sin(100t + 0.524 - 1.466) = 5\sin(100t - 0.942) \tag{10.88}$$

- At $\omega = 1000$ rad/s, the gain is $-20$ dB (i.e. $10^{\frac{-20}{20}} = 0.1$ in absolute value) and the phase is $-90° = -\frac{\pi}{2}$ rad. So, if the input is

$$u(t) = 5\sin(1000t + \frac{\pi}{6}) \tag{10.89}$$

the steady-state output will be

$$y(t) = 5 \times 0.1\sin(1000t + \frac{\pi}{6} - \frac{\pi}{2}) = 0.5\sin(1000t - \frac{\pi}{3}) \tag{10.90}$$

- The system is linear. So, if the input is

$$u(t) = 0.5\sin(0.1t + \frac{\pi}{6}) + 25\sin(1000t + \frac{\pi}{6}) \tag{10.91}$$

the steady-state output will be

$$y(t) = 0.5 \times 10\sin(0.1t + \frac{\pi}{6}) + 25 \times 0.1\sin(1000t + \frac{\pi}{6} - \frac{\pi}{2}) = 5\sin(0.1t + \frac{\pi}{6}) + 2.5\sin(1000t - \frac{\pi}{3} \tag{10.92}$$

Notice how the frequency with the largest amplitude in the input now has the smallest. □

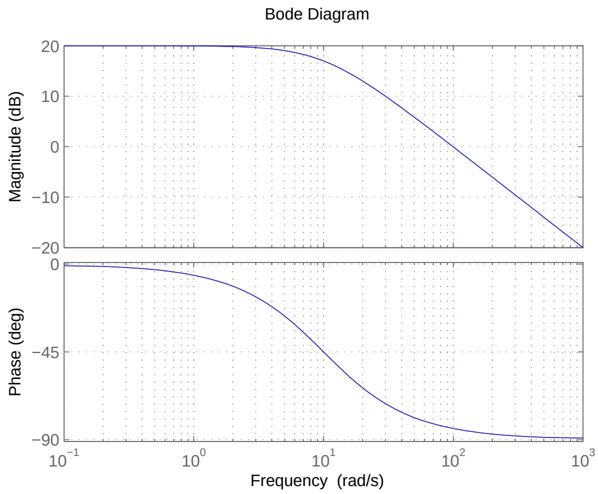



Figure 10.16: Bode diagram of Example 10.17.

## 10.5 Time and frequency responses of a first-order system without zeros

*Still to appear.*

## 10.6 Time and frequency responses of a second-order system without zeros

*Still to appear.*

## 10.7 Systems with more zeros and poles: frequency responses

*Still to appear.*

## 10.8 Systems with more zeros and poles: stability

*Still to appear.*

## 10.9 Systems with more zeros and poles: time responses

*Still to appear.*

## Exercises

1. For each of the following pairs of a transfer function and an input:

   - find the Laplace transform of the input;
   - find the Laplace transform of the output;
   - find the value of the output for $t \gg 1$ without using the inverse Laplace transform;
   - find the output as a function of time;
   - separate that function of time into a transient and a steady state;
   - confirm the value of the output for $t \gg 1$ found previously.

   (a) $G(s) = \dfrac{10}{s^2 + 21s + 20}$ and $u(t) = 0.4,\ t > 0$

   (b) $G(s) = \dfrac{5}{s + 0.1}$ and $u(t) = 2t,\ t > 0$

   (c) $G(s) = \dfrac{s}{s^2 + s + 1}$ and $u(t) = \delta(t)$

   (d) $G(s) = \dfrac{s}{s^2 + s + 1}$ and $u(t) = 0.4,\ t > 0$

   (e) $G(s) = \dfrac{7}{s}$ and $u(t) = 0.4,\ t > 0$

2. From the poles of the transfer functions of Exercise 1 of Chapter 9, explain which of them are stable, unstable, or marginally stable.

3. Figure 10.17 shows the Bode diagrams of some transfer functions. For each of them, read in the Bode diagram the values from which you can calculate the transfer function's steady state response to the following inputs:

   - $u(t) = \sin(2t)$
   - $u(t) = \sin(2t + \frac{\pi}{2})$
   - $u(t) = \sin(1000t)$
   - $u(t) = 10\sin(1000t)$
   - $u(t) = \frac{1}{3}\sin(0.1t - \frac{\pi}{4}) + \sin(2t + \frac{\pi}{2})10\sin(1000t)$

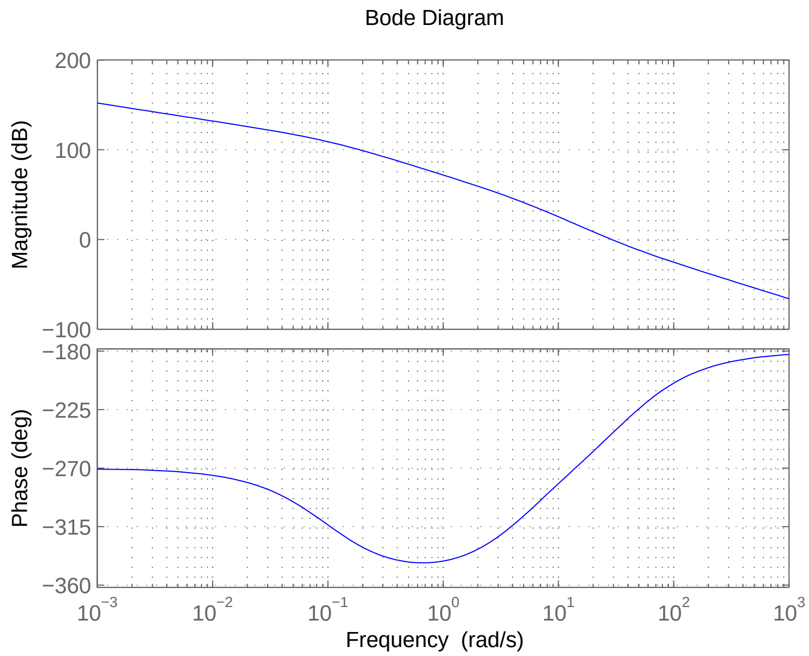
4. For each of the following transfer functions:

   - find the corresponding Fourier transform;
   - find the gain (both in absolute value and in decibel) and the phase (in radians or degrees, as you prefer) at the indicated frequencies.

   (a) $G(s) = \dfrac{5}{s + 0.1}$ and $\omega = 0.01, 0.1, 1$ rad/s

   (b) $G(s) = \dfrac{s}{s^2 + s + 1}$ and $\omega = 0.1, 1, 10$ rad/s

   (c) $G(s) = \dfrac{7}{s}$ and $\omega = 1, 10, 100$ rad/s



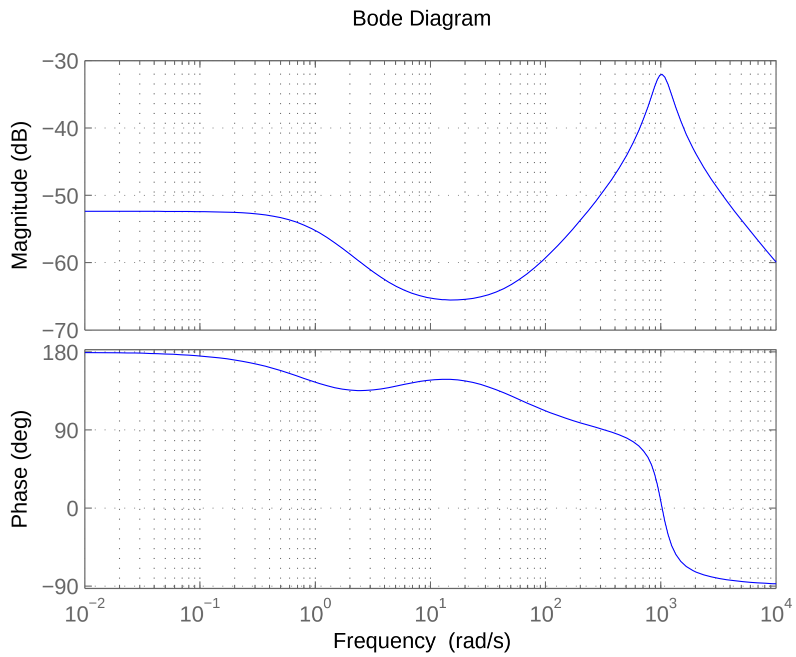




Figure 10.17: Bode diagrams of Exercise 3.

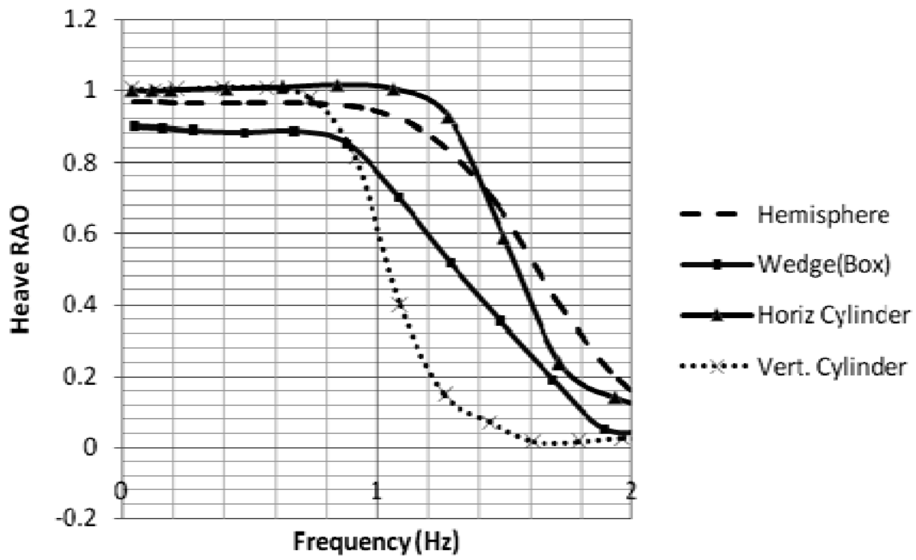



Figure 10.18: RAO of four heaving buoys of Exercise 5 (source: `http://marine-eng.ir/article-1-80-en.pdf`).

5. In naval and ocean engineering it is usual to call **Response Amplitude Operator** (RAO) to what we called gain. It is often represented in absolute value in a linear plot as a function of frequency. Figure 10.18 shows the RAO of four different heaving buoys. Suppose that each of them is subject to waves with an amplitude of 2 m and a frequency of $2\pi$ rad/s. What will be the amplitude of the oscillation of each buoy? *RAO*

6. Use the Routh-Hurwitz criterion to find how many unstable poles each of the following transfer functions has, and classify each system as stable, marginally stable, or unstable.

   (a) $\dfrac{s^2 + \frac{5}{7}s - 10}{s^4 - 2s^3 - 13s^2 + 14s + 24}$

   (b) $\dfrac{s+2}{s^4 - 2s^3 - 13s^2 + 14s + 24}$

   (c) $\dfrac{s+2}{s^6 - 2s^5 - 13s^4 + 14s^3 + 24s^2}$ *Hint:* can you put anything in evidence in the denominator?

   (d) $\dfrac{s^3 + 2s^2 + s}{s^4 + 4s^3 + 4s + 5}$

   (e) $\dfrac{s^3 + 2s^2 + s}{s^5 + 4s^4 + 4s^2 + 5s}$

   (f) $\dfrac{s^3 + 2s^2 + s}{2s^3 - 6s + 4}$

7. Find the ranges of values of $K_1, K_2 \in \mathbb{R}$ for which the systems with the following characteristic equations are stable.

   (a) $s^3 + 3s^2 + 10s + K_1$

(b) $s^3 + K_2 s^2 + 10s + 5$

(c) $s^3 + 2s^2 + (K_1 + 1)s + K_2$

8. Consider transfer function $G(s) = \dfrac{10}{10s+1}$.

   (a) When the input is a unit step, what will the steady state response be?

   (b) When the input is a step with amplitude 3, what will the steady state response be?

   (c) Without computing an expression for the output, give a rough estimate of how long it takes for the output to reach 20, when the input is a step with amplitude 3.

   (d) Without computing an expression for the output, give a rough estimate of the 2% settling time, when the input is a step with amplitude 3.

   (e) Calculate the output as a function of time, using an inverse Laplace transform, and find the exact values of the estimations from the last two questions.

   (f) Suppose that the input is now a unit step again. What will the new value of the 2% settling time be? *Hint:* is the system linear or non-linear?

9. Sketch the following step responses, marking, whenever they exist,

   - the settling time according to the 5% criterion,
   - the settling time according to the 2% criterion,
   - the steady-state value.

   (a) $G(s) = \dfrac{15}{s+5}$, for input $u(t) = 4H(t)$

   (b) $G(s) = \dfrac{10}{s-1}$, for input $u(t) = H(t)$

   (c) $G(s) = \dfrac{1}{2s+1}$, for input $u(t) = -H(t)$

   (d) $G(s) = \dfrac{-2}{4s+1}$, for input $u(t) = 10H(t)$

   (e) $G(s) = \dfrac{10}{s}$, for input $u(t) = H(t)$

10. Sketch the Bode diagrams of the following transfer functions, indicating

   - the gain for low frequencies,
   - the frequency at which the gain is 3 dB below the gain for low frequencies,
   - the slope of the gain for high frequencies,
   - the phase for low frequencies,
   - the phase for high frequencies,

- the frequency at which the phase is the average of those two values.

*Hint:* you do not need to draw the exact evolution of the phase; approximate it by three straight lines: a horizontal one for low frequencies, another horizontal one for high frequencies, and then connect these two by a straight line two decades wide. You can also approximate the gain by two straight lines, but do not forget to mark the frequency at which the gain has decreased 3 dB.

(a) $G(s) = \dfrac{15}{s+5}$

(b) $G(s) = \dfrac{1}{s+10}$

(c) $G(s) = \dfrac{1}{2s+1}$

(d) $G(s) = \dfrac{2}{4s+1}$

(e) $G(s) = \dfrac{10}{s}$

11. Find analytically the unit step responses of $G_1(s) = \dfrac{100}{s+10}$ and $G_2(s) = \dfrac{s+100}{s+10}$. Sketch them both in the same plot, marking the 5% settling time for each. Plot separately the difference between them. Then do the same for $G_3(s) = \dfrac{8}{s+12}$ and $G_4(s) = \dfrac{s+8}{s+12}$.

12. Let $G(s) = \dfrac{1}{s+1}$.

    (a) Consider the unit step response of $G(s)$. What is the settling time, according to the 5% criterion?

    (b) Find analytically the unit ramp response $y(t)$ of $G(s)$.

    (c) Find the analytical expression of the steady-state $y_{ss}(t)$ of that response $y(t)$.

    (d) How long does it take for $\left|\frac{y_{ss}(t)-y(t)}{y(t)}\right|$ to be less than 5%? In other words, find how long it takes for the unit ramp response to be within a 5% wide band around its steady state.

13. A first order system $\frac{K}{s+p}$ has the response tabulated in Table 10.2, when its input is a unit step applied at instant $t = 0.5$ s. Find the gain $K$ and the pole $p$. *Hint:* subtract the response from the steady state value; you should now have an exponential with a negative power. Plot its logarithm and adjust a straight line.

14. Prove that, if $G(s) = \dfrac{b_0}{s^2 + a_1 s + a_0}$ is stable, its step response has derivative zero at $t = 0$. Do this as follows:

    (a) Use a table of Laplace transforms to find the unit step response of $\frac{1}{(s+a)(s+b)}$. Calculate its derivative, proving thus the thesis for the case of two real poles.

Table 10.2: Unit step response of Exercise 13.

| time | output |
|---|---|
| 0.0 | 0.0000 |
| 0.1 | 0.0000 |
| 0.2 | 0.0000 |
| 0.3 | 0.0000 |
| 0.4 | 0.0000 |
| 0.5 | 0.0000 |
| 0.6 | 0.0632 |
| 0.7 | 0.0865 |

| time | output |
|---|---|
| 0.8 | 0.0950 |
| 0.9 | 0.0982 |
| 1.0 | 0.0993 |
| 1.1 | 0.0998 |
| 1.2 | 0.0999 |
| 1.3 | 0.1000 |
| 1.4 | 0.1000 |
| 1.5 | 0.1000 |

(b) Use a table of Laplace transforms to find the unit step response of $\frac{1}{(s+a)^2}$. Calculate its derivative, proving thus the thesis for the case of a double real pole.

(c) Use a table of Laplace transforms to find the unit step response of $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$. Calculate its derivative, proving thus the thesis for the case of a double real pole.

15. For each of the transfer functions below, and for the corresponding step input, find, if they exist:

- the natural frequency $\omega_n$ and the damping factor $\xi$,
- the steady state value $y_{ss}$,
- the delay time $t_d$ and the rise time $t_r$,
- the peak time $t_p$ and the maximum overshoot $M_p$ (expressed in percentage),
- the 5% and the 2% settling times (use the expressions for the exponential envelope of the oscillations),
- the location of the poles,

and sketch the step response.

(a) $G(s) = \dfrac{7}{s^2 + 0.4s + 1}$, for input $u(t) = 0.1H(t)$

(b) $G(s) = \dfrac{1}{s^2 + 5.1s + 9}$, for input $u(t) = 18H(t)$

(c) $G(s) = \dfrac{1}{2s^2 + 8}$, for input $u(t) = H(t)$

(d) $G(s) = \dfrac{10}{s^2 - s + 1}$, for input $u(t) = 2H(t)$

(e) $G(s) = \dfrac{0.3}{s^2 + 4s - 1}$, for input $u(t) = 15H(t)$

16. Sketch the Bode diagrams of the following transfer functions, indicating

- the gain for low frequencies,
- the resonant peak value, and the frequency at which it is located, if indeed there is one,

- the slope of the gain for high frequencies,
- the phase for low frequencies,
- the phase for high frequencies,
- the frequency at which the phase is the average of those two values.

*Hint:* you do not need to draw the exact evolution of the phase; approximate it by three straight lines: a horizontal one for low frequencies, another horizontal one for high frequencies, and then connect these two by a straight line two decades wide. You can also approximate the gain by two straight lines, but if there is a resonant peak mark it in your plot.

(a) $G(s) = \frac{1}{s^2 + 20s + 100}$

(b) $G(s) = \frac{7}{s^2 + 0.4s + 1}$

(c) $G(s) = \frac{1}{s^2 + 5.1s + 9}$

(d) $G(s) = \frac{1}{2s^2 + 8}$

17. Find the second order transfer functions that, for a unit step input, have:

    (a) $t_p = 0.403$ s, $M_p = 16.3\%$, $y_{ss} = 0.8$

    (b) $t_p = 0.907$ s, $y(t_p) = 11.63$, $y_{ss} = 10$

    (c) $t_r = 0.132$ s, $t_{s2\%} = 2.0$ s, $y_{ss} = 0.5$

18. Consider the mechanical system in Figure 10.19. When $f(t) = 8.9$ N, $t \geq 0$, the output has $t_p = 1$ s, $M_p = 9.7\%$, and $y_{ss} = 3 \times 10^{-2}$ m.

    (a) Find the values of mass $M$, viscous damping coefficient $B$, and spring stiffness $K$.

    (b) Suppose we want the same steady-state regime and the same settling time, but a maximum overshoot of 0.15%. What should the new values of $M$, $B$ and $K$ be?

19. Plot the Bode diagrams of the following plants:

    (a) $G(s) = \frac{-4s + 20}{s^3 + 0.4s^2 + 4s}$

    (b) $\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 16\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 65\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 50y(t) = 100\frac{\mathrm{d}u(t)}{\mathrm{d}t} + 50u(t)$

    (c) $G(s) = \frac{120(s+1)}{s(s+2)^2(s+3)}$

    (d) $G(s) = \frac{s^2}{(s+0.5)(s+10)}$

    (e) $G(s) = \frac{10s}{(s+10)(s^2+s+2)}$

    (f) $G(s) = \frac{(s+4)(s+20)}{(s+1)(s+80)}$

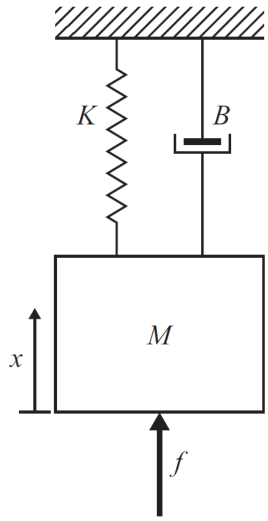



Figure 10.19: System of Exercise 18.

20. Establish a correspondence between the three Bode diagrams and the three unit step responses in Figure 10.20.

21. Establish a correspondence between the three Bode diagrams and the three unit step responses in Figure 10.21.

22. Find the transfer functions corresponding to the Bode diagrams in Figure 10.22.

23. Consider the following transfer functions:

$$G_1(s) = \frac{5050s + 10000}{s^2 + 101s + 100} \tag{10.93}$$

$$G_2(s) = \frac{100s + 10000}{s^2 + 101s + 100} \tag{10.94}$$

    (a) Find their poles.
    (b) Which pole is faster? Why?
    (c) Which of the two transfer functions will respond faster to a unit step? Why?

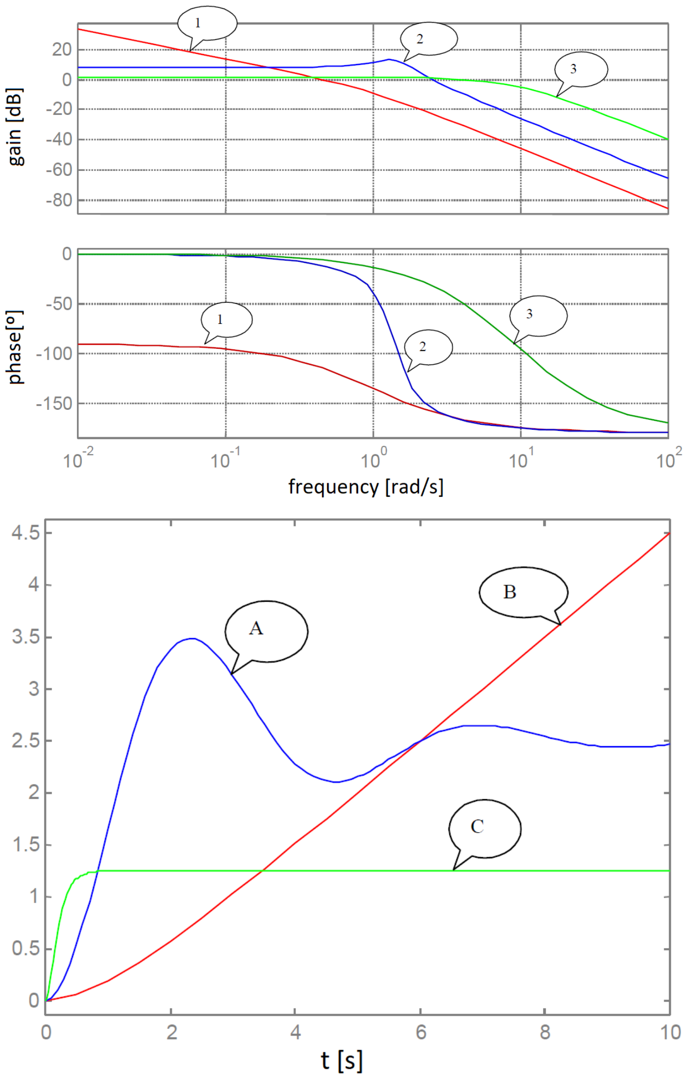



Figure 10.20: Bode diagrams and unit step responses of Exercise 20.

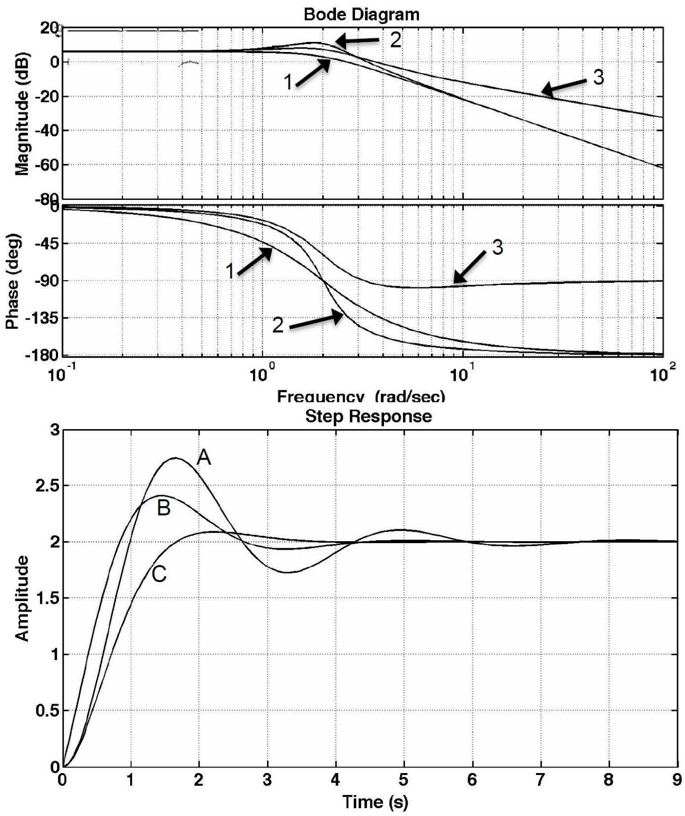



Figure 10.21: Bode diagrams and unit step responses of Exercise 21.

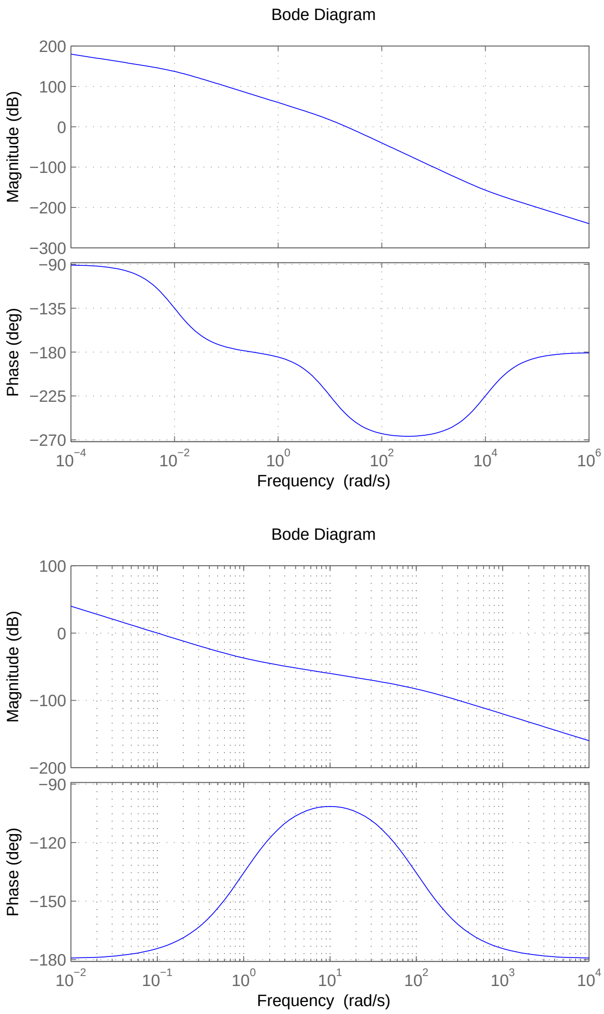



Figure 10.22: Bode diagrams of Exercise 22.