HYDRODYNAMIC BEHAVIOR OF SSEP WITH CURRENT RESERVOIRS

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ABSTRACT. The purpose of this article is to provide a simple proof of the hydrodynamic and hydrostatic behavior of the symmetric simple exclusion process in contact with reservoirs whose current is fixed. More precisely, the reservoirs inject/remove particles at/from any point of a window of size K placed at each extremity of the bulk and particles are injected/removed to the first open/occupied position. The proof of the Hydrodynamic Limit is based on the entropy method of [17], while the proof of the Hydrostatic Limit is based on the recent method developed in [25] and [21].

1. INTRODUCTION

We consider the Symmetric Simple Exclusion Process (SSEP) in the discrete box $\{1,\dots,N-1\}$ coupled with slow reservoirs, placed at $x=0$ and $x=N$, whose role is to inject and and remove particles in a window of size $K\geqslant 1$. A particle may enter to the first free site and leave from the first occupied site in its respective window (*i.e.,* $\{1,\dots K\},\{N-K,\dots,N-1\}$). We control the action of the reservoirs by fixing the rates of injection/removal as proportional to $jN^{-\theta}$. We address here the characterization of the hydrodynamic and hydrostatic behavior for values $\theta\geqslant 1$. We show that the spatial density of particles is given by a weak solution of the heat equation with non-linear (resp. linear) Robin boundary conditions, if $\theta=1$ and $K\geqslant 2$ (resp. $K=1$, in which we recover the results of [1]), and Neumann boundary conditions, if $\theta>1$ for any $K\geqslant 1$. For the case $\theta=1$, the irreversibility of the boundary dynamics reflects on a non-linear macroscopic boundary evolution for $K\geqslant 1$. The model where particles may enter only through the right and leave only through the left with rates $\frac{j}{2}$ was first introduced by De Masi *et al* in [5], and the reservoirs were termed "current reservoirs". In [5], the dynamics was shown to have the Propagation of Chaos property, which was shown by providing sharp estimates on the $v-$functions. As a consequence, the Fick's Law was shown to hold and the Hydrostatic Limit was proved in [7] and [6], respectively.

When $K=1$, we are reduced to the SSEP with "linear" reservoirs, where the hydrodynamic and hydrostatic scenario were investigated in [1] for $\theta\geqslant 0$ and for $\theta<0$, the hydrodynamic behavior was studied in [12]. For $\theta\geqslant 0$, in [1], Baldasso *et al* showed the Hydrodynamic Limit by the application of the Entropy method, first presented in [17]. In their case, which corresponds here to a particular case when $K=1$ case, they are able to use an auxiliary measure which is product and given by a suitable profile and for that reason, the entropy production at the boundaries is small enough to enable them to show a replacement lemma at the boundaries. In the present paper, we apply a similar strategy for $\theta\geqslant 1$, but with an extra difficulty due to the explicit correlation terms at the boundaries, which makes us to use another replacement lemma. Unfortunately, that is not possible for $\theta<1$ since the measure that we could compare with, a measure close to the stationary state of the system, is quite far from being product. Due to the boundary terms of the dynamics, we are not able to control the entropy of the initial measure wrt any product measure and therefore we cannot apply the entropy method except in the case where the boundary is quite slow, that is, when $\theta\geqslant 1$. Having the hydrodynamic limit proved, it is simple to obtain the hydrostatic limit, by showing that the stationary correlations of the system, vanish as the system size grows to infinity. When $K=1$ that is exactly the strategy pursued in [1]. In our case, when $K\geqslant 2$ we do not have

Date: April 22, 2019.

any information about the stationary correlations of the system and for that reason we have to do it in a different way. Therefore, here the hydrostatic behavior is investigated through the methods developed in [25] and [21]. In particular, we will follow essentially [25], where the Hydrostatic Limit was shown for $K = 1$. The proof presented in [21] is robust enough for the Hydrostatic Limit to follow directly from the Hydrodynamic Limit when $\theta = 1$, thus we will focus on the case $\theta > 1$ and refer the interested reader to the aforementioned work and references therein. Our main interest is when $\theta > 1$, where the macroscopic evolution is governed by a Neumann Laplacian on $[0,1]$. In contrast to the arguments in [1], where the Hydrostatic Limit was shown through estimates on the density and correlation fields, the method in [25] is based on the study of the system's evolution under a "sub-diffusive" time scale. This allows us to show replacement lemmas that under a different time scale were not possible. In this sense, our results regarding the Hydrostatic Limit also extend the ones obtained in [1] for $\theta \geqslant 1$ by the application of a simpler method and when correlation estimates are not easy to obtain.

Regarding the results of the present paper, as already mentioned, the model expresses a macroscopic phase transition from non-linear Robin to Neumann boundary conditions. In particular, we derive the following hydrodynamic equation when $\theta = 1$

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u), \quad (t,u) \in [0,T] \times (0,1), \\ \partial_u \rho_t(0) = -j \sum_{x=1}^{K} (\alpha_x (1-\rho_t(0)) \rho_t^{x-1}(0) - \gamma_x \rho_t(0)(1-\rho_t(0))^{x-1}), \quad t \in [0,T], \\ \partial_u \rho_t(1) = j \sum_{x=1}^{K} (\beta_x (1-\rho_t(1)) \rho_t^{x-1}(1) - \delta_x \rho_t(1)(1-\rho_t(1))^{x-1}), \quad t \in [0,T], \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u), \quad (t,u) \in [0,T] \times (0,1), \\ \partial_u \rho_t(0) = \partial_u \rho_t(1) = 0, \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases} \tag{2}$$

when $\theta > 1$. We also remark that our results generalize those of [5] and our proof is much simpler and does not require any knowledge on the v-functions. Our proof is quite simple and relies on good estimes between Dirichlet forms and carré du champ operator and a fez replacement lemmas which allow to control the boundary terms. Throught the paper we will state the results for $K > 1$, but we do the proofs in detail for $K = 2$ only, since for $K > 2$ the techniques are the same and the biggest change is in the notation. Nevertheless, we will state some appropriate remarks regarding the general case $K > 2$. For $\beta_x = \gamma_x = \frac{1}{2}$ and $\delta_x = \alpha_x = 0$, the uniqueness for the Cauchy problem (1) was shown in [7]. For $K = 2$ with $\alpha_2 = \gamma_2$ and $\beta_2 = \delta_2$ the proof reduces to the case of linear Robin boundary conditions, studied in [1]. In the present work we will assume uniqueness of the weak solution for the remaining parameters in (1).

There are several important issues that are left open. The first one is to obtain the hydrodynamic limit in the case $\theta < 1$ for which the arguments of [5] do not work. We believe that in this regime we should get the heat equation with Dirichlet boundary conditions. Hydrostatic limit is in this case is also open. Another important issue is related to the fluctuations around the hydrodynamic limit, for which we need to obtain very sharp estimates on the space-time correlations of the system. Large deviations from the stationary state is also another challenge to look at in the near future. Finally, in order to get information about the stationary state of the system, we also want to analyze the matriz ansazt method of Derrida [**?**], which does not straightforwardly applies to these dynamics. The article is divided as follows. In section 2 we present the model, notation, the weak formulation for the solution of the Cauchy problem and the main results, namely, the Hydrodynamic Limit (Theorem 2.8) and Hydrostatic Limit (Theorem 4.1). In section 3 we show the Hydrodynamic Limit. We start by an heuristic proof, identifying the main difficulties and tools to solve them. Then we proceed with the entropy method: in Proposition 3.1 we show tightness of the empirical measures, which shows that there are convergent subsequences. With the assumption of the uniqueness of the solution of (1), we proceed with the characterization of limit points. In particular, in Theorem 19 we show that the spatial density of particles converge to the solution of (1), where we use the Replacement Lemmas and Energy Estimate (which we postpone the proof to the Appendix A and B, respectively). The section 4 is organized similarly. First, we introduce some notation, then we proceed with the heuristics. Instead of studying the empirical measure, we will focus on the mass of the system, and derive an integral equation for this quantity. Then, we use the machinery developed for the Hydrodynamic Limit: in Proposition 4.4 we show tightness of the mass, and in Lemmas 4.5 and 4.6 we show the Replacement and Moving Particle Lemmas, respectively. We then show an analogous of Theorem 19 for the mass, and conclude the proof with a concentration result, whose proof is identical to [25], but short enough for us to present it here for completness.

2. The model

Denote by N a scaling parameter, which will be taken to infinity later on. For $N \geqslant 2$ we denote by $\Lambda_N = \{1,\dots,N-1\}$ the discrete set of points which we call the bulk. The exclusion process in contact with stochastic reservoirs is a Markov process, that we denote by $\{\eta_t : t \geqslant 0\}$, which has state space $\Omega_N := \{0,1\}^{\Lambda_N}$. The configurations of the state space Ω_N are denoted by η, so that for $x \in \Lambda_N$, $\eta(x) = 0$ means that the site x is vacant while $\eta(x) = 1$ means that the site x is occupied. Before defining the generator, let us introduce some notation. For fixed $K \in \mathbb{N}^+$, let $I_-^K = \{1,\dots,K\}$, $I_+^K = \{N-1-K,\dots,N-1\}$. Now for each $x \in \mathbb{N}^+$ define the subsets $I_-^K(x) = \{1,\dots,x\} \cap I_-^K$, $I_+^K(x) = \{x,\dots,N-1\} \cap I_+^K$, and for $g : \Lambda_N \to \mathbb{R}$ define

$$(\tau_\pm g)(x) := \prod_{y \in I_\pm^K(x)} g(y).$$

The infinitesimal generator

$$\mathscr{L}_N = \mathscr{L}_{N,0} + \tfrac{1}{N^\theta} \mathscr{L}_{N,b} \tag{3}$$

is given on functions $f : \Omega_N \to \mathbb{R}$ by

$$(\mathscr{L}_{N,0} f)(\eta) = \sum_{x=1}^{N-2} \Big(f(\eta^{x,x+1}) - f(\eta) \Big),$$

$$(\mathscr{L}_{N,b} f)(\eta) = (\mathscr{L}_{N,-} f)(\eta) + (\mathscr{L}_{N,+} f)(\eta)$$

where

$$(\mathscr{L}_{N,-} f)(\eta) = \sum_{x \in I_-^K} \{\alpha_x (1-\eta(x))(\tau_- \eta)(x-1) + \gamma_x \eta(x)(\tau_-(1-\eta))(x-1)\}(f(\eta^x) - f(\eta))$$

and

$$(\mathscr{L}_{N,b} f)(\eta) = \sum_{x \in I_+^K} \{\beta_x (1-\eta(x))(\tau_- \eta)(x+1) + \delta_x \eta(x)(\tau_-(1-\eta))(x+1)\}(f(\eta^x) - f(\eta)),$$

with $(1-\eta)(y) \equiv 1 - \eta(y)$ and

$$\eta^{x,y}(z) = \begin{cases} \eta(z),\, z \neq x, y, \\ \eta(y),\, z = x, \\ \eta(x),\, z = y \end{cases} \quad , \quad \eta^x(z) = \begin{cases} \eta(z),\, z \neq x, \\ 1-\eta(x),\, z = x. \end{cases} \tag{4}$$

For the sake of exposition, from now on let $\beta_{N-y} \equiv \beta_y$ and $\delta_{N-y} \equiv \delta_y$. We consider the Markov process speeded up in the time scale N^2 and we note that the process $(\eta_{tN^2})_{t \geqslant 0}$ has infinitesimal generator given by $N^2 \mathscr{L}_N$.

2.1. **Hydrodynamic equation.** From now on we fix a finite time horizon $[0,T]$. We denote by $\langle\cdot,\cdot\rangle_\mu$ the inner product in $L^2([0,1])$ with respect to a measure μ defined in $[0,1]$ and $\|\cdot\|_{L^2(\mu)}$ is the corresponding norm. We note that when μ is the Lebesgue measure we write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_{L^2}$ for the corresponding norm.

We denote by $C^{m,n}([0,T]\times[0,1])$ the set of functions defined on $[0,T]\times[0,1]$ that are m times differentiable on the first variable and n times differentiable on the second variable, wit continuous derivatives. For a function $G:=G(s,u)\in C^{m,n}([0,T]\times[0,1])$ we denote by $\partial_s G$ its derivative with respect to the time variable s and and by $\partial_u G$ its derivative with respect to the space variable u. For simplicity of notation we set $\Delta G:=\partial_u^2 G$.

Now we want to define the space where the solutions of the hydrodynamic equations will live on, namely the Sobolev space $\mathscr{H}_1$ on $[0,1]$. For that purpose, we define the semi inner-product $\langle\cdot,\cdot\rangle_1$ on the set $C^\infty([0,1])$ by

$$\langle G,H\rangle_1=\int_0^1(\partial_u G)(u)(\partial_u H)(u)\,du \tag{5}$$

and the corresponding semi-norm is denoted by $\|\cdot\|_1$.

Definition 2.1. *The Sobolev space $\mathscr{H}^1$ on $[0,1]$ is the Hilbert space defined as the completion of $C^\infty([0,1])$ for the norm*

$$\|\cdot\|^2_{\mathscr{H}^1}:=\|\cdot\|^2_{L^2}+\|\cdot\|^2_1.$$

Its elements elements coincide a.e. with continuous functions.

The space $L^2(0,T;\mathscr{H}^1)$ is the set of measurable functions $f:[0,T]\to\mathscr{H}^1$ such that

$$\int_0^T\|f_s\|^2_{\mathscr{H}^1}\,ds<\infty.$$

We can now give the definition of the weak solutions of the hydrodynamic equations that will be derived for the process described above when $\theta=1$.

Definition 2.2. *Let $\rho_0:[0,1]\to[0,1]$ be a measurable function. We say that $\rho:[0,T]\times[0,1]\to[0,1]$ is a weak solution of the heat equation with the non-linear Robin boundary conditions*

$$\begin{cases}\partial_t\rho_t(u)=\Delta\rho_t(u), & (t,u)\in[0,T]\times(0,1),\\ \partial_u\rho_t(0)=-j\sum_{x=1}^K(\alpha_x(1-\rho_t(0))\rho_t^{x-1}(0)-\gamma_x\rho_t(0)(1-\rho_t(0))^{x-1}), & \\ \partial_u\rho_t(1)=j\sum_{x=1}^K(\beta_x(1-\rho_t(1))\rho_t^{x-1}(1)-\delta_x\rho_t(1)(1-\rho_t(1))^{x-1}), & t\in[0,T]\\ \rho(0,\cdot)=\rho_0(\cdot), & \end{cases} \tag{6}$$

if the following two conditions hold:

1. $\rho\in L^2(0,T;\mathscr{H}^1)$,

2. ρ satisfies the weak formulation:

$$\begin{aligned}F_{\mathscr{R}}:=&\langle\rho_t,G_t\rangle-\langle\rho_0,G_0\rangle-\int_0^t\langle\rho_s,\Big(\Delta+\partial_s\Big)G_s\,ds\rangle\\ &+\int_0^t\Big\{\rho_s(1)\partial_u G_s(1)-\rho_s(0)\partial_u G_s(0)\Big\}\,ds\\ &-j\sum_{x=1}^K\int_0^t\Big\{G_s(1))(\beta_x(1-\rho_t(1))\rho_t^{x-1}(1)-\delta_x\rho_t(1)(1-\rho_t(1))^{x-1})\,ds\Big\}\\ &-j\sum_{x=1}^K\int_0^t\Big\{G_s(0)(\alpha_x(1-\rho_t(0))\rho_t^{x-1}(0)-\gamma_x\rho_t(0)(1-\rho_t(0))^{x-1})\Big\}\,ds=0,\end{aligned} \tag{7}$$

for all $t\in[0,T]$, any function $G\in C^{1,2}([0,T]\times[0,1])$.

Remark 2.3. *Observe that in the case $j=0$ the equation above is the heat equation with Neumann boundary conditions.*

Remark 2.4. *For $\beta_x=\frac{1}{2}=\gamma_x$ and $\delta_x=\alpha_x=0$ with $x=1,\dots,K$ we recover the boundary conditions obtained in* [5].

Remark 2.5. *Observe that for $K=2$ the equation (6) can be rewritten as*

$$\begin{cases} \partial_t\rho_t(u)=\Delta\rho_t(u), \quad (t,u)\in[0,T]\times(0,1),\\ \partial_u\rho_t(1)=j(\beta_1-(\beta_1+\delta_1)\rho_t(1)+(\delta_2-\beta_2)(\rho_t^2(1)-\rho_t(1))),\\ \partial_u\rho_t(0)=j(\rho_t(0)(\alpha_1+\gamma_1)-\alpha_1-(\gamma_2-\alpha_2)(\rho_t^2(0)-\rho_t(0)), \quad t\in[0,T]\\ \rho(0,\cdot)=\rho_0(\cdot), \end{cases} \tag{8}$$

and for $\alpha_2=\gamma_2$ and $\beta_2=\delta_2$ we recover the linear Robin boundary conditions as in [1] *and when $\alpha_2=\gamma_2=\beta_2=\delta_2=0$ and $\beta_1=1-\delta_1=\beta$ and $\alpha_1=1-\alpha_1=\alpha$ we deal with exactly the same model of* [1] *and we recover their result.*

Remark 2.6. *We observe that the weak solution of* (7) *when $j=0$ (corresponding to Neumann boundary conditions) and when $j\neq 0$ and $\alpha_2=\gamma_2$ and $\beta_2=\delta_2$ (corresponding to linear Robin boundary conditions), is unique. We refer the reader to* [1] *for a proof. In what follows we assume uniqueness of the weak solution in the other range of the parameters which are not treated in the previous articles.*

2.2. **Hydrodynamic Limit.** In this section we want to state the hydrodynamic limit of the process $\{\eta_{tN^2}\}_{t\geqslant 0}$ with state space Ω_N and with infinitesimal generator $N^2\mathcal{L}_N$ defined in (3). For that purpose, let $\mathcal{M}^+$ be the space of positive measures on $[0,1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta\in\Omega_N$ we define the empirical measure $\pi^N(\eta,du)$ on $[-1,1]$ by

$$\pi^N(\eta,du)=\frac{1}{N-1}\sum_{x\in\Lambda_N}\eta(x)\delta_{\frac{x}{N}}(dq), \tag{9}$$

where δ_a is a Dirac mass on $a\in[0,1]$, and

$$\pi_t^N(\eta,du):=\pi^N(\eta_{tN^2},du).$$

This measure gives weight $\frac{1}{N}$ to each occupied site of the configuration η.

Fix $T>0$ and $\theta\in\mathbb{R}$. We denote by $\mathbb{P}_{\mu_N}$ the probability measure in the Skorohod space $\mathcal{D}([0,T],\Omega_N)$ induced by the Markov process $\{\eta_{tN^2}\}_{t\geqslant 0}$ and the initial probability measure μ_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$. Let $\{\mathbb{Q}_N\}_{N\geqslant 1}$ be the sequence of probability measures on $\mathcal{D}([0,T],\mathcal{M}^+)$ induced by the Markov process $\{\pi_t^N\}_{t\geqslant 0}$ and by $\mathbb{P}_{\mu_N}$.

At this point we need to fix an initial profile ρ_0 and an initial probability measure μ_N. Therefore, let $\rho_0:[0,1]\to[0,1]$ be a measurable function.

Definition 2.7. *We say that a sequence of probability measures $\{\mu_N\}_{N\geqslant 1}$ in Ω_N is associated to the profile ρ_0 if for any continuous function $G:[0,1]\to\mathbb{R}$ and every $\delta>0$*

$$\lim_{N\to\infty}\mu_N\left(\eta\in\Omega_N:\left|\frac{1}{N-1}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\eta(x)-\int_0^1 G(u)\rho_0(u)du\right|>\delta\right)=0. \tag{10}$$

Note that (10) can be written as

$$\int_{\Omega_N}G(u)\pi^N(\eta,du)\longrightarrow_{N\to\infty}\int_0^1 G(u)\rho_0(u)du \quad \text{wrt } \mu_N, \tag{11}$$

which means that the empirical measure at time $t=0$ converges in probability with respect to μ_N, as $N\to\infty$, to the deterministic measure $\rho_0(u)du$.

The hydrodynamic limit that we want to derive states that the previous result is also true for any $t \in [0,T]$, that is, the empirical measure at time t converges in probability with respect to the distribution of the system at time t, as $N \to \infty$, to the deterministic measure $\rho_t(u)du$, where ρ_t is a solution (here in the weak sense) to the partial differential equation given above.

Theorem 2.8. *Let $\rho : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geqslant 1}$ be a sequence of probability measures in Ω_N associated to ρ_0, see (10). Then, for any $t \in [0,T]$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\eta^N_\cdot : \Big|\frac{1}{N-1}\sum_{x\in\Lambda_N} G\big(\tfrac{x}{N}\big)\eta_{tN^2}(x) - \int_0^1 G(u)\rho_t(u)du\Big| > \delta\Big) = 0,$$

where $\rho_t(\cdot)$ is the unique weak solution of (6).

The proof of Theorem 2.8 splits into showing first the tightness of the sequence $\{\mathbb{Q}_N\}_{N\geqslant 1}$ and then characterizing the limiting point $\mathbb{Q}$. These two results combined imply the convergence of $\{\mathbb{Q}_N\}_{N\geqslant 1}$ to $\mathbb{Q}$ as $N \to \infty$. The characterization of the limit point $\mathbb{Q}$ is done according to the following steps: first, we prove that all limiting points of the sequence $\{\mathbb{Q}_N\}_{N\geqslant 1}$ are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure (this is a consequence of the exclusion dynamics), whose density $\rho_t(u)$ is a weak solution of the hydrodynamic equation. From the assumption of the uniqueness of the weak solutions of this equation, we conclude that $\{\mathbb{Q}_N\}_{N\geqslant 1}$ has a unique limiting point $\mathbb{Q}$, and therefore we conclude the convergence to this limit point.

2.3. **Hydrostatic Limit.** We denote by μ_N^{ss} the unique invariant measure, whose existence comes from the cardinality of the state space Ω_N being finite, and the Markov process corresponding to $\mathscr{L}_N$ being irreducible. The Hydrostatic Limit states that, as the evolution of the spatial density of particles is associated to the solution of the Hydrodynamic Equation, under the stationary state the density of particles is associated to the *stationary* solution of the Hydrodynamic Equation.

Theorem 2.9 (Hydrostatic Limit)**.** *Let μ_N^{ss} be the probability measure in Ω_N invariant for the Markov process with infinitesimal generator $N^2\mathscr{L}_N$. Then, for any $H \in C[0,1]$ and $\delta > 0$ the sequence μ_N^{ss} is associated to the profile ρ_θ, that is*

$$\lim_{N\to\infty} \mu_N^{ss}\left(\eta : \left|\frac{1}{N-1}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\eta(x) - \int_0^1 H(u)\rho_\theta(u)du\right| \geqslant \delta\right) = 0,$$

where for $\theta = 1$, ρ_θ is the unique stationary solution for the heat equation with Robin boundary conditions (1), while for $\theta > 1$, ρ_θ is a solution of the heat equation with Neumann boundary conditions (2).

3. Hydrodynamic Limit

3.1. **Heuristics.** Let us fix a test function $G \in C^{1,2}([0,T]\times[0,1])$. A simple computation shows that for $s \in [0,T]$ it holds that

$$\begin{aligned} N^2\mathscr{L}_{N,0}\langle \pi_s^N, G_s\rangle = \langle \pi_s^N, \Delta_N G_s\rangle \\ + \nabla_N^+ G_s(0)\eta_{sN^2}(1) - \nabla_N^- G_s(1)\eta_{sN^2}(N-1). \end{aligned} \tag{12}$$

For the boundary action, note that

$$\begin{aligned} \tfrac{j}{N^\theta} N^2 \mathscr{L}_{N,b}^L\langle \pi_s^N, G_s\rangle = jN^{1-\theta} G_s(\tfrac{1}{N})(\alpha_1 - (\alpha_1+\gamma_1)\eta(1)) \\ + jN^{1-\theta} G_s(\tfrac{N-1}{N})(\beta_1 - (\beta_1+\delta_1)\eta(N-1)) \end{aligned}$$

and

$$\tfrac{j}{N^\theta}N^2\mathscr{L}^{NL}_{N,b}\langle\pi^N_s,G_s\rangle = jN^{1-\theta}G_s(\tfrac{2}{N})(\alpha_2\eta(1)-\gamma_2\eta(2)-(\alpha_2-\gamma_2)\eta(1)\eta(2))$$
$$+jN^{1-\theta}G_s(\tfrac{N-2}{N})(\delta_2\eta(N-2)-\beta_2\eta(N-1)-(\delta_2-\beta_2)\eta(N-1)\eta(N-2)).$$

From the previous computations, we obtain that

$$\begin{aligned}
&\mathscr{M}^N_t(G) = \langle\pi^N_t,G_t\rangle-\langle\pi^N_0,G_0\rangle-\int_0^t\langle\pi^N_s,(\partial_s+\Delta_N)G_s\rangle\,ds\\
&-\int_0^t \nabla^+_N G_s(0)\eta_{sN^2}(1)-\nabla^-_N G_s(1)\eta_{sN^2}(N-1)\,ds\\
&-\int_0^t jN^{1-\theta}G_s(\tfrac{1}{N})(\alpha_1-(\alpha_1+\gamma_1)\eta_{sN^2}(1))+jN^{1-\theta}G_s(\tfrac{N-1}{N})(\beta_1-(\beta_1+\delta_1)\eta_{sN^2}(N-1))\,ds \qquad (13)\\
&-\int_0^t jN^{1-\theta}G_s(\tfrac{2}{N})(\alpha_2\eta_{sN^2}(1)-\gamma_2\eta_{sN^2}(2)-(\alpha_2-\gamma_2)\eta_{sN^2}(1)\eta_{sN^2}(2))ds\\
&-\int_0^t jN^{1-\theta}G_s(\tfrac{N-2}{N})(\delta_2\eta_{sN^2}(N-2)-\beta_2\eta_{sN^2}(N-1)-(\delta_2-\beta_2)\eta_{sN^2}(N-1)\eta_{sN^2}(N-2))ds
\end{aligned}$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$, where for each $t\geqslant 0$, $\mathscr{F}_t := \sigma(\eta(s): s<t)$. Above the notation $\langle\pi^N_s,G\rangle$ represents the integral of G with respect the measure π^N_s. In due course , we will show that the $\mathbb{E}_{\mu_N}\Big[\big(M^N_t(G)\big)^2\Big]$ vanishes as $N\to\infty$. Now we focus on the integral term above.

Let us start with the boundary term coming from the bulk, that is, the term on the second line of the previous display. Note that by Theorem A.5, with the choice $\psi\equiv 1$, since we are able to replace $\eta_{sN^2}(1)$ (resp. $\eta_{sN^2}(N-1)$) by the average in a box of size $\lfloor\varepsilon N\rfloor$ around 1 (resp. $N-1$):

$$\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1) := \frac{1}{\varepsilon N}\sum_{x=2}^{1+\varepsilon N}\eta_{sN^2}(x), \quad \overleftarrow{\eta}^{\varepsilon N}_{sN^2}(N-1) := \frac{1}{\varepsilon N}\sum_{x=N-2}^{N-1-\varepsilon N}\eta_{sN^2}(x), \qquad (14)$$

then, since

$$\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(1)\sim\rho_s(0) \quad \text{(resp.} \quad \overrightarrow{\eta}^{\varepsilon N}_{sN^2}(N-1)\sim\rho_s(1))$$

we arrive at

$$\int_0^t \partial_u G_s(0)\rho_s(0)-\partial_u G_s(1)\rho_s(1)\,ds.$$

Note that this term appears in the weak formulation. By abuse of notation, above and below εN denotes $\lfloor\varepsilon N\rfloor$, otherwise it makes no sense to talk about the average in a microscopic box of that size.

Now we analyse the terms coming from the boundary. We start with the terms on the third line on the right hand-side of (13). Note that when $\theta>1$, since G and η are bounded, these terms are of order $O(N^{1-\theta})$ and so they vanish as $N\to+\infty$. When $\theta=1$ and using again Theorem A.5, with the choice $\psi\equiv 1$, those terms are going to contribute to the integral formulation with

$$\int_0^t jG_s(0)(\alpha_1-(\alpha_1+\gamma_1)\rho_s(0))+jG_s(1)(\beta_1-(\beta_1+\delta_1)\rho_s(1))\,ds$$

Now we look at the fourth and fifth terms at the right hand-side of (13). We focus on the terms on the fourth line, but we note that the analysis is completely analogous for the terms in the fifth line. As before, for $\theta>1$ those terms are of order $O(N^{1-\theta})$ and so they vanish as $N\to+\infty$. When $\theta=1$, the terms that do not involve the product of $\eta(1)$ and $\eta(2)$ can be treated in the following way.

From Theorem A.4, with the choice $\varphi\equiv 1$, we can replace $\eta(2)$ by $\eta(1)$ and from Theorem A.5, with the choice $\psi\equiv 1$, we can then replace $\eta(1)$ by $\eta^{\varepsilon N}(1)$. Finally the term $\eta(1)\eta(2)$ can be treated as follows. First, from Theorem A.5, with the choice $\psi(\eta)=\eta(1)$, we can replace $\eta(2)$ by $\eta^{\varepsilon N}(1)$. So now, in the term $\eta(1)\eta^{\varepsilon N}(1)$, we replace $\eta(1)$ by $\eta^{\varepsilon N}(1)$ by applying Theorem A.5 with the choice $\psi(\eta)=\eta^{\varepsilon N}(1)$, which is invariant for all the exchanges $\eta^{z,z+1}$ for $z\in\{2,\cdots,\varepsilon N-1\}$. Form this we conclude that the terms on the fourth line of (13) contributes to the integral formulation with

$$\int_0^t jG_s(0)(\alpha_2-\gamma_2)(\rho_s^2(0)-\rho_s(0))\,ds.$$

3.2. **Proof of the Hydrodynamic Limit.**

3.2.1. *Tightness.* To show the tightness of $\{\mathbb{Q}^N\}_{N\geqslant 1}$ we will use *Aldous conditions* as in [18], Chapter 4.

Proposition 3.1. *The sequence $\{\mathbb{Q}^N\}_{N\geqslant 1}$ is tight under the Skorohod topology of $\mathscr{D}([0,T],\mathscr{M})$.*

Proof. By [18] (Chapter 4) we know that $\{\mathbb{Q}^N\}_{N\geqslant 1}$ is *relatively compact* if $\{\mathbb{Q}^{N,H}\}_N$ is *relatively compact* on $\mathscr{D}([0,T],\mathbb{R})$, where $H\in C^2[0,1]$ and $\mathbb{Q}^{N,H}$ is the probability measure induced by the map

$$(\mathscr{D}([0,T],\mathscr{M}),\mathbb{Q}^N)\ni\pi^N_\cdot\mapsto\psi(\pi^N_\cdot)=\langle\pi^N_\cdot,H\rangle\in(\mathscr{D}([0,T],\mathbb{R}),\mathbb{Q}^{N,H}).$$

Now we proceed to show the Aldous' conditions.

(1) $\forall t\in[0,T],\epsilon>0$, $\exists K_t(\epsilon)\subset\mathscr{S}$ compact such that

$$\sup_{N\geqslant 1}\mathbb{Q}^{N,H}\big(\langle\pi^N_\cdot,H\rangle\in\mathscr{D}([0,T],\mathbb{R})\backslash K_t(\epsilon)\big)<\epsilon.$$

Take $H\in C[0,1]$ and $\epsilon>0$. Then

$$\big|\langle\pi^N_t,H\rangle\big|=\left|\frac{1}{N}\sum_{x\in\Lambda_N}H(\tfrac{x}{N})\eta_t(x))\right|\leqslant\sup_{u\in[0,1]}|H(u)|=\|H\|_\infty.$$

In this way, choosing $K_t(\epsilon)=\overline{B_r(0)}$ with $r>\|H\|_\infty$ is enough.

$$\text{(2)}\quad\lim_{\gamma\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_N,\theta\leqslant\gamma}\mathbb{Q}\big(\langle\pi^N_\cdot,H\rangle\in\mathscr{D}_{\mathscr{R}}[0,T]:\big|\langle\pi^N_{\tau+\lambda},H\rangle-\langle\pi^N_\tau,H\rangle\big|>\epsilon\big)=0,\qquad(15)$$

with $\mathscr{T}_N$ the set of stopping times bounded by T. Note that by definition of $\mathbb{Q}^{N,H}$:

$$\begin{aligned}&\mathbb{Q}^{N,H}\big(\langle\pi^N_\cdot,H\rangle\in\mathscr{D}([0,T],\mathbb{R}):\big|\langle\pi^N_{\tau+\lambda},H\rangle-\langle\pi^N_\tau,H\rangle\big|>\epsilon\big)\\&=\mathbb{Q}^N\big(\pi^N_\cdot\in\mathscr{D}([0,T],\mathscr{M}):\big|\langle\pi^N_{\tau+\lambda},H\rangle-\langle\pi^N_\tau,H\rangle\big|>\epsilon\big)\\&=\mathbb{P}_{\mu^N}\big(\eta_\cdot\in\mathscr{D}([0,T],\Omega_N):\big|\langle\pi^N(\eta_{\tau+\lambda}),H\rangle-\langle\pi^N(\eta_\tau),H\rangle\big|>\epsilon\big).\end{aligned}\qquad(16)$$

Fixed H, and recalling that M_t^H is a martingal with respect to the natural filtration of η, summing and subtracting the appropriate terms we have that

$$\langle\pi^N_\tau,H\rangle-\langle\pi^N_{\tau+\lambda},H\rangle=M_\tau^{N,H}-M_{\tau+\lambda}^{N,H}-\int_\tau^{\tau+\lambda}N^2\mathscr{L}_N\langle\pi^N_s,H\rangle\,ds.$$

Thus we can bound the last term in (16) as follows

$$
\begin{aligned}
&\mathbb{P}_{\mu}^{N}\left(\eta_{\cdot}\in\mathscr{D}([0,T],\Omega_N):\left|\langle\pi_{\tau+\lambda}^{N},H\rangle-\langle\pi_{\tau}^{N},H\rangle\right|>\epsilon\right)\\
&\leqslant\mathbb{P}_{\mu}^{N}\left(\eta_{\cdot}\in\mathscr{D}([0,T],\Omega_N):\left|\int_{\tau}^{\tau+\lambda}N^2\mathscr{L}_N\langle\pi_s^N,H\rangle ds\right|+\left|M_{\tau}^{N,H}-M_{\tau+\lambda}^{N,H}\right|>\epsilon\right)\\
&\leqslant\mathbb{P}_{\mu}^{N}\left(\eta_{\cdot}\in\mathscr{D}([0,T],\Omega_N):\left|\int_{\tau}^{\tau+\lambda}N^2\mathscr{L}_N\langle\pi_s^N,H\rangle ds\right|>\epsilon/2\right)+\\
&+\mathbb{P}_{\mu}^{N}\left(\eta_{\cdot}\in\mathscr{D}([0,T],\Omega_N):\left|M_{\tau}^{N,H}-M_{\tau+\lambda}^{N,H}\right|>\epsilon/2\right).
\end{aligned}
\tag{17}
$$

Applying Chebyshev's inequality in the term on the third line of the previous display and Markov's inequality on the fourth line, (17) is bounded from above by:

$$
\leqslant\frac{1}{\epsilon}\mathbb{E}_{\mu^N}\left[\left|\int_{\tau}^{\tau+\lambda}N^2\mathscr{L}_N\langle\pi_s^N,H\rangle ds\right|\right]+\frac{1}{\epsilon^2}\mathbb{E}_{\mu^N}\left[\left(M_{\tau}^{N,H}-M_{\tau+\lambda}^{N,H}\right)^2\right].
$$

Now we work with the first term in the previous sum. Note that $\left|\triangle_N H(\frac{x}{N})\right|\leqslant 2\left\|H''\right\|_\infty$ and $\left|\nabla_N^{\pm}H(\frac{x}{N})\right|\leqslant\left\|H'\right\|_\infty$, where we used that $H\in C^2[0,1]$. Since there is at most one particle per site, we can bound

$$
\int_{\tau}^{\tau+\lambda}N^2\mathscr{L}_n\langle\pi_s^N,H\rangle ds\lesssim\int_{\tau}^{\tau+\lambda}\tfrac{1}{N^{\theta-1}}\left\|H'\right\|_\infty+\tfrac{1}{N^{\theta-1}}\left\|H''\right\|_\infty ds,
$$

where the notation $\lesssim$ means "less than a constant times". In this way, we have for $\theta\geqslant 1$

$$
\lim_{\gamma\to\infty}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\lambda\leqslant\gamma}\mathbb{E}_{\mu^N}\left[\left|\int_{\tau}^{\tau+\lambda}N^2\mathscr{L}_N\langle\pi_s^N,H\rangle ds\right|\right]=0.
$$

Now we need to show that

$$
\lim_{\gamma\to\infty}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\lambda\leqslant\gamma}\mathbb{E}_{\mu^N}\left[\left(M_{\tau}^{N,H}-M_{\tau+\lambda}^{N,H}\right)^2\right]=0.
\tag{18}
$$

It is well known that $(M_t^{N,H})^2-[M^{N,H}]_t$ is a (zero mean) martingale with respect to the natural filtration $\mathscr{F}_t^{\eta}$. The trick here is to write the expression (18) as a function of the quadratic variation and then bound it by a constant, similarly to what we have just done. From [20] (Appendix 1.6) we have that $[M^{N,H}]_t:=\int_0^t B_s^{N,H}ds$ is the quadratic variation of $M_t^{N,H}$, with

$$
B_s^{N,H}:=N^2\left(\mathscr{L}_N\langle\pi^N(\eta_s),H\rangle^2-2\langle\pi^N(\eta_s),H\rangle\mathscr{L}_N\langle\pi^N(\eta_s),H\rangle\right)
$$

where $B_s^{H,N}:=B_{s,-}^{H,N}+B_{s,0}^{H,N}+B_{s,+}^{H,N}$, each term corresponding to $\mathscr{L}_{N,-},\mathscr{L}_{N,0},\mathscr{L}_{N,+}$, respectively. In this way, we have that

$$
\mathbb{E}_{\mu^N}\left[\left(M_{\tau}^{N,H}-M_{\tau+\lambda}^{N,H}\right)^2\right]=\mathbb{E}_{\mu^N}\left[\int_{\tau}^{\tau+\lambda}B_s^{N,H}ds\right].
$$

Now we prooceed to bound $B_s^{H,N}$. The contribution from the bulk dynamics can be bounded as follows:

$$
\begin{aligned}
B_{s,0}^{N,H}&=N^2\sum_{x\in\{1,N-2\}}\left(\langle\pi^N(\eta_s^{x,x+1}),H\rangle-\langle\pi^N(\eta_s),H\rangle\right)^2\\
&=\sum_{x\in\{1,N-2\}}(\eta_s(x)-\eta_s(x+1))^2(H(\tfrac{x+1}{N})-H(\tfrac{x}{N}))^2\leqslant\tfrac{N-1}{N^2}\left\|(H')^2\right\|_\infty.
\end{aligned}
$$

We will only make the computations for the left boundary, since for the right it is analogous. Bounding the rates in the generator by a constant, we have

$$\begin{aligned}B^{N,H}_{s,-}&\lesssim j\frac{N^2}{N^\theta}\sum_{x\in I^-}\Big((\langle\pi^N(\eta^{(x)}_s),H\rangle^2-\langle\pi^N(\eta_s),H\rangle^2)-\\&-2\langle\pi^N(\eta_s),H\rangle(\langle\pi^N(\eta^{(x)}_s),H\rangle^2-\langle\pi^N(\eta_s),H\rangle^2)\Big)\\&=j\frac{N^2}{N^\theta}\sum_{x\in I^-}\Big(\langle\pi^N(\eta^{(x)}_s),H\rangle-\langle\pi^N(\eta_s),H\rangle\Big)^2\\&\lesssim\frac{j}{N^{\theta-1}}\|H\|^2_\infty.\end{aligned}$$

Analagoulsy, we have that $B^{N,H}_{s,+}\lesssim\frac{j}{N^{\theta-1}}\|H\|^2_\infty$, and

$$\lim_{\gamma\to\infty}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\lambda\leqslant\gamma}\mathbb{E}_{\mu^N}\int_\tau^{\tau+\lambda}B^{N,H}_s ds=0.$$

We conclude that $\{\mathbb{Q}^{N,H}\}_{N\geqslant1}$ is tight $\forall H\in C^2[0,1]$, and therefore also $\{\mathbb{Q}^N\}_{N\geqslant1}$ is in $\mathcal{D}([0,T],\mathcal{M})$. □

3.2.2. *Characterization of limit points.* Now that we know that $\{\mathbb{Q}^N\}_{N\geqslant1}$ has limit points, in this subsection we will characterize them. We will start by showing that every limit point is concentrated on absolutely continuous trajectories with respect to the Lebesgue measure (that is, $\pi^N_t(du)\longrightarrow\pi_t(du)=\rho_t(u)du$). To see this, we will apply Portmanteau's theorem. We also recall the following lemma.

Lemma 3.2. *If a measure μ is such that $\forall G\in C[0,1]$ we have $|\langle\mu,G\rangle|\leqslant\int_0^t|G(u)|du$, then μ is absolutely continuous with respect to the Lebesgue measure.*

After that, we will show that the $\mathbb{Q}$—measure gives full weight to trajectories $\pi_\cdot\in\mathcal{D}([0,T],\mathcal{M})$, where $\rho_t(u)$ is weak solution to the heat equation with either Robin or Neumann boundary conditions, depending on the value of θ.

Proposition 3.3. *Let $\lim_{N\to\infty}\{\mathbb{Q}^N\}_{N\geqslant1}=\mathbb{Q}$. Then $\mathbb{Q}$ is concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure.*

Proof. For fixed $H\in C[0,1]$, define the map $\pi^N_\cdot\mapsto\Theta(\pi^N_\cdot)=\sup_{0\leqslant t\leqslant T}\left|\langle\pi^N_t,H\rangle\right|$. Bounding $\eta_t(x)\leqslant1$ we have

$$\Theta(\pi^N_\cdot)\leqslant\frac{1}{N}\sum_{x\in\Lambda_N}\left|H(\tfrac{x}{N})\right|\Leftrightarrow\mathbb{Q}^N\left(\pi_\cdot\in\mathcal{D}([0,T],\mathcal{M}):\Theta(\pi_\cdot)\leqslant\frac{1}{N}\sum_{x\in\Lambda_N}\left|H(\tfrac{x}{N})\right|\right)=1.$$

By continuity of H, for N large enough the sum above is close to an integral, and we may write

$$\mathbb{Q}^N\left(\pi_\cdot\in\mathcal{D}([0,T],\Omega_N):\Theta(\pi_\cdot)\leqslant\int_0^t|H(u)|du+\epsilon\right)=1.$$

If we show that for fixed ϵ the set $A_\epsilon:=\{\pi_\cdot\in\mathcal{D}([0,T],\Omega_N):\Theta(\pi_\cdot)\leqslant\int_0^t|H(u)|du+\epsilon\}$ is closed with respect to the *Skorohod* topology, then we can apply *Portmanteau's theorem* to get $\mathbb{Q}(A_\epsilon)\geqslant\limsup_{N\to\infty}\mathbb{Q}^N(A_\epsilon)=1$, which clearly implies that $\mathbb{Q}(A_\epsilon)=1$. To check that A_ϵ is closed we will show that any sequence in A_ϵ has limit in A_ϵ. Thus, let $\pi^N_\cdot\xrightarrow{N\to\infty}\pi_\cdot$ in the *Skorohod topology*, where $\{\pi^N_\cdot\}_{N\geqslant1}\in A_\epsilon$ and $\pi_\cdot\in\mathcal{D}([0,T],\mathcal{M})$.

In particular, by [?] we have that $\forall s < T \quad \pi^N_{\cdot} \xrightarrow{N\to\infty} \pi_{\cdot} \Rightarrow \pi^N_s \xrightarrow{N\to\infty} \pi_s$. Taking a sequence $(t_k)_k \searrow t$ such that $\forall k \geqslant 1 \quad \pi^N_{t_k} \xrightarrow{N\to\infty} \pi_{t_k}$ we have

$$\epsilon + \int_0^1 |H(u)|du \geqslant \left|\langle \pi_{t_k}, H\rangle\right| \xrightarrow{k\to\infty} |\langle \pi_t, H\rangle|.$$

Thus, A_ϵ is closed, and by *Portmanteau's theorem*, $\mathbb{Q}(A_\epsilon) = 1$. □

Theorem 3.4. *Let $\mathbb{Q}$ be a limit point of $\{\mathbb{Q}^N\}_{N\geqslant 1}$, whose existence follows from the fact that the sequence $\{\mathbb{Q}^N\}_{N\geqslant 1}$ is tight. Then we have*

$$\mathbb{Q}\Big(\pi_{\cdot} \in \mathscr{D}([0,T],\mathscr{M}) : F_\theta = 0\Big) = 1 \tag{19}$$

where F_θ is given in (33) for $\theta = 1$ and (34) for $\theta > 1$.

Proof. Recall that we already showed that the limit point of π^N_t is absolutely continuous with respect to the Lebesgue measure, that is $\pi_t(du) = \rho_t(u)du$. The idea to show (19) is the following: we will use *Portmanteau's theorem* (after a technicality, where we will use that $\rho_t(u)$ lives in $L^2(0,T;\mathscr{H}^1(0,1))$, where $\mathscr{H}^1(0,1)$ is the *Sobolev Space* on $[0,1]$), to work with the measure $\mathbb{Q}^N$. Then we will take advantage of the Replacement Lemmas A.5 and A.4 to exchange $\eta_{N^2 s}(1)$ by its average; $\eta(2)$ by $\eta(1)$, and then by its average again. Afterwards, we use Dynkin's martingale and Doob's inequality to show the correct convergence. We will do the proof for $\theta = 1$ only, since for $\theta > 1$ it is analogous. For the argument's sake, let us take H independent of time. It is enough to show that $\forall \delta > 0$:

$$\begin{aligned}
&\mathbb{Q}\Big(\pi_{\cdot} \in \mathscr{D}_{\mathscr{M}}[0,t] : \rho \in L^2(0;T,\mathscr{H}^1(0,1)), \pi_t(u) = \rho_t(u)du \,| \\
&\sup_{0\leqslant t\leqslant T} |\, \langle \rho_t, H_t\rangle - \langle \rho_0, H_0\rangle + \int_0^t \langle \rho_s, (\partial_s + \triangle)H_s\rangle\, ds - \\
&+ \int_0^t \Big\{\rho_s(1)\partial_u H_s(1) - \rho_s(0)\partial_u H_s(0)\Big\}\, ds \\
&- j\int_0^t \Big\{H_s(1)(\beta_1 - (\beta_1+\delta_1)\rho_s(1) + (\delta_2 - \beta_2)(\rho_s^2(1) - \rho_s(1)))\Big\} ds \\
&- j\int_0^t \Big\{H_s(0)(\alpha_1 - (\alpha_1+\gamma_1)\rho_s(0) + (\gamma_2 - \alpha_2)(\rho_t^2(0) - \rho_t(0)))\Big\} ds \,| > \delta\Big).
\end{aligned} \tag{20}$$

The condition $\rho \in L^2(0;T,\mathscr{H}^1(0,1))$ will be shown on the next section, and we will assume at the moment to hold. For simplicity, we will take H to be time independent, but we remark that the arguments for H time dependent are the same. We note that we cannot apply *Portmanteau's theorem* directly. From [10] we know that the maps

$$\pi_{\cdot} \mapsto \int_0^T \langle \pi_s, H_1(s)\rangle ds \quad \text{and} \quad \pi_{\cdot} \mapsto \sup_{0\leqslant t\leqslant T}\left|\langle \pi_t, H_2(t)\rangle - \langle \pi_0, H_3(0)\rangle + \int_0^t \langle \pi_s, H_4(s)\rangle ds\right|,$$

for any $H_i \in C[0,1]$ with $i = 1,2,3,4$, are continuous with respect to the Skohorod topology. In this way, the problem lies with the terms coming from the boundary conditions, thus making the set inside the probability above not an open set in the Skorohod space. To solve this problem, we take the following functions:

$$\overleftarrow{\iota}^u_\epsilon(v) = \frac{1}{\epsilon}1_{(u-\epsilon,u]}(v) \quad \text{and} \quad \overrightarrow{\iota}^u_\epsilon(v) = \frac{1}{\epsilon}1_{[u,u+\epsilon)}(v),$$

and we define the inner product as

$$\langle \pi_s, \overleftarrow{\iota_\epsilon^u} \rangle = \frac{1}{\epsilon}\int_{u-\epsilon}^{u} \rho_s(v)dv \quad \text{and} \quad \langle \pi_s, \overrightarrow{\iota_\epsilon^u} \rangle = \frac{1}{\epsilon}\int_{u}^{u+\epsilon} \rho_s(v)dv.$$

To not overload the notation, we will omit the conditions $\rho \in L^2(0;T,\mathscr{H}^1(0,1)), \pi_t(u) = \rho_t(u)du$ from (20), since it is clear from the context. Now we "replace" the terms that are not continuous to get

$$\begin{aligned}
&\mathbb{Q}\Bigg(\sup_{0\leqslant t\leqslant T} | \langle \rho_t, H\rangle - \langle \rho_0, H\rangle - \int_0^t \langle \rho_s, \Delta H\rangle\, ds + \\
&+ \int_0^t \Big\{ \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle \partial_u H(1) - \langle \pi_s, \overrightarrow{\iota_\epsilon^0}\rangle \partial_u H(0)\Big\}\, ds - \\
&- j\int_0^t \Big\{ H(1)(\beta_1 - (\beta_1+\delta_1)\langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle + (\delta_2-\beta_2)\langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle(\langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle - 1)\Big\} ds - \\
&- j\int_0^t \Big\{ H(0)(\alpha_1 - (\alpha_1+\gamma_1)\langle \pi_s, \overrightarrow{\iota_\epsilon^0}\rangle + (\gamma_2-\alpha_2)\langle \pi_s, \overrightarrow{\iota_\epsilon^0}\rangle(\langle \pi_s, \overrightarrow{\iota_\epsilon^\epsilon}\rangle - 1)\Big\} ds \,| > \delta' \Bigg)
\end{aligned} \tag{21}$$

plus the sum of the following terms

$$\begin{aligned}
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| \int_0^t \{(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\partial_u H(1)\} \Big| > \delta' \Bigg), \\
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| \int_0^t \{(\rho_s(0) - \langle \pi_s, \overrightarrow{\iota_\epsilon^0}\rangle)\partial_u H(0)\} ds \Big| > \delta' \Bigg), \\
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| j\int_0^t \{H(1)((\beta_1+\delta_1)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle))\} ds \Big| > \delta' \Bigg), \\
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| j\int_0^t \{H(1)(\delta_2-\beta_2)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle)\} ds \Big| > \delta' \Bigg), \\
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| j\int_0^t \{H(1)((\delta_2-\beta_2)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle)\} ds \Big| > \delta' \Bigg), \\
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| j\int_0^t \{H(1)((\delta_2-\beta_2)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle)\langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\} ds \Big| > \delta' \Bigg), \\
&\mathbb{Q}\Bigg(\pi_\cdot : \sup_{0\leqslant t\leqslant T} \Big| j\int_0^t (\delta_2-\beta_2)\{H(1)(\rho_s(1) - \langle \pi_s, \overleftarrow{\iota_\epsilon^1}\rangle)\} ds \Big| > \delta' \Bigg),
\end{aligned}$$

plus the terms from the left boundary. To see that all the terms above vanish in the limit it remains to apply the proposition below, which also holds for $\langle \pi_s, \overrightarrow{\iota}_\epsilon^u\rangle$.

Proposition 3.5. *If $\rho \in L^2(0,T;\mathscr{H}^1(0,1))$, then for all $\epsilon > 0$ we have*

$$\Big|\rho_s(u) - \langle \pi_s, \overleftarrow{\iota}_\epsilon^u\rangle\Big| \leqslant \frac{1}{2}\epsilon \|\partial_u \rho\|_2^2, \tag{22}$$

Proof. Since for now we are assuming that $\rho \in L^2(0,T;\mathscr{H}^1(0,1))$, the norm above is finite and we have for $u \in [0,1]$

$$\frac{1}{\epsilon}\int_0^\epsilon \rho_s(0) - \rho_s(u)du = -\frac{1}{\epsilon}\int_0^\epsilon \int_0^u \partial_u \rho_s(v)dvdu. \tag{23}$$

By *Cauchy-Schwarz's inequality* we have that the absolute value of the previous expression is bounded from above by

$$\left|\frac{1}{\epsilon}\int_0^\epsilon\left[\int_0^u(\partial_u\rho_s(v))^2dv\int_0^u 1dv\right]du\right|=\left|\frac{1}{\epsilon}\int_0^\epsilon\left(u\int_0^u(\partial_u\rho_s(v))^2dv\right)du\right|. \tag{24}$$

Since $\int_0^u(\partial_u\rho_s(v))^2dv\leqslant\|\partial_u\rho\|_2^2<\infty$, we can bound the previous expression by

$$\frac{1}{\epsilon}\|\partial_u\rho\|_2^2\int_0^\epsilon udu=\frac{1}{2}\epsilon\|\partial_u\rho\|_2^2\xrightarrow{\epsilon\to 0}0. \tag{25}$$

□

Remark 3.6. *For the general case $K\geqslant 2$, the main problem are the terms of the form $\rho_s^{K-1}(0)$ and $(1-\rho_s(0))^{K-1}$ (and similar for the right boundary). A simple way to solve this is to proceed by induction. Since $a^2=(a+b_1-b_1)(a+b_2-b_2)=(a-b_1)(a-b_2)+b_1(a-b_1)+b_2(a-b_2)+b_1b_2$ and we have that $b_1b_2a=b_1b_2(a+b_3-b_3)=b_1b_2(a-b_3)+b_1b_2b_3$, taking $a\equiv\rho_s(0)$ and $b_j\equiv\langle\pi_s,\overrightarrow{\iota_\epsilon^{(j-1)\epsilon}}\rangle$ for $j\geqslant 0$, we can replace $\rho_s^{K-1}(0)$ by $\prod_{j=0}^{K-2}\langle\pi_s,\overrightarrow{\iota_\epsilon^{j\epsilon}}\rangle$ plus a sum of terms that vanish when $\epsilon\to 0$ in the limit. For the right boundary the argument is analogous.*

To finally apply Portmanteau's theorem, we argue that we can approximate $\overleftarrow{\iota}_\epsilon^u$, $\overrightarrow{\iota}_\epsilon^u$ by continuous functions in such a way that the error vanishes as $\epsilon\to 0$. Now we apply Portmanteau's theorem and, recalling the definition of $\mathbb{Q}^N$ we bound from above (21) by

$$\begin{aligned}
&\liminf_{N\to\infty}\mathbb{P}_{\mu^N}\Bigg(\eta_\cdot\in\mathcal{D}([0,T],\Omega_N):\sup_{0\leqslant t\leqslant T}|\langle\pi_t^N,H\rangle-\langle\pi_0^N,H\rangle-\int_0^t\langle\pi_s^N,\Delta H\rangle ds\\
&+\int_0^t\left\{\langle\pi_s^N,\overleftarrow{\iota_\epsilon^1}\rangle\partial_uH(1)-\langle\pi_s^N,\overrightarrow{\iota_\epsilon^0}\rangle\partial_uH(0)\right\}ds-\\
&-j\int_0^t\Big\{H(1)(\beta_1-(\beta_1+\delta_1)\langle\pi_s^N,\overleftarrow{\iota_\epsilon^1}\rangle+(\delta_2-\beta_2)\langle\pi_s^N,\overleftarrow{\iota_\epsilon^1}\rangle(\langle\pi_s^N,\overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle-1)\Big\}ds-\\
&-j\int_0^t\Big\{H(0)(\alpha_1-(\alpha_1+\gamma_1)\langle\pi_s^N,\overrightarrow{\iota_\epsilon^0}\rangle+(\gamma_2-\alpha_2)\langle\pi_s^N,\overrightarrow{\iota_\epsilon^0}\rangle(\langle\pi_s^N,\overrightarrow{\iota_\epsilon^\epsilon}\rangle-1)\Big\}ds\,|>\delta'\Bigg).
\end{aligned} \tag{26}$$

Summing and subtracting $\int_0^t \mathscr{L}_N\langle \pi_s^N, H\rangle ds$ in (26), and recalling our expression for the Dynkin's Martingale in (13), we can bound the last probability by the sum of the following terms:

$$
\begin{aligned}
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T} \big|M_t^N\big| \geqslant \delta''\Big),\\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T} \Big|\int_0^t \langle \pi_s^N, \Delta_N H\rangle - \langle \pi_s^N, \Delta H\rangle ds\Big| > \delta''\Big),\\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T} \Big|\eta(N-1)\nabla_N^- H(1) - \langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle \partial_u H(1)\Big| > \delta''\Big),\\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T} \Big|\eta(1)\nabla_N^+ H(0) - \langle \pi_s^N, \overrightarrow{\iota_\epsilon^0}\rangle \partial_u H(0)\Big| > \delta''\Big),\\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T} \Big|H(\tfrac{N-1}{N})(\beta_1 - (\beta_1+\delta_1)\eta_{sN^2}(N-1)) - H(1)\beta_1 - (\beta_1+\delta_1)\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle\Big| > \delta''\Big),\\
&\liminf_{N\to\infty} \mathbb{P}_{\mu^N}\Big(\sup_{0\leqslant t\leqslant T} \big| H(\tfrac{N-2}{N})(\delta_2-\beta_2)\eta_{sN^2}(N-1)\eta_{sN^2}(N-2) -\\
&-H(1)(\delta_2-\beta_2)\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle(\langle \pi_s^N, \overleftarrow{\iota_\epsilon^{1-\epsilon}}\rangle - 1)\big| > \delta''\Big),
\end{aligned}
\tag{27}
$$

plus the terms from the left boundary. The second term on the last display vanishes as $N\to\infty$, since $\langle \pi_s^N, \Delta_N H\rangle - \langle \pi_s^N, \Delta H\rangle \xrightarrow{N\to\infty} 0$. To bound the first, we will use Doob's inequality. For the others we will apply the replacement lemmas as follows. Let

$$
\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) := \frac{1}{\epsilon N}\sum_{x=2}^{1+\epsilon N} \eta_{sN^2}(x), \qquad \overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) := \frac{1}{\epsilon N}\sum_{x=N-1-\epsilon N}^{N-2} \eta_{sN^2}(x) \tag{28}
$$

Then, since $\overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1) = \langle \pi_s^N, \overleftarrow{\iota}_\epsilon^1\rangle$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1) = \langle \pi_s^N, \overrightarrow{\iota}_\epsilon^0\rangle$) we have $\overleftarrow{\eta_{sN^2}}^{\epsilon N}(N-1) \sim \rho_s(1)$ (resp $\overrightarrow{\eta_{sN^2}}^{\epsilon N}(1) \sim \rho_s(0)$) , we show in Appendix A that we can exchange $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$) by the averages above, and $\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(2)$) by $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$). Thus, we now proceed in the following way:

(1) For the bulk terms in (27), we apply Lemma A.5 with the choice $\psi(\eta)=1$ and replace $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$) by $\overleftarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ (resp. $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(1)$), paying a price $\mathscr{O}(N^{-1})$.
(2) For the *linear* boundaries terms in (27), we apply again Lemma A.5 with $\psi(\eta)=1$.
(3) We treat $\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(2)$) by first applying Lemma A.4 with $\psi(\eta)=1$ to replace $\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(2)$) by $\eta_{sN^2}(N-1)$ (resp. $\eta_{sN^2}(1)$). Again, we apply Lemma A.5 with $\varphi(\eta)=\eta_{sN^2}(N-1)$ (resp. $\varphi(\eta)=\eta_{sN^2}(1)$) with a cummulative error of $\mathscr{O}(N^{-1})$.
(4) For the correlation terms $\eta_{sN^2}(N-1)\eta_{sN^2}(N-2)$ (resp. $\eta_{sN^2}(1)\eta_{sN^2}(2)$), we first replace $\eta_{sN^2}(N-2)$ by $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ with the choice $\psi(\eta)=\eta_{sN^2}(N-1)$ (similar for the left). Now that we have the term $\eta_{sN^2}(N-1)\overrightarrow{\eta}_{sN^2}(N-1)^{\epsilon N}$, then we replace $\eta_{sN^2}(N-1)$ by $\overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ by Lemma A.5 with $\psi(\eta)=\eta_{sN^2}^{\epsilon N}(N-1)$ with a cummulative error of $\mathscr{O}(N^{-1})$.
(5) Observing that $\langle \pi_s^N, \overrightarrow{\iota}_\epsilon^0\rangle = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(1)$, $\langle \pi_s^N, \overleftarrow{\iota}_\epsilon^1\rangle = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(N-1)$ and

$$
\langle \pi_s^N, \overrightarrow{\iota}_\epsilon^0\rangle\langle \pi_s^N, \overrightarrow{\iota}_\epsilon^\epsilon\rangle = \overrightarrow{\eta}_{sN^2}^{\epsilon N}(1)\overleftarrow{\eta}_{sN^2}^{\epsilon N}(\epsilon N+1) + \mathscr{O}((\epsilon N)^{-1}), \tag{29}
$$

we are done.

Now we are left with the contribution from the *Dynkin's martingale*, that we shall bound as follows:

$$
\mathbb{P}_{\mu^N}\left(\sup_{0\leqslant t\leqslant T} \big|M_t^N\big| > \delta\right) \leqslant \frac{2}{\delta}\mathbb{E}_{\mu^N}\left(\big|M_T^N\big|^2\right)^{\frac12} = \frac{2}{\delta}\mathbb{E}_{\mu^N}\left(\int_0^T B_s^{N,H} ds\right)^{\frac12} \tag{30}
$$

where we applied *Doobs's inequality* in the first step. Recalling that $\int_0^T B_s^{N,H} ds$ is the quadratic variation of *Dynkin's martingale*, one can proceed as we did in order to show tightness and we are done.

For $\theta > 1$, by the same arguments we arrive at

$$\begin{aligned}\liminf_{N\to\infty}\mathbb{P}_{\mu^N}\Bigg(&\sup_{0\leqslant t\leqslant T} |\langle \pi_t^N, H\rangle - \langle \pi_0^N, H\rangle - \int_0^t \langle \pi_s^N, \Delta H\rangle\, ds \\ &+ \int_0^t \{\langle \pi_s^N, \overleftarrow{\iota_\epsilon^1}\rangle \partial_u H(1) - \langle \pi_s^N, \overrightarrow{\iota_\epsilon^0}\rangle \partial_u H(0)\}\, ds\Bigg).\end{aligned} \tag{31}$$

We proceed exactly the same way, summing and subtracting $\int_0^t \mathscr{L}_N \langle \pi_s^N, H\rangle ds$ and bounding each term as in (27), and applying the Replacement Lemma A.5 to exchange $\eta_{sN^2}(1)$ and $\eta_{sN^2}(N-1)$ by respective boxes. Afterwards, the procedure is identical. □

4. Hydrostatic Limit

We start by recalling the Heat equation with either Robin or Neumann boundary conditions, depending on the value of θ.

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u), & (t,u) \in [0,T] \times (0,1), \\ \partial_u \rho_t(0) = b_\theta(\rho_t(0)), & t \in [0,T], \\ \partial_u \rho_t(1) = b_\theta(\rho_t(1)), & t \in [0,T], \\ \rho(0,\cdot) = \rho_0(\cdot), \end{cases} \tag{32}$$

where for $\theta = 1$ we have Robin boundary conditions

$$\begin{cases} b_1(\rho_t(0)) = -j\sum_{x=1}^K \{\alpha_x(1-\rho_t(0))\rho_t^{x-1}(0) - \gamma_x \rho_t(0)(1-\rho_t(0))^{x-1}\}, \\ b_1(\rho_t(1)) = j\sum_{x=1}^K \{\beta_x(1-\rho_t(1))\rho_t^{x-1}(1) - \delta_x \rho_t(1)(1-\rho_t(1))^{x-1}\}, \end{cases} \tag{33}$$

while for $\theta > 1$ we have Neumann boundary conditions

$$b_\theta(\rho_t(0)) = b_\theta(\rho_t(1)) = 0. \tag{34}$$

For each $\theta \geqslant 1$ let $\rho_\theta : [0,1] \to [0,1]$ be defined as the stationary solution of equation (32). As mentioned in section 2, the main result of this section is the following theorem, which can also be seen as a law of large numbers for the empirical measure with respect to the stationary state.

Theorem 4.1 (Hydrostatic Limit). *Let μ_N^{ss} be the probability measure in Ω_N invariant for the Markov process with infinitesimal generator $N^2\mathscr{L}_N$. Then, for any $H \in C[0,1]$ and $\delta > 0$ the sequence μ_N^{ss} is associated to the profile ρ_θ, that is*

$$\lim_{N\to\infty} \mu_N^{ss}\left(\eta : \left|\frac{1}{N-1}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\eta(x) - \int_0^1 H(u)\rho_\theta(u)du\right| \geqslant \delta\right) = 0,$$

where for $\theta = 1$, ρ_θ is the unique stationary solution for the heat equation with Robin boundary conditions (33), while for $\theta > 1$, ρ_θ is a solution of the heat equation with Neumann boundary conditions (34)

For the case $\theta = 1$, the theorem above is consequence of two facts: the existence of a (unique) classical solution of the Hydrodynamic Equation and its convergence to the stationary solution as time goes to infinity, plus the establishment of a Hydrodynamic Limit. For the proof we refer the reader to both [21] (Theorem 2.2), and [25]. The main ingredient for the proof is the concentration result, that we state below. For that, define the probability $\mathscr{P}_N \in \mathscr{M}^+$ as

$$\mathscr{P}_N := \mu_N^{ss} \circ (\pi_N)^{-1}$$

and let $\epsilon_\theta := \{\tilde{\pi} \in \mathcal{M}^+ : \tilde{\pi}(du) = \rho_\theta(u)du\}$ such that ρ_θ solves 33. Now let d be the distance defined on the Skorohod space $\mathcal{D}([0,T],\mathcal{M}^+)$ under which this space is a Polish space (see [18], Chapter 4 for an example). Then, in [21] it is shown that $\mathcal{P} := \lim_{N\to\infty} \mathcal{P}_N$ is concentrated on ϵ_θ.

Proposition 4.2. *$\{\mathcal{P}_N\}_{N\in\mathbb{N}}$ is concentrated in ϵ_θ, i.e., $\forall \delta > 0$,*

$$\mathcal{P}_N\left(\pi \in \mathcal{M}^+ : \inf_{\tilde{\pi}\in\epsilon_\theta} d(\pi,\tilde{\pi}) \geqslant \delta\right) \xrightarrow{N\to\infty} 0.$$

From results established on the previous chapters we know that $\epsilon_\theta = \{\rho_\theta(u)du, \theta < 1\}$, and thus the Hydrostatic Limit for $\theta = 1$ is derived from the result above exactly as in [25]. Due to the non-reversible dynamics and the non-uniqueness of stationary solutions for the Cauchy problem with Neumann boundary conditions, our main interest is on the behaviour of the system (under the stationary state) for $\theta > 1$. On this section, we exploit the method developed by Tsunoda in [25], which consists in the study of the evolution of the process in a non-diffusive time scale $N^{1+\theta}$, which enables us to see the evolution of the system in the macroscopic scale. Our first step is to derive an integral equation for total mass of the system, then show that when time goes to infinity, the solution for such equation stabilizes to our constant of interest. The main following steps are analogous to the proof of the Hydrodynamic Limit, in the sense that we show tightness for the induced probability measure under the $N^{1+\theta}$−time scale followed by the characterization of the limit points. Here, we will show the aforementioned replacement lemmas in a similar fashion to Lemmas A.4 and A.5.

4.1. **Heuristics for** $\theta > 1$. Let $\eta^N = \{\eta_t^N\}_{t\geqslant 0}$ be the markov process associated to the generator $N^{1+\theta}\mathcal{L}_N$ with initial distribution μ_N^{ss}, and define the mass of the system at time t as follows.

Definition 4.3. *For each $t \geqslant 0$ and θ fixed, we define*

$$m_t^N = \frac{1}{N-1}\sum_{x\in\Lambda_N} \eta_t^N(x),$$

and for each $T > 0$ we let $\mathcal{D}([0,T],\mathbb{R})$ be the set of càdlàg trajectories $m_\cdot : [0,T] \to \mathbb{R}$ with respect to the Skorokhod topology. For each $N \in \mathbb{N}$, let $\mathcal{Q}_N \equiv \mathcal{Q}_{N,T}$ be the distribution of $\{m_\cdot^N : t \geqslant 0\}$ on $\mathcal{D}([0,T],\mathbb{R})$, i.e., $\mathcal{Q}_N = Law(m_\cdot^N)$. Moreover, we define

$$G(\eta) := \frac{1}{N-1}\sum_{x\in\Lambda_N} \eta^N(x).$$

Then, Dynkin's Martingale for G takes the form

$$M_t^{N,G} = G(\eta_t^N) - G(\eta_0^N) - N^{1+\theta}\int_0^t (\mathcal{L}_N G)(\eta_s^N)ds,$$

and by definition we have $G(\eta_t^N) = m_t^N$. Since the bulk dynamics conserves the number of particles, it is not difficult to see that

$$\int_0^t (\mathcal{L}_{N,0}G)(\eta_s^N)ds = 0.$$

Simple computations show that the term $-N^{1+\theta}\int_0^t (\mathcal{L}_{N,\pm}G)(\eta_s^N)ds$ equals

$$j\frac{N}{N-1}\int_0^t \big(\alpha_1+\beta_1+\eta_s^N(1)(\alpha_2-(\alpha_1+\gamma_1))+\eta_s^N(N-1)(\beta_2-(\beta_1+\delta_1))\big)ds$$
$$+j\frac{N}{N-1}\int_0^t \big(\eta^N(1)\eta_s^N(2)(\gamma_2-\alpha_2)+\eta_s^N(N-1)\eta_s^N(N-2)(\delta_2-\beta_2)\big)ds.$$

By the definition of $G(\eta)$ and m_t, we can rewrite Dynkin's Martingale to obtain an expression for m_t.

$$m_t^N = m_0 + M_t^N + j\int_0^t \Big(\alpha_1+\beta_1+\eta_s^N(1)(\alpha_2-(\alpha_1+\gamma_1))+\eta_s^N(N-1)(\beta_2-(\beta_1+\delta_1))\Big)ds$$
$$+j\int_0^t \Big(\eta_s^N(1)\eta_s^N(2)(\gamma_2-\alpha_2)+\eta_s^N(N-1)\eta_s^N(N-2)(\delta_2-\beta_2)\Big)ds + O(\tfrac{1}{N}),$$

where for the term $O(\frac{1}{N})$, we summed and subtracted 1 in the numerator of $\frac{N}{N-1}$. If we are able to exchange $\eta_s^N(1)$ and $\eta_s^N(2)$ (resp. $\eta_s^N(N-1)$ and $\eta_s^N(N-2)$) by m_s^N, then, since under this time-scale our martingale M_t^N vanishes as $N\to\infty$, we have the following integral equation for m_t:

$$m_t = m_0 + j\int_0^t \Big(\alpha_\ell + (\alpha_{n\ell}-(\alpha_\ell+\gamma_\ell))m_s - (\alpha_{n\ell}-\gamma_{n\ell})m_s^2\Big)ds, \tag{35}$$

where we write $\alpha_\ell = \alpha_1+\beta_1, \alpha_{n\ell}=\alpha_2+\beta_2, \gamma_\ell=\gamma_1+\delta_1$ and $\gamma_{n\ell}=\gamma_2+\delta_2$. The result above is the content of the Replacement Lemma 4.5.

Intuitively, we have that for large t the density of particles should stabilize to some constant, *i.e.,* $\partial_t m_t = 0$. This leads us to look for stable solutions of the differential form of equation (35):

$$\partial_t m_t = j\Big(\alpha_\ell + (\alpha_{n\ell}-(\alpha_\ell+\gamma_\ell))m_t - (\alpha_{n\ell}-\gamma_{n\ell})m_t^2\Big). \tag{36}$$

Due to the quadratic term, we have two stable solutions. In this way, fixing the parameters such that the discriminant vanishes

$$(\alpha_\ell+\gamma_\ell-\alpha_{n\ell})^2 + 4\alpha_\ell(\alpha_{n\ell}-\gamma_{n\ell}) = 0, \tag{37}$$

as $t\to\infty$, we are then left with

$$m_\infty = \begin{cases} \Big(\frac{\alpha_\ell}{\gamma_{n\ell}-\alpha_{n\ell}}\Big)^{\frac{1}{2}}, & (\alpha_\ell+\gamma_\ell-\alpha_{n\ell})^2+4\alpha_\ell(\alpha_{n\ell}-\gamma_{n\ell})=0 \wedge \alpha_{n\ell}\neq\gamma_{n\ell} \\ \frac{\alpha_\ell}{\alpha_\ell+\gamma_\ell-\alpha_{n\ell}}, & \alpha_{n\ell}=\gamma_{n\ell}. \end{cases} \tag{38}$$

Showing that the sequence $\{\mathcal{Q}_N\}_{N\geqslant 1}$ is tight (Proposition 4.4) and that $\mathcal{Q} := \lim_{N\to\infty}\mathcal{Q}_N$ is concentrated on solutions of (35) (Proposition 4.9), we are left with showing that $\mathcal{P} := \lim_{N\to\infty}\mathcal{P}_N$ gives full weight to the function $\rho_\theta(u) = m_\infty 1_{\{u\in[0,1]\}}$ to conclude the proof of Theorem 4.1.

4.2. Proof of the Hydrostatic Limit for $\theta > 1$.

Proposition 4.4. *The sequence of probability measures $\{\mathcal{Q}_N\}_{N\geqslant 1}$ is relatively compact in $\mathcal{D}([0,T],\mathbb{R})$.*

Proof. We now apply Aldous' criterion is the same fashion as in Proposition 3.1. For that, fix T and let $\mathcal{Q}_{N,T}^H \equiv \mathcal{Q}_N^H$ be the probability measure induced by the map

$$\psi : m_\cdot^N \mapsto \psi(m_\cdot^N) = \langle m_\cdot^N, H\rangle \in (\mathcal{D}([0,T],\mathbb{R}), \mathcal{Q}_N^H),$$

with $H\in C^2[0,1]$. Then, it is enough to show that the sequence $(\mathcal{Q}_N^H)_{N\geqslant 1}$ is relatively compact.

To show the condition (1), since we have $\big|\langle m_t^N, H\rangle\big| \leqslant \|H\|_\infty$, it is also enough to take $K(t,\epsilon) = \overline{B_r(0)}$ for any radius $r > \|H\|_\infty$. For the condition (2), from the definition of G we have

$$\mathcal{Q}_N^H\Big(\langle m_\cdot^N, H\rangle \in \mathcal{D}([0,T],\mathbb{R}) : \big|\langle m_{\tau+\lambda}^N, H\rangle - \langle m_\tau^N, H\rangle\big| > \epsilon\Big)$$
$$= \mathcal{P}_N^{\mu^{ss}}\Big(\eta_\cdot^N \in \mathcal{D}([0,T],\Omega_N) : \big|\langle G(\eta_{\tau+\lambda}^N), H\rangle - \langle G(\eta_\tau^N), H\rangle\big| > \epsilon\Big),$$

where $\mathscr{P}_N^{\mu^{ss}}$ is the measure induced by the process η^N, which starts from the stationary measure μ_N^{ss}. Now we sum and subtract Dynkin's martingale for G, and thus proceed to bound

$$\mathscr{P}_N^{\mu^{ss}}\left(\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N): \left|\int_\tau^{\tau+\lambda} N^{1+\theta}\mathscr{L}_N\langle G(\eta_s^N),H\rangle\right| > \epsilon/2\right)$$
$$+\mathscr{P}_N^{\mu^{ss}}\left(\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N): \left|M_\tau^{N,G}-M_{\tau+\lambda}^{N,G}\right| > \epsilon/2\right).$$

Applying Chebyshev's and Markov's inequality in the first and second lines respectively, in the previous display, and using the formula for the quadratic variation for the second term, the desired result is obtained proceeding exactly as in Proposition 3.1. □

Lemma 4.5 (Replacement Lemma). *Let $V \equiv V_N$ be defined as*

$$V(\eta)=\eta(1)(\alpha_2-(\alpha_1+\gamma_1))+\eta(N-1)(\beta_2-(\beta_1+\delta_1))-(\alpha_2+\beta_2-(\alpha_1+\gamma_1+\beta_1+\delta_1))G(\eta)$$
$$+\eta(1)\eta(2)(\gamma_2-\alpha_2)+\eta(N-1)\eta(N-2)(\delta_2-\beta_2)-(\gamma_2+\delta_2-(\alpha_2+\beta_2))G(\eta)^2,$$

and denote by $\mathbb{E}_N$ the expectation with respect to the process $\eta_\cdot^N$. Then, $\forall t \geqslant 0$ we have

$$\lim_{N\to\infty}\mathbb{E}_N\left|\int_0^t V(\eta_s^N)ds\right|=0.$$

Proof. Proceeding exactly as the other replacement lemmas (from (56) to (58) for example), from the Entropy, Jensen's and Feynmann-Kac's inequality, plus noticing that $e^{|x|}\leqslant e^x+e^{-x}$, we are again reduced to the variational expression

$$\frac{C_\gamma}{B}+t\sup_f\left\{\langle V,f\rangle_{\nu_\gamma^N}+\frac{1}{B}N^\theta\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\gamma^N}\right\} \tag{39}$$

with the supremum taken over all densities. As seen in (49), we have

$$\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\gamma^N}\lesssim -D_N(\sqrt{f},\nu_\gamma^N)+\mathscr{O}(N^{-\theta}),$$

and from the Lemma below, (39) is smaller than

$$\frac{C_\gamma}{B}+t\sup_f\left\{C(ND_{N,0}(\sqrt{f}))^{1/2}-\frac{1}{B}N^\theta D_N(\sqrt{f},\nu_\gamma^N)+\frac{1}{B}\right\}.$$

Computing the supremum above and taking the respective limits we are done. □

Lemma 4.6 (Moving Particle lemma). *For any f density we have*

$$\left|\langle V,f\rangle_{\nu_\gamma^N}\right|\lesssim(ND_{N,0}(f,\nu_\gamma^N))^{1/2}.$$

Proof. Fixed f, by definition we have

$$\langle V,f\rangle_{\nu_\gamma^N}=\frac{\alpha_2-(\alpha_1+\gamma_1)}{N-1}\sum_{x\in\Lambda_N}\int_{\Omega_N}(\eta(1)-\eta(x))f(\eta)\nu_\gamma^N(d\eta)$$
$$+(\gamma_2-\alpha_2)\int_{\Omega_N}(\eta(1)\eta(2)-G(\eta)^2)f(\eta)\nu_\gamma^N(d\eta)+RB$$

with RB the terms arising from the right boundary. The proof for the linear terms can be found both in [20] and [25], while for the non linear terms, as we will see, after simple computations the proof is identical. In this way will present only the computations for the latter. We consider the left boundary terms only, since for the right boundary the computations are analogous. Thus, bounding the rates by some constant, we have

$$\lesssim \frac{1}{(N-1)^2}\sum_{x,y\in\Lambda_N}\int_{\Omega_N}(\eta(1)\eta(2)-\eta(x)\eta(y))f(\eta)\,\nu_\gamma^N(d\eta).$$

Since we have at most one particle per site, $\eta(x)\eta(y)\leqslant\eta(x)$ for any $x,y\in\Lambda_N$ and we can bound the expression in the previous display by

$$\lesssim \frac{1}{N-1}\sum_{x\in\Lambda_N}\int_{\Omega_N}(\eta(1)-\eta(x))f(\eta)\,\nu_\gamma^N(d\eta),$$

thus, we are back to the "linear" case. Breaking the sum in two times its halves, and summing and subtracting $\frac{1}{2}\sum_{x\in\Lambda_N}f(\eta^{1,x})$, we have

$$\begin{aligned}\lesssim&\frac{1}{2(N-1)}\sum_{x\in\Lambda_N}\int_{\Omega_N}(\eta(1)-\eta(x))(f(\eta)+f(\eta^{1,x}))\,\nu_\gamma^N(d\eta)\\&+\frac{1}{2(N-1)}\sum_{x\in\Lambda_N}\int_{\Omega_N}(\eta(1)-\eta(x))(f(\eta)-f(\eta^{1,x}))\,\nu_\gamma^N(d\eta).\end{aligned}$$

The first term can be shown to vanish by performing the change of variables $\eta\mapsto\eta^{1,x}$ and recalling that the Bernoulli product are invariant for such change of variables. We can also manipulate the second term similarly to what was done in Lemma A.2, by writing the identity $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$, bounding $\eta(1)-\eta(x)$ by 1, and applying the Cauchy-Schwartz inequality on the remaining term. This leads to

$$\lesssim \frac{1}{N-1}\sum_{x\in\Lambda_N}\left[\int_{\Omega_N}\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{1,x})}\right)^2\nu_\gamma^N(d\eta)\right]^{\frac{1}{2}},$$

where we also used the fact that f is a density. Applying the inequality

$$\sum_{i\in\Lambda_N}a_i^{1/2}\leqslant[(N-1)\sum_{i\in\Lambda_N}a_i]^{1/2},$$

we then need to bound

$$\left[\frac{1}{N-1}\sum_{x\in\Lambda_N}\int_{\Omega_N}\left(\sqrt{f(\eta)}-\sqrt{f(\eta^{1,x})}\right)^2\nu_\gamma^N(d\eta)\right]^{1/2}. \tag{40}$$

Since exchanging sites $1\leftrightarrow x$ is the same as exchanging sites $1\leftrightarrow 2\leftrightarrow 3\leftrightarrow 4\leftrightarrow\cdots$, until x, and then exchanging backwards the sites from $x-1$, *i.e.*,

$$\eta^{1,x}=\left(\cdots\left(\left(\left(\cdots\left(\eta^{1,2}\right)^{2,3}\cdots\right)^{x-1,x}\right)^{x-2,x-1}\right)\cdots\right)^{2,1},$$

summing and subtracting each term of the composition of transformations above inside the square in (40), each time applying the triangular inequality (in a total of N times), we have that this expression is then bounded by some constant times $(ND_{N,0}(f;\nu_\gamma^N))^{1/2}$, which completes the proof. □

Remark 4.7. *With the same reasoning we can show Lemma 4.5 for $K>2$, which leads to the following extension of (35):*

$$m_t=m_0+j\sum_{x=1}^{K}\int_0^t(\alpha_x+\beta_x)(1-m_s)m_s^{x-1}-(\gamma_x+\delta_x)m_s(1-m_s)^{x-1}ds. \tag{41}$$

Corollary 4.8. *The limit point $\mathscr{Q}$ is concentrated on continuous trajectories.*

Proof. Since we already have tightness of the sequence $\{m_t^N : t \in [0,T]\}_{N\geqslant 0}$, thanks to the exclusion process and the probability of more than one jump happening at the same time being zero, the result follows from Proposition 14 in [13]. □

Proposition 4.9. *Let $\mathscr{Q}$ be a limit point of $\{\mathscr{Q}_N\}_{N\geqslant 1}$, whose existence follows from the fact that the sequence $\{\mathscr{Q}_N\}_{N\geqslant 1}$ is tight. Then, we have*

$$\mathscr{Q}\left(m_\cdot : m_t = m_0 + j\int_0^t \left(\alpha_\ell + (\alpha_{n\ell} - (\alpha_\ell + \gamma_\ell))m_s - (\alpha_{n\ell} - \gamma_{n\ell})m_s^2\right) ds\right) = 1$$

Proof. Although the map $m_\cdot \mapsto m_t$ is not continuous with respect to the Skorokhod topology, by Proposition 4.8 we know that $\mathscr{Q}$ is concentrated in continuous trajectories. In this way, it is enough to bound

$$\limsup_{k\to\infty} \mathscr{P}_{N_k}^{\mu^{ss}}\Bigg(\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N) : \sup_{0\leqslant t\leqslant T} | G(\eta_t^N) - G(\eta_0^N) \\ - j\int_0^t (\alpha_\ell + (\alpha_{n\ell} - (\alpha_\ell + \gamma_\ell))G(\eta_s^N) - (\alpha_{n\ell} - \gamma_{n\ell})G(\eta_s^N)^2) ds \,|> \delta\Bigg).$$

Summing and subtracting Dynkin's martingale for G, the expression above is no larger than

$$\limsup_{k\to\infty} \mathscr{P}_{N_k}^{\mu^{ss}}\left(\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N) : \sup_{0\leqslant t\leqslant T}\left|M_t^{N,G}\right| > \delta/2\right) \\ + \limsup_{k\to\infty} \mathscr{P}_{N_k}^{\mu^{ss}}\left(\eta_\cdot^N \in \mathscr{D}([0,T],\Omega_N) : \sup_{0\leqslant t\leqslant T}\left|\int_0^t V(\eta_s^N) ds\right| > \delta/2\right).$$

Applying Markov's inequality on both terms, and using the Replacement Lemma 4.5 we are done. □

Proposition 4.10. *Letting $\Delta = -((\alpha_\ell + \gamma_\ell - \alpha_{n\ell})^2 + 4\alpha_{n\ell}(\alpha_{n\ell} - \gamma_{n\ell}))$, the differential equation*

$$\partial_t m_t = j\left(\alpha_\ell + (\alpha_{n\ell} - (\alpha_\ell + \gamma_\ell))m_t - (\alpha_{n\ell} - \gamma_{n\ell})m_t^2\right).$$

has solution

$$m_t = \frac{1}{2(\gamma_{n\ell} - \alpha_{n\ell})}\left\{\alpha_\ell + \gamma_\ell - \alpha_{n\ell} \pm \Delta^{1/2}\tan\left(\frac{1}{2}\Delta^{1/2}(t + m_0)\right)\right\}$$

if $\alpha_{n\ell} \neq \gamma_{n\ell}$, or solution

$$\frac{\alpha_\ell}{(\alpha_\ell + \gamma_\ell) - \alpha_{n\ell}} + m_0 e^{-j((\alpha_\ell+\gamma_\ell)-\alpha_{n\ell})t}$$

if $\alpha_{n\ell} = \gamma_{n\ell}$.

Proof of the Hydrostatic Limit, $\theta > 1$. This is done exactly as in [25], but since the proof is short enough we present it here in some detail for completness. From the Theorem 1.3 in [4] we know that the sequence $\{\mathscr{P}_N\}_{N\geqslant 1}$ is tight. In this way, let $\{\mathscr{P}_{N_k}\}_{k\geqslant 1}$ and $\{\mathscr{Q}_{N_k}\}_{k\geqslant 1}$ be subsequences that converge to $\mathscr{P}$ and $\mathscr{Q}$, respectively. Let $c := m_\infty$, for a fixed $\delta > 0$ let $B_\delta(c) \subset [0,1]$ be the ball centered in c with radius δ and define the set

$$I_\delta(c) = [0,1] \backslash \overline{B_\delta(c)}.$$

Moreover, define the function $\rho_\theta(u) = c\mathbb{1}_{\{u\in[0,1]\}}$. Since Proposition 4.2 states that $\mathscr{P}$ is concentrated on stationary solutions of the hydrodynamic equation, we then need to show that

$$\mathscr{P}(\{\rho_\theta(u)du\}) = 1.$$

Denoting by π the limit point of π_N (which we know from the Hydrodynamic Limit), from the definition of $\mathscr{P}_N$, and since η^N starts from the stationary measure μ_N^{ss}, for any fixed t and T we have

$$\begin{aligned}\mathscr{Q}_N(m_\cdot^N \in \mathscr{D}([0,T],\mathbb{R}) : m_t^N \in I_\delta(c)) &= \mu_N^{ss}(\eta \in \Omega_N : G(\eta) \in I_\delta(c)) \\ &= \mathscr{P}_N(\pi^N \in \mathscr{M}^+ : \langle \pi^N, 1\rangle \in I_\delta(c)),\end{aligned} \tag{42}$$

with 1 the identity function on $[0,1]$. By Portmanteau's Theorem we have

$$\begin{aligned}\mathscr{P}(\pi : \langle \pi, 1\rangle \in I_\delta(c)) &\leqslant \liminf_{k\to\infty} \mathscr{P}_{N_k}(\pi^{N_k} \in \mathscr{M}^+ : \langle \pi^{N_k}, 1\rangle \in I_\delta(c)) \\ &= \liminf_{k\to\infty} \mathscr{Q}_{N_k}(m_\cdot^{N_k} : m_t \in I_\delta(c)),\end{aligned}$$

with the last equality due to (42). Also by Portmanteau's Theorem, we may write

$$\limsup_{k\to\infty} \mathscr{Q}_{N_k}(m_\cdot^{N_k} : m_t^{N_k} \in \overline{I_\delta(c)}) \leqslant \mathscr{Q}(m_\cdot : m_t \in \overline{I_\delta(c)})$$

justifying that $\mathscr{Q}$ is concentrated in continuous paths (Proposition 4.8). Observing that

$$\liminf_{k\to\infty} \mathscr{Q}_{N_k}(m_\cdot^{N_k} : m_t^{N_k} \in I_\delta(c)) \leqslant \limsup_{k\to\infty} \mathscr{Q}_{N_k}(m_\cdot^{N_k} : m_t^{N_k} \in \overline{I_{\delta/2}(c)}),$$

we can assert that

$$\mathscr{P}(\pi : \langle \pi, 1\rangle \in I_\delta(c)) \leqslant \mathscr{Q}(m_\cdot : m_t \in \overline{I_{\delta/2}(c)}). \tag{43}$$

By Proposition 4.10, $m_t \notin \overline{I_{\delta/2}(c)}$ for t large enough under the restrictions on the parameters therein mentioned, and by Proposition 4.9 the right hand side of (43) vanishes for times larger than such t, which completes the proof. □

Remark 4.11. *We end with a final observation for $K > 2$. The proof is identical to when $K = 2$ by defining $c = m_\infty$, with m_t as in (41). Similarly, under some restriction on the parameters this must be a root of the following polynomial in m_∞*

$$0 = \sum_{x=1}^{K} (\alpha_x + \beta_x)(1 - m_\infty) m_\infty^{x-1} - (\gamma_x + \delta_x) m_\infty (1 - m_\infty)^{x-1}.$$

Considering the original model, introduced in [5], i.e., letting $\beta_x = \gamma_x = 1/2$ and $\alpha_x = \delta_x = 0$ for $x = \{1, \dots, K\}$, a simple computation show that the root of the polynomial above is in fact $\frac{1}{2}$.

Appendix A. Replacement Lemmas

In this section we prove the replacements lemmas that are needed in order to get the weak formulation of solutions to the hydrodynamic equation.

A.1. **Dirichlet forms.** Let $\rho : [0,1] \to [0,1]$ be a measurable profile and let $\nu_{\rho(\cdot)}^n$ be the Bernoulli product measure on Ω_N defined by

$$\nu_{\rho(\cdot)}^N(\eta : \eta(x) = 1) = \rho(\tfrac{x}{N}). \tag{44}$$

For a probability measure μ on Ω_N and a density $f : \Omega_N \to \mathbb{R}$ with respect to μ, the Dirichlet form is defined as

$$\langle f, -\mathscr{L}_N f\rangle_\mu = \langle f, -\mathscr{L}_{N,0} f\rangle_\mu + \tfrac{j}{N^\theta}\langle f, -L_{N,b}^L f\rangle_\mu + \tfrac{j}{N^\theta}\langle f, -\mathscr{L}_{N,b}^{NL} f\rangle_\mu, \tag{45}$$

where

$$\langle f, g\rangle_\mu = \int_{\Omega_N} f(\eta) g(\eta)\, d\mu,$$

for all functions $f,g:\Omega_N\to\mathbb{R}$. Our first computation is a comparison between the Dirichlet form just defined and the next quantity:

$$D_N(\sqrt{f},\mu):=D_{N,0}(\sqrt{f},\mu)+\tfrac{j}{N^\theta}D^L_{N,b}(\sqrt{f},\mu)+\tfrac{j}{N^\theta}D^{NL}_{N,b}(\sqrt{f},\mu), \tag{46}$$

where

$$D_{N,0}(\sqrt{f},\mu):=\sum_{x=1}^{n-2}\int_{\Omega_N}\Big[\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\Big]^2d\mu,$$

$$\begin{aligned}D^L_{N,-}(\sqrt{f},\mu)&=\frac{j}{N^\theta}\int(\alpha_1(1-\eta(1))+\gamma_1\eta(1))\Big[\sqrt{f(\eta^1)}-\sqrt{f(\eta)}\Big]^2d\mu,\\ D^L_{N,+}(\sqrt{f},\mu)&=\frac{j}{N^\theta}\int(\beta_1(1-\eta(N-1))+\delta_1\eta(N-1))\Big[\sqrt{f(\eta^{N-1})}-\sqrt{f(\eta)}\Big]^2d\mu,\end{aligned} \tag{47}$$

and

$$\begin{aligned}D^{NL}_{N,-}(\sqrt{f},\mu)&=\frac{j}{N^\theta}\int(\alpha_2\eta(1)(1-\eta(2))+\gamma_2(1-\eta(1))\eta(2))\Big[\sqrt{f(\eta^2)}-\sqrt{f(\eta)}\Big]^2d\mu,\\ D^{NL}_{N,+}(\sqrt{f},\mu)&=\frac{j}{N^\theta}\int(\beta_2\eta(N-1)(1-\eta(N-2))+\delta_2(1-\eta(N-1))\eta(N-2))\Big[\sqrt{f(\eta^{N-2})}-\sqrt{f(\eta)}\Big]^2d\mu.\end{aligned} \tag{48}$$

We claim that for $\theta\geqslant 1$ and for $\rho:[0,1]\to[0,1]$ a constant profile equal to, for example, α, the following bound holds

$$\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_\alpha}\lesssim -D_N(\sqrt{f},\nu_\alpha)+O(\tfrac{1}{N}). \tag{49}$$

From Lemma 5.1 and Lemma 5.2 of [1] it is only necessary to control the contribution from $\mathcal{L}^{NL}_{N,b}$. For that purpose we recall the following lemma from [2]:

Lemma A.1. *Let f be a density with respect to a finite positive measure μ on Ω_N and for $x,y\in\Lambda_N$ let $c_{x,y}(\eta)$ be a positive local function. Then, we have that*

$$\begin{aligned}&\int c_{x,y}(\eta)\Big[\sqrt{f(\eta^{x,y})}-\sqrt{f(\eta)}\Big]\sqrt{f(\eta)}\,d\mu\\ &\lesssim-\int c_{x,y}(\eta)\Big[\sqrt{f(\eta^{x,y})}-\sqrt{f(\eta)}\Big]^2d\mu\\ &+\int\frac{1}{c_{x,y}(\eta)}\Big[c_{x,y}(\eta)-c_{x,y}(\eta^{x,y})\frac{\mu(\eta^{x,y})}{\mu(\eta)}\Big]^2\Big[\sqrt{f(\eta^{x,y})}+\sqrt{f(\eta)}\Big]^2d\mu.\end{aligned} \tag{50}$$

From this lemma it is simple to check that the claim follows.

A.2. **Replacement Lemmas.** We start this section by proving the next lemma which will allow us to prove one of the replacement lemmas that is needed in the proof of hydrodynamics.

Lemma A.2. *Let $x<y\in\Lambda_N$ and let $\varphi:\Omega\to\Omega$ be a positive and bounded function which satisfies $\varphi(\eta)=\varphi(\eta^{z,z+1})$ for any $z=x,\cdots,y-1$. For any density f with respect to ν_α and any positive constant A, it holds that*

$$\Big|\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\alpha}\Big|\lesssim\tfrac{1}{A}D_N(\sqrt{f},\nu_\alpha)+A.$$

Proof. By summing and subtracting appropriate terms, we have that

$$|\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\alpha}|\leqslant\frac{1}{2}\sum_{z=x}^{y-1}\left|\int\varphi(\eta)(\eta(z)-\eta(z+1))[f(\eta)-f(\eta^{z,z+1})]\,d\nu_\alpha\right|$$
$$+\frac{1}{2}\sum_{z=x}^{y-1}\left|\int\varphi(\eta)(\eta(z)-\eta(z+1))[f(\eta)+f(\eta^{z,z+1})]\,d\nu_\alpha\right|.$$

Note that since φ satisfies $\varphi(\eta)=\varphi(\eta^{z,z+1})$ for any $z=x,\cdots,y-1$, by a change of variables, we conclude that the last term in the previous display is equal to zero. Now, we treat the remaining term. Using the equality $(a-b)=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and then Young's inequality, the first term at the right side of last display is bounded from above by a constant times

$$A\int(\varphi(\eta)(\eta(z)-\eta(z+1)))^2\left(\sqrt{f(\eta^{z,z+1})}+\sqrt{f(\eta)}\right)^2d\nu_\alpha+\frac{1}{A}D_N(\sqrt{f},\nu_\alpha).$$

The fact that φ is bounded, $|\eta(x)|\leqslant 1$ and f is a density, the term on the left-hand side of last expression is bounded from above by a constant. This ends the proof. □

Before proceeding with the proof of the Replacement Lemmas, we introduce some notation. From [20] we know that if μ and ν are measures in countable space E , and μ is absolutely continuous with respect to ν, the entropy $H(\mu\,|\,\nu)$ of μ with respect to ν is given by the formula

$$H(\mu\,|\,\nu)=\sum_{x\in E}\mu(x)\log\frac{\mu(x)}{\nu(x)},\tag{51}$$

and is equal to ∞ otherwise. This leads to the also well known Entropy Inequality.

$$\int f(\eta)\mu(d\eta)\leqslant\frac{1}{\gamma}H(\mu\,|\,\nu)+\frac{1}{\gamma}\int e^{\gamma f(\eta)}\nu(d\eta)\tag{52}$$

for $\gamma>0$. We will also use the Feynman-Kac inequality, which can be found for example in [1].

Theorem A.3 (Feynman-Kac inequality)**.** *Let $\{X_t\}_{t\geqslant 0}$ be a Markov process in the countable space E, with infinitesimal generator $\mathscr{L}$. Let ν be a probability measure in E and $V:[0,\infty)\times E\to\mathbb{R}$ be a bounded function. Denote $\mathscr{L}_t=\mathscr{L}+V(t)$ where $V_t=V(t,\cdot)$. Define*

$$\Gamma_t=\sup_{\|f\|_2=1}\left\{\langle V_t,f^2\rangle_\nu+\langle\mathscr{L}f,f\rangle_\nu\right\},\tag{53}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(E,\nu)$ and $\|\cdot\|_2=\langle\cdot,\cdot\rangle_\nu^{\frac{1}{2}}$. Then

$$\mathbb{E}_\nu\left[e^{\int_0^t V_r(X_r)dr}\right]\leqslant e^{\int_0^t\Gamma_s ds}.\tag{54}$$

We are now able to show the first Replacement Lemma.

Theorem A.4. *Let $\varphi:\Omega\to\Omega$ be a positive and bounded function which satisfies $\varphi(\eta)=\varphi(\eta^{z,z+1})$ for any $z=x,\cdots,y-1$. For any $t\in[0,T]$ we have that*

$$\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t\varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|\right]=0.\tag{55}$$

Proof. The expectation above can be written as

$$\int\mathbb{E}_\eta\left[\left|\int_0^t\varphi(\eta_{sn^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|\right]d\mu_n.$$

From entropy inequality, for any $B>0$, last display is bounded from above by

$$\frac{H(\mu_N|\nu_\alpha)}{BN}+\frac{1}{BN}\log\int e^{B\mathbb{E}_\eta\left[\left|\int_0^t \varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|\right]}d\nu_\alpha, \tag{56}$$

and at this point we use Jensen's inequality and we bound the previous display by

$$\frac{H(\mu_N|\nu_\alpha)}{BN}+\frac{1}{BN}\log\int \mathbb{E}_\eta\left[e^{BN\left|\int_0^t \varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|}\right]d\nu_\alpha,$$

which is equal to

$$\frac{H(\mu_N|\nu_\alpha)}{BN}+\frac{1}{BN}\log\mathbb{E}_{\nu_\alpha}\left[e^{BN\left|\int_0^t \varphi(\eta_{sN^2})(\eta_{sN^2}(x)-\eta_{sN^2}(y))ds\right|}\right]. \tag{57}$$

Since $e^{|x|}\leqslant e^x+e^{-x}$ and

$$\limsup_{N\to+\infty}N^{-1}\log(a_N+b_N)=\max\{\limsup_{N\to+\infty}N^{-1}\log(a_N),\limsup_{N\to+\infty}N^{-1}\log(b_N)\},$$

we are able to remove the absolute value inside the exponential in the expectation above. Moreover, a simple computation shows that $H(\mu_N|\nu_\alpha)\leqslant NC_{\alpha,\beta}$. By Feynman-Kac's formula, (57) is bounded from above by

$$\tfrac{C_{\alpha,\beta}}{B}+t\sup_f\Big\{\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\alpha}+\tfrac{N}{B}\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\alpha}\Big\}. \tag{58}$$

The supremum above is over densities f with respect to ν_α. By Lemma A.2 with the choice $A=\frac{B}{N}$ we have that

$$\Big|\langle\varphi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\alpha}\Big|\lesssim\tfrac{N}{B}D_N(\sqrt{f},\nu_\alpha)+\tfrac{B}{N}$$

From (49) and the inequality above, the term on the right-hand side of (58), is bounded from above by

$$\tfrac{B}{N}+\tfrac{1}{N}.$$

Taking $N\to\infty$ and then $B\to+\infty$ we are done. □

Theorem A.5. *Let $\psi:\Omega\to\Omega$ be a positive and bounded function which satisfies $\psi(\eta)=\psi(\eta^{z,z+1})$ for any $z=x+1,\cdots,x+\varepsilon N-1$. For any $t\in[0,T]$ and $x\in\Lambda_N$ such that $x\in\{1,\cdots,N-\varepsilon N-2\}$ we have that*

$$\limsup_{N\to+\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t\psi(\eta_{sN^2})(\eta_{sN^2}(x)-\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x))ds\right|\right]=0. \tag{59}$$

Note that for $x\in\Lambda_N$ such that $x\in\{N-\varepsilon N-1,N-1\}$ the previous result is also true, but we replace in the previous expectation $\overrightarrow{\eta}^{\varepsilon N}_{sN^2}(x)$ by $\overleftarrow{\eta}^{\varepsilon N}_{sN^2}(x)$.

Proof. We present the proof for the case when $x\in\{1,\cdots,N-\varepsilon N-2\}$ but we note that the other case is completely analogous. By applying the same arguments as in the proof of the previous theorem, we can bound from above the previous expectation by

$$\tfrac{C_{\alpha,\beta}}{B}+t\sup_f\Big\{\langle\psi(\eta)(\eta(x)-\overrightarrow{\eta}^{\varepsilon N}(x)),f\rangle_{\nu_\alpha}+\tfrac{N}{B}\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\alpha}\Big\}. \tag{60}$$

where B is a positive constant. The supremum above is over densities f with respect to ν_α. The first term in the supremum above can be written as

$$\frac{1}{\varepsilon n}\sum_{y=x+1}^{x+\varepsilon N}\langle\psi(\eta)(\eta(x)-\eta(y)),f\rangle_{\nu_\alpha}$$

By Lemma A.2 with the choice $A=\frac{B}{N}$ and from (49), the term on the right-hand side of (60), is bounded from above by

$$\tfrac{B}{N}+\tfrac{1}{N}.$$

Taking $N\to\infty$ and then $B\to+\infty$ we are done. □

Appendix B. Energy Estimate

Now we prove that the density $\rho(t,u)$ belongs to the space $L^2(0,T;\mathscr{H}^1(0,1))$, see Definition 2.1. To prove it, we consider the linear functional ℓ_ρ defined in $C_c^{0,1}([0,T]\times(0,1))$ by

$$\ell_\rho(G)=\int_0^T\int_0^1\partial_q G_s(u)\rho(s,u)duds=\int_0^T\int_0^1\partial_q G_s(u)d\pi(s,u)ds.$$

From the next result, we can conclude that ℓ_ρ is $\mathbb{Q}$ almost surely continuous and therefore we can extend this linear functional to $L^2([0,T]\times(0,1))$. As a consequence of the Riesz's Representation Theorem there exists $H\in L^2([0,T]\times(0,1))$ such that

$$\ell_\rho(G)=-\int_0^T\int_0^1 G_s(u)H_s(u)duds$$

for all $G\in C_c^{0,1}([0,T]\times(0,1))$. From this we conclude that $\rho\in L^2(0,T;\mathscr{H}^1(0,1))$.

Proposition B.1. *There exist positive constants C and c such that*

$$\mathbb{E}\Big[\sup_{G\in C_c^{0,1}([0,T]\times(0,1))}\Big\{\ell_\rho(G)-c\|G\|_2^2\Big\}\Big]\leqslant C<\infty.$$

Above $\|G\|_2$ denotes the norm of a function $G\in L^2([0,T]\times(0,1))$.

Proof. By density and by the Monotone Convergence Theorem it is enough to prove that for a countable dense subset $\{G_m\}_{m\in\mathbb{N}}$ on $C_c^{0,2}([0,T]\times(0,1))$ it holds that

$$\mathbb{E}\Big[\max_{k\leqslant m}\{\ell_\rho(G^k)-c\|G^k\|_2^2\}\Big]\leqslant C_0,$$

for any m and for C_0 independent of m. Note that the function that associates to a trajectory $\pi_\cdot\in\mathscr{D}([0,T],\mathscr{M}^+)$ the number $\max_{k\leqslant m}\big\{\ell_\rho(G^k)-c\|G^k\|_2^2\big\}$, is continuous and bounded wrt the Skorohod topology of $\mathscr{D}([0,T],\mathscr{M}^+)$ and for that reason, the expectation in the previous display is equal to the next limit

$$\lim_{N\to\infty}\mathbb{E}_{\mu_N}\Bigg[\max_{k\leqslant m}\Bigg\{\int_0^T\frac{1}{N-1}\sum_{x=1}^{N-1}\partial_u G_s^k(\tfrac{x}{N})\eta_s(x)ds-c\|G^k\|_2^2\Bigg\}\Bigg].$$

By entropy and Jensen's inequalities plus the fact that $e^{\max_{k\leqslant m}a_k}\leqslant\sum_{k=1}^m e^{a_k}$ the previous display is bounded from above by

$$C_0+\frac{1}{N}\log\mathbb{E}_{\nu_\alpha}\Bigg[\sum_{k=1}^m e^{\int_0^T\sum_{x\in\Lambda_N}\partial_u G_s^k(\frac{x}{N})\eta_s(x)ds-cN\|G^k\|_2^2}\Bigg],$$

By linearity of the expectation, to treat the second term in the previous display it is enough to bound the term

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}_{\nu_\alpha^N}\Big[e^{\int_0^T\sum_{x\in\Lambda_N}\partial_u G_s(\frac{x}{N})\eta_s(x)ds-cN\|G\|_2^2}\Big],$$

for a fixed function $G\in C_c^{0,2}([0,T]\times(0,1))$, by a constant independent of G. By Feynman-Kac's formula, the expression inside the limsup is bounded from above by

$$\int_0^T\sup_f\Big\{\frac{1}{N}\int_{\Omega_N}\sum_{x\in\Lambda_N}\partial_u G_s(\tfrac{x}{N})\eta(x)f(\eta)d\nu_\alpha-c\|G\|_2^2+N\langle\mathscr{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_\alpha}\Big\}ds \tag{61}$$

where the supremum is carried over all the densities f with respect to ν_α. Note that by a Taylor expansion on G, it is easy to see that we can replace its space derivative by the discrete gradient $\nabla_N G_s(\frac{x-1}{N})$ by paying an error of order $O(\frac{1}{N})$. Then, from a summation by parts, we obtain

$$\int_{\Omega_N}\sum_{x=1}^{N-2}G_s(\tfrac{x}{N})(\eta(x)-\eta(x+1))f(\eta)d\nu_\alpha$$

By writing the previous term as one half of it plus one half of it and in one of the halves we swap the occupation variables $\eta(x)$ and $\eta(x+1)$, for which the measure ν_α is invariant, last display becomes equal to

$$\frac{1}{2}\int_{\Omega_N}\sum_{x=1}^{N-2}G_s(\tfrac{x}{N})(\eta(x)-\eta(x+1))(f(\eta)-f(\eta^{x,x+1}))d\nu_\alpha. \tag{62}$$

Repeating similar arguments to those used in the proof of Lemma A.2, last term is bounded from above by

$$\begin{aligned}&\frac{1}{4N}\int_{\Omega_N}\sum_{x=1}^{N-2}(G_s(\tfrac{x}{N}))^2(\sqrt{f(\eta)}+\sqrt{f(\eta^{x,x+1})})^2d\nu_\alpha\\ +\;&\frac{1}{4N}\int_{\Omega_N}\sum_{x=1}^{N-2}(\sqrt{f(\eta)}-\sqrt{f(\eta^{x,x+1})})^2d\nu_\alpha\\ \leqslant\;&\frac{C}{N}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2+\frac{1}{4N}D_{0,N}(\sqrt{f},\nu_\alpha)\end{aligned}$$

for some $C>0$. From (49) we get that (61) is bounded from above by

$$C\int_0^T\Big[1+\frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2\Big]ds-c\|G\|_2^2$$

plus an error of order $O(\frac{1}{N})$. Above C is a positive constant independent of G. Since $\frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\frac{x}{N}))^2$ converges, as $N\to+\infty$, to $\|G\|_2^2$, then it is enough to choose $c>C$ to conclude that

$$\limsup_{N\to\infty}\Big\{C\int_0^T\Big[1+\frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2\Big]ds-c\|G\|_2^2\Big\}\lesssim 1$$

and we are done. □

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