Statistical Distributions in Telecommunications

Basic Notions

- The use of statistical models is essential to describe:
  - non-guided propagation in random environments;
  - user's mobility;
  - phone calls and data connections;
  - users' influence in network performance.

Basic Notions

- Given the probability density, $p(x)$, and cumulative distribution functions, $P(x)$, i.e., PDF and CDF, one has

$$p(x) = \frac{dP}{dx}$$

or

$$P(x) = \int_{-\infty}^{x} p(t)\, dt$$

where

$$P(x) = \mathrm{Prob}(X \leq x)$$

- One has

$$\mathrm{Prob}(|X| \leq x) = P(x) - P(-x)$$

Basic Notions

- The main parameters are:
  - average, $\overline{x}$

$$\overline{x} = \int_{-\infty}^{\infty} x \; p(x)\, dx$$

  - mean square, $\overline{x^2}$

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 \; p(x)\, dx$$

  - median, x_m

$$\int_{-\infty}^{x_m} p(x)\, dx = \int_{x_m}^{\infty} p(x)\, dx = 1/2$$

Basic Notions

- mode, m_x

$$p(m_x) = \max[p(x)]$$

- moments, μ_n

$$\mu_n = \int_{-\infty}^{\infty} \left(x - \bar{x}\right)^n \, p(x) \, dx$$

- variance, $\sigma_x^{\,2}$, and standard deviation, σ_x,

$$\sigma_x^{\,2} = \mu_2$$

- Chebyshev's inequality allows to quantify the dispersion of the random variable:

$$\mathrm{Prob}\left(\left|x - \bar{x}\right| \geq k\sigma_x\right) \leq \frac{1}{k^2}$$

Uniform Distribution

- CDF and PDF:

$$p(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0 \quad, & \text{o.c.} \end{cases}$$

$$P(x) = \begin{cases} 0 \quad, & x < a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1 \quad, & b < x \end{cases}$$

Uniform Distribution

- Parameters:

$$\overline{x} = x_m = \frac{a+b}{2}$$

$$\sigma_x = \frac{b-a}{\sqrt{12}}$$

$$\overline{x^2} = \frac{a^2+ab+b^2}{3}$$

- It is used to describe the phase of a signal.

Normal (Gauss) Distribution

- PDF and CDF:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\left(\frac{x-\overline{x}}{\sqrt{2}\sigma_x}\right)^2} \quad , \; x \in \mathrm{IR}$$

$$P(x) = \frac{1 + \mathrm{erf}\left(\frac{x-\overline{x}}{\sqrt{2}\sigma_x}\right)}{2}$$

- Parameters:

$$\overline{x} = x_m = m_x$$

$$\overline{x^2} = \overline{x}^2 + \sigma_x^2$$

Normal (Gauss) Distribution

- Approximation for $P(x)$

$$P(u) \cong 1 - \frac{e^{-u^2/2}}{\sqrt{2\pi}\left(0.661u + 0.339\sqrt{u^2 + 5.51}\right)}$$

where

$$u = \frac{x - \overline{x}}{\sigma_x}$$

for $u > 0$.

Alternatively,

$$\operatorname{erf}(v) \cong 1 - \sqrt{\pi}\,\frac{e^{-v^2}}{(\pi - 1)v + \sqrt{v^2 + \pi}} \quad , \quad v > 0$$

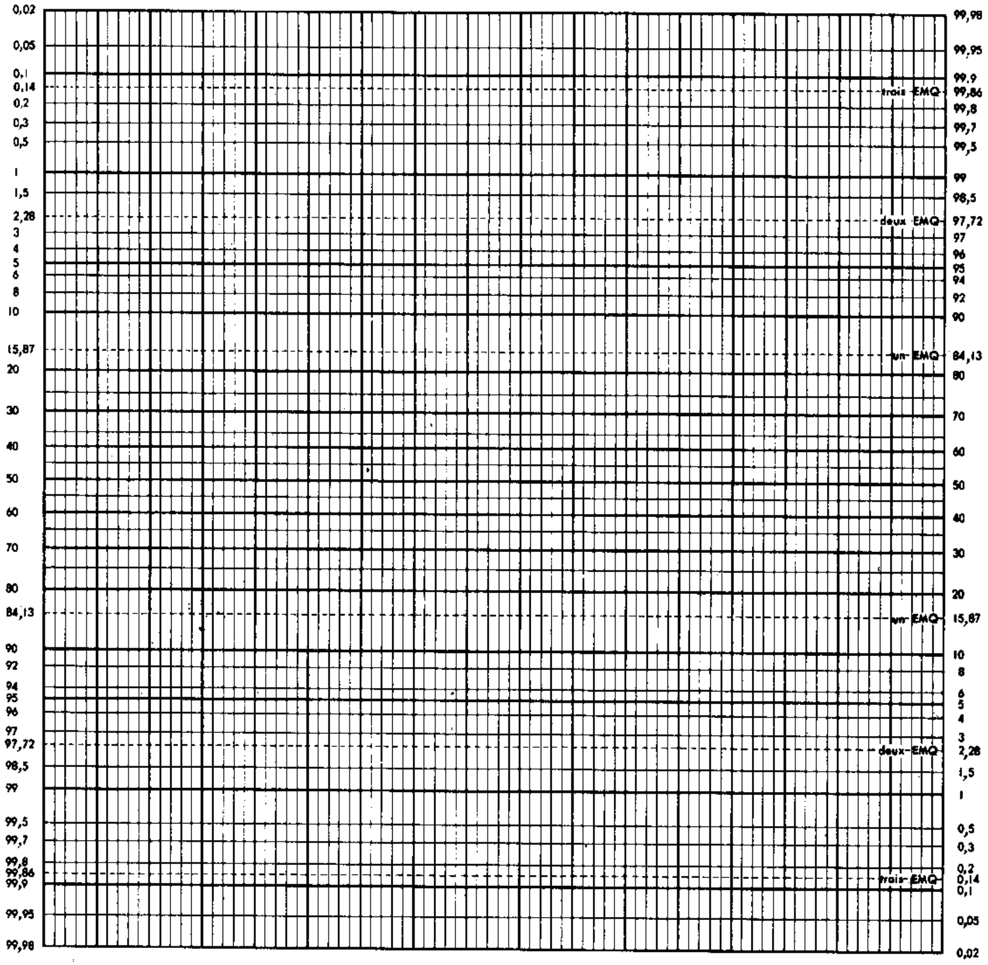
Normal (Gauss) Distribution

- Occurrence intervals

u	$1 - P(u)$	u	$1 - P(u)$
0	$5.000 \cdot 10^{-1}$	1.282	10^{-1}
1	$1.587 \cdot 10^{-1}$	2.326	10^{-2}
2	$2.275 \cdot 10^{-2}$	3.090	10^{-3}
3	$1.350 \cdot 10^{-3}$	3.719	10^{-4}
4	$3.167 \cdot 10^{-5}$	4.265	10^{-5}
5	$2.867 \cdot 10^{-7}$	4.753	10^{-6}

Normal (Gauss) Distribution

- Diagrams in Gauss Scale



[*Source: Boithias, 1983*]

Normal (Gauss) Distribution

NoDi (5/5)

- It is used to describe fluctuations around a mean value.
- It cannot be used to describe entities that cannot be negative.

Log-Normal Distribution

LNDi (1/3)

- PDF and CDF:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_\ell} \frac{1}{x} e^{-\left(\frac{\ln(x) - \overline{x_\ell}}{\sqrt{2}\sigma_\ell}\right)^2} \;,\; x > 0$$

$$P(x) = \frac{1 + \operatorname{erf}\left(\frac{\ln(x) - \overline{x_\ell}}{\sqrt{2}\sigma_\ell}\right)}{2}$$

Log-Normal Distribution

- Parameters:

$$\overline{x} = e^{\overline{x_\ell} + \sigma_\ell^2/2}$$

$$\overline{x^2} = e^{2\left(\overline{x_\ell} + \sigma_\ell^2\right)}$$

$$x_m = e^{\overline{x_\ell}}$$

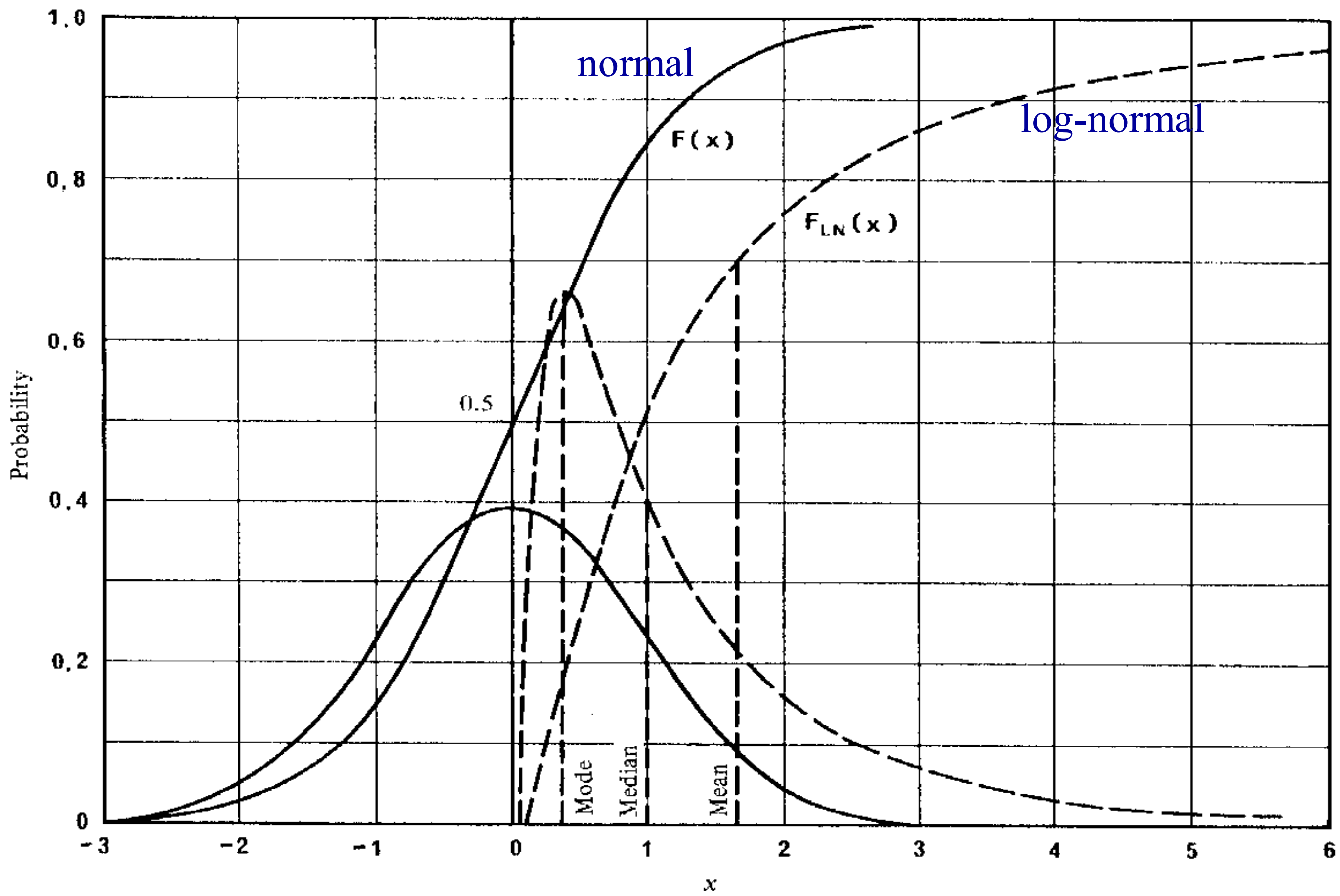
$$m_x = e^{\overline{x_\ell} - \sigma_\ell^2}$$

$$\sigma_x = \sqrt{e^{\sigma_\ell^2} - 1}\, e^{\overline{x_\ell} + \sigma_\ell^2/2}$$

- It is used to describe entities signal power or amplitude, namely slow fading.

Log-Normal Distribution

- PDF and CDF:



[*Source: ITU-R, Vol. V, Rep. 1007*]

Rayleigh Distribution

- PDF and CDF:

$$p(x) = \frac{2x}{\overline{\overline{x^2}}} \, \mathrm{e}^{-x^2/\overline{x^2}} \quad , \quad x > 0$$

$$P(x) = 1 - e^{-x^2/\overline{x^2}}$$

- Parameters:

$$\overline{x} = \sqrt{\pi \, \overline{\overline{x^2}}} \Big/ 2$$

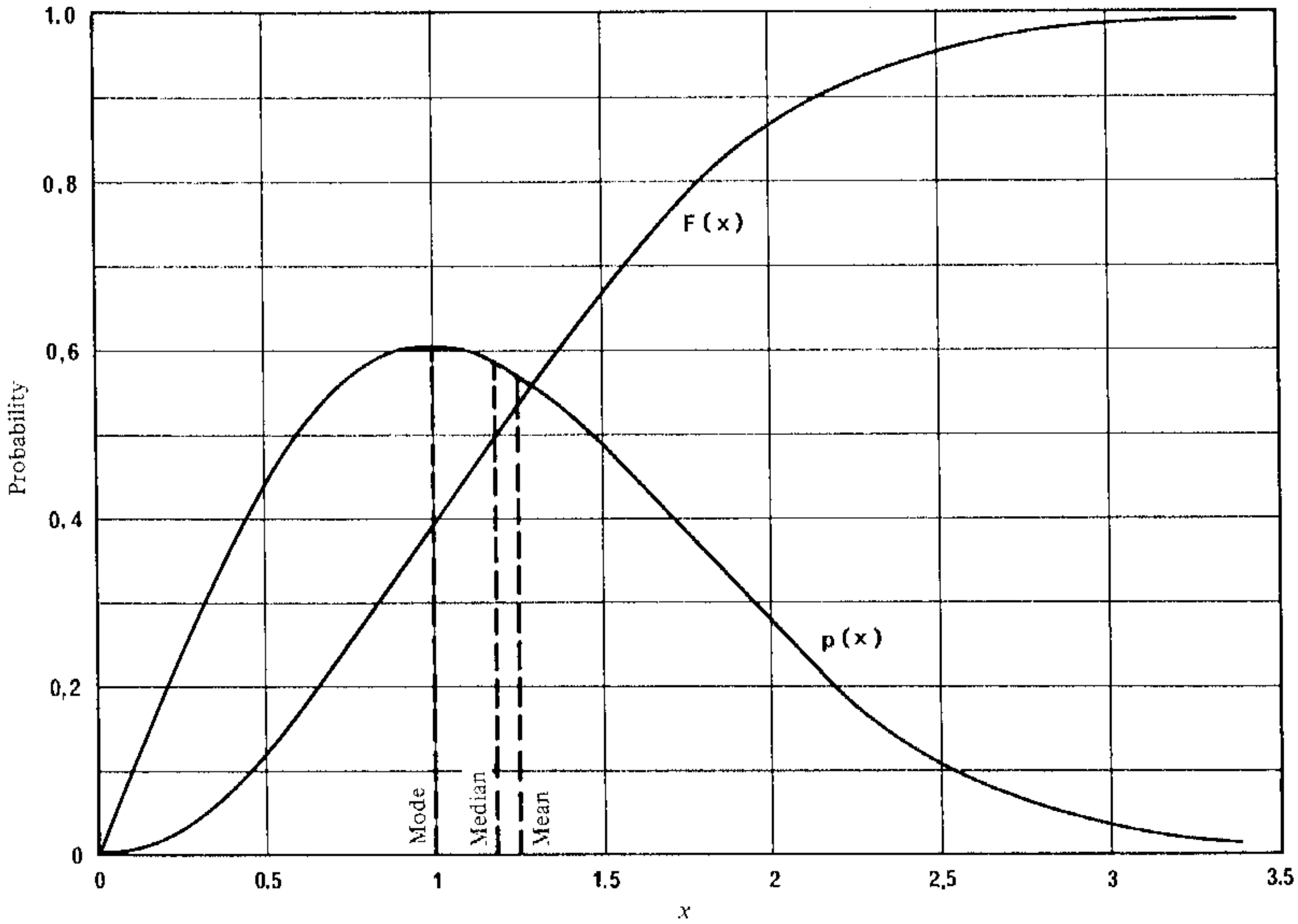
$$m_x = \sqrt{\overline{\overline{x^2}} \Big/ 2}$$

$$x_m = \sqrt{\ln(2) \, \overline{\overline{x^2}}}$$

$$\sigma_x = \sqrt{(4-\pi) \, \overline{\overline{x^2}}} \Big/ 2$$

Rayleigh Distribution

- PDF and CDF:



[*Source: ITU-R, Vol. V, Rep. 1007*]

Rayleigh Distribution

- It is associated to the magnitude of the sum of vectors having amplitudes with a normal distribution and phases with a uniform one.
- It is used to describe intense fast fading.

Suzuki Distribution

- PDF and CDF:

$$p_R(x,v) = \frac{x}{v^2/2} e^{-x^2/v^2}$$

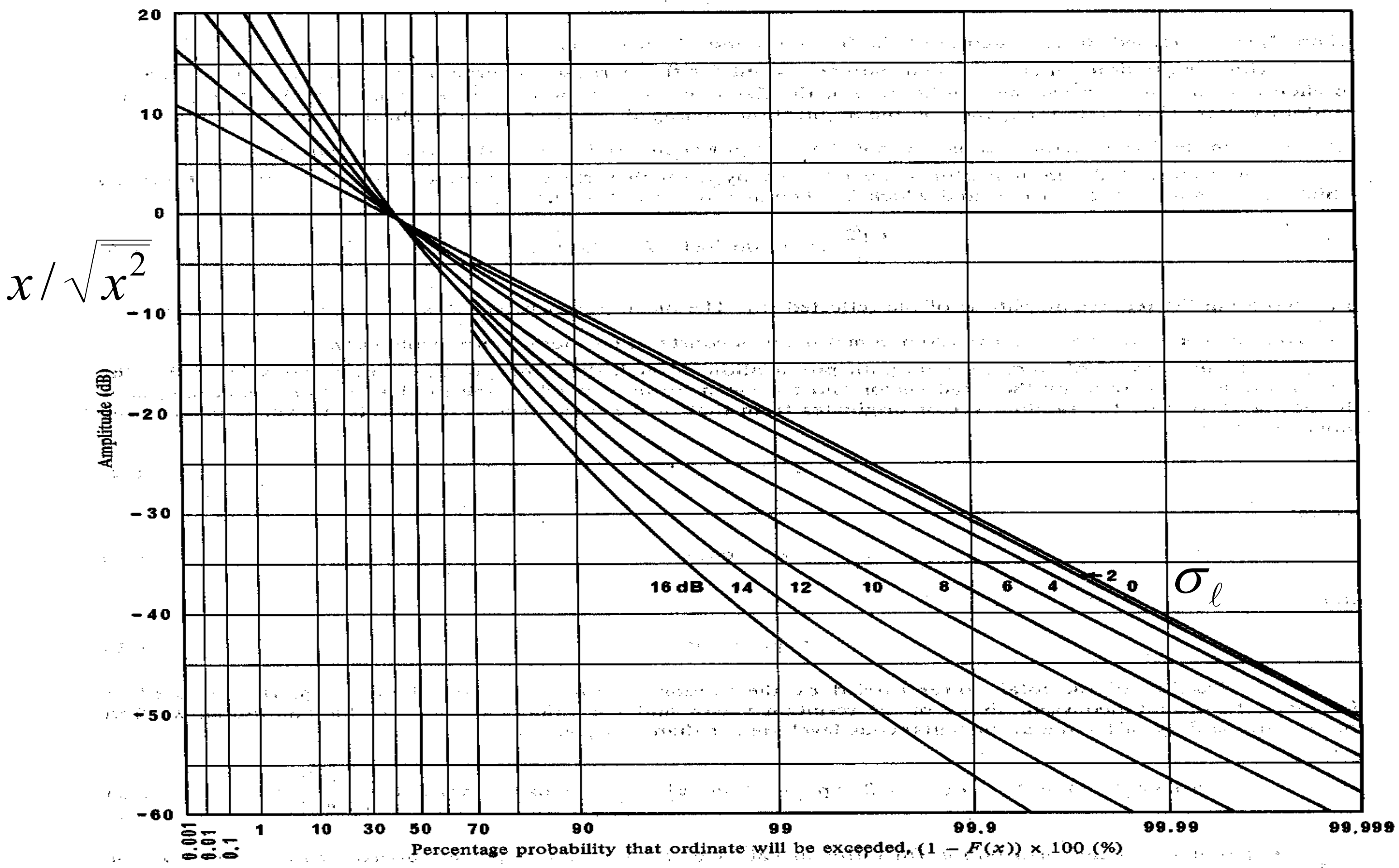
$$p_{LN}(v) = \frac{1}{\sqrt{2\pi}\,\sigma_\ell} \frac{1}{v} e^{-\left(\frac{\ln(v)-\overline{v_\ell}}{\sqrt{2}\,\sigma_\ell}\right)^2}$$

$$P(x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2 e^{-2\sigma_\ell t} - t^2/2} \, dt \quad (\overline{v_\ell} = 0)$$

- It describes joint slow and fast fading.

Suzuki Distribution

- CDF:



[*Source: ITU-R, Vol. V, Rep. 1007*]

Rice Distribution

RiDi (1/3)

- PDF:

$$p(x) = \frac{2x}{\overline{x_R^2}}\, e^{-\left(x^2 + x_0^2\right)/\overline{x_R^2}}\ \mathrm{I}_0\!\left(\frac{x\,x_0}{\overline{x_R^2}/2}\right)$$

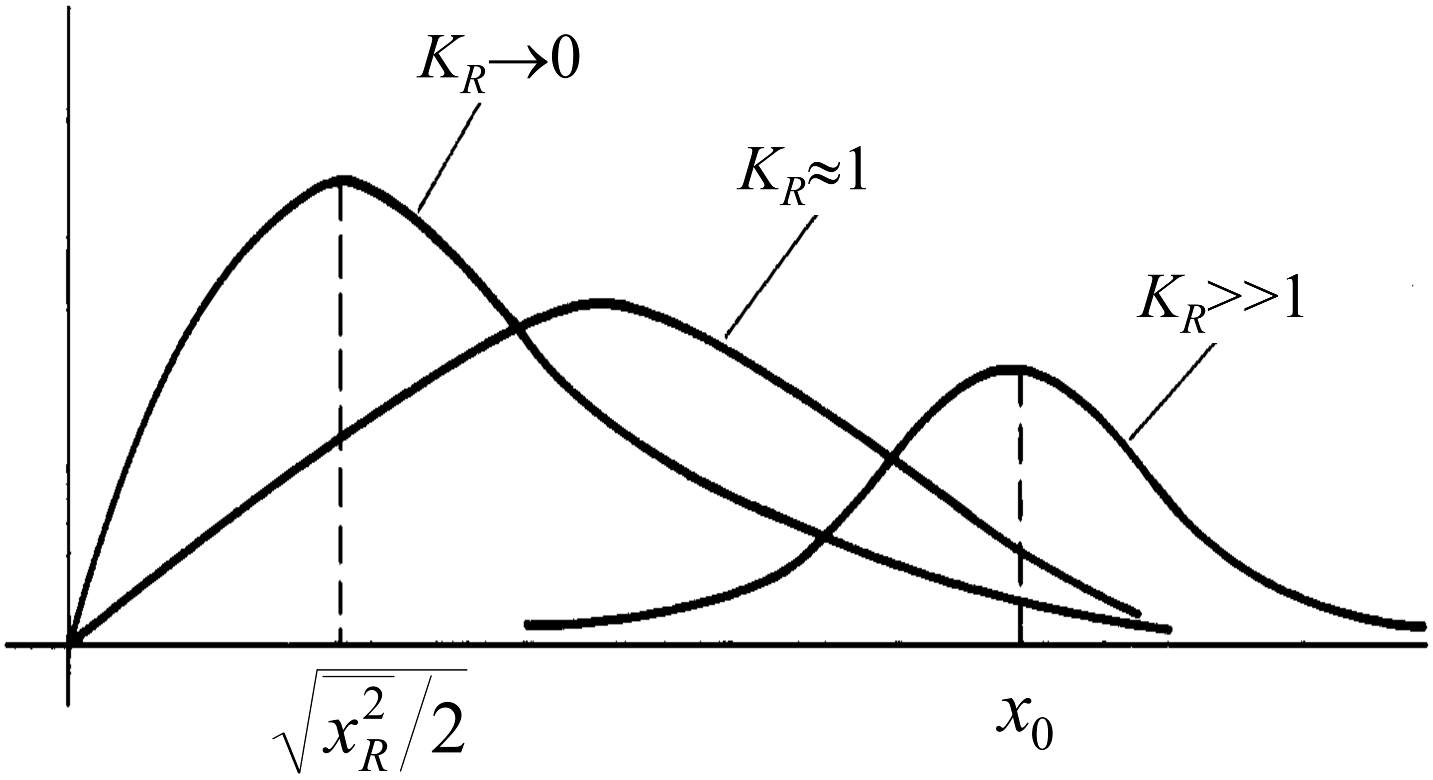
describing the sum of a fixed vector (amplitude x_0) with a Rayleigh distributed vector (power $\overline{x_R^2}$).

- It is used to describe non-intense fast fading.
- Usually, the Rice parameter is used

$$K_{R\,[\mathrm{dB}]} = 10\log\left(x_0^2 \big/ \overline{x_R^2}\right)$$

Rice Distribution

- PDF:

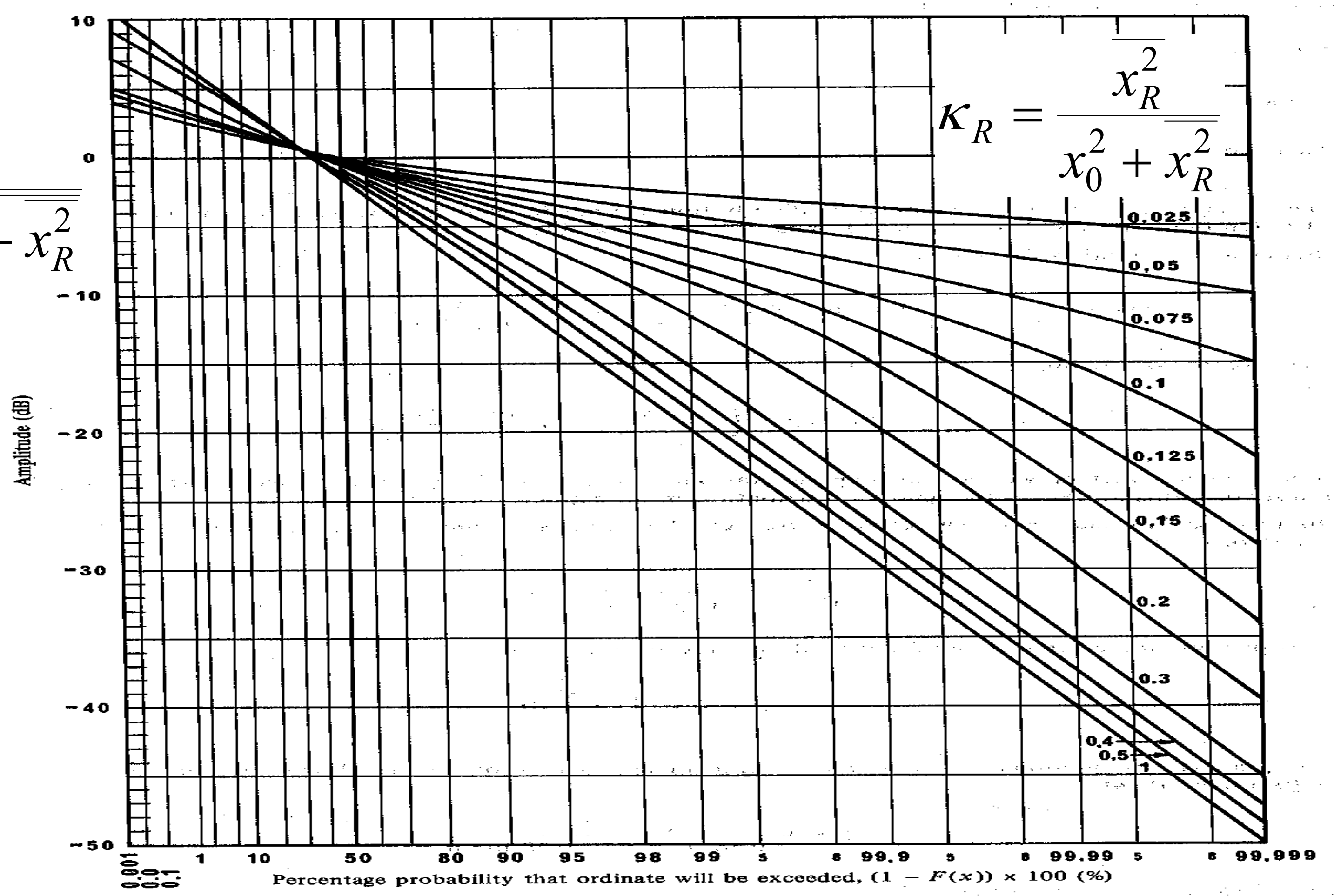


[*Source: Parsons, 1992*]

Rice Distribution

- CDF:

$$\frac{x}{\sqrt{x_0^2 + \overline{x_R^2}}}$$



[*Source: ITU-R, Vol. V, Rep. 1007*]

Exponential Distribution

- PDF and CDF:

$$p(x) = \frac{1}{\bar{x}} \mathrm{e}^{-x/\bar{x}} \quad , \quad x > 0$$

$$P(x) = 1 - e^{-x/\bar{x}}$$

- Parameters:

$$x_m = \bar{x} \ln(2)$$

$$m_x = 0$$

$$\overline{x^2} = 2\,\bar{x}^2$$

$$\sigma_x^2 = \bar{x}^2$$

Exponential Distribution

ExDi (2/2)

- It is used to describe the duration of various phenomena, namely associated to signal fading and phone calls.

Bernouli Distribution

- Mass probability function

$$p(s) = q^s\,(1-q)^{1-s} \quad , \quad s = 0, 1$$

where q is the probability of occurring 1.

- Parameters are

$$\bar{s} = q$$

$$\sigma_s^2 = q\,(1-q)$$

- It is used to describe the occupation of a telecommunications channel.

Binomial Distribution

BiDi (1/2)

- Mass probability function:

$$p(k)=\binom{n}{k}q^k\,(1-q)^{n-k}\quad,\ k=0,\ldots,n$$

where q is the occurrence probability for each of the n times.

- Parameters

$$\overline{k}=q$$

$$m_k=\text{int}[(n+1)]\,q$$

$$\sigma_k^2=n\,q\,(1-q)$$

Binomial Distribution

BiDi (2/2)

- It is used to describe phone calls, where $q = \lambda t/n$, t being the sampling time interval and λ the average phone calls generation.

Poisson Distribution

- Mass probability function:

$$p(k) = \frac{\alpha^k}{k!} e^{-\alpha}$$

- Parameters:

$$\overline{k} = \alpha$$

$$\sigma_k^2 = \alpha$$

$$m_k = \begin{cases} 0 & , \alpha < 1 \\ \alpha & , \alpha > 1 \end{cases}$$

- It is used to describe the generation of phone calls ($\alpha = \lambda t$).

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