DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

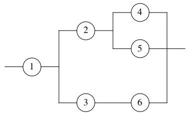
Reliability and Quality Control

MMA

Exame Época Especial
2016/2017
24/07/2017 – **09:00AM**

- Duration: 3 hours
- Please justify your answers.
- This test has **three pages** and **seven questions**. The total of points is **40.0**.

1. A lifting station for wastewater containing sewage depends on 6 components set according to the following system block diagram:



(a) Identify the minimal path sets and minimal cut sets of this system, and provide an expression (do not simplify it!) for its structure function. **(1.5)**

- **Minimal path sets**

$$\begin{aligned} \mathscr{P}_1 &= \{1,2,4\} \\ \mathscr{P}_2 &= \{1,2,5\} \\ \mathscr{P}_3 &= \{1,3,6\} \\ p^* &= 3 \text{ minimal path sets} \end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned} \mathscr{K}_1 &= \{1\} \\ \mathscr{K}_2 &= \{2,3\} \\ \mathscr{K}_3 &= \{2,6\} \\ \mathscr{K}_4 &= \{4,5,3\} = \{3,4,5\} \\ \mathscr{K}_5 &= \{4,5,6\} \\ q &= 5 \text{ minimal cut sets} \end{aligned}$$

- **Structure function**

$$\begin{aligned} \phi(\underline{X}) &\stackrel{Th.1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathscr{P}_j} X_i\right) \\ &= 1-(1-X_1X_2X_4)\times(1-X_1X_2X_5)\times(1-X_1X_3X_6) \end{aligned}$$

Obs. — Equivalently,

$$\begin{aligned} \phi(\underline{X}) &\stackrel{Th.1.30}{=} \prod_{j=1}^{q}\left[1-\prod_{i\in\mathscr{K}_j}(1-X_i)\right] \\ &= [1-(1-X_1)]\times[1-(1-X_2)(1-X_3)]\times[1-(1-X_2)(1-X_6)] \\ &\quad \times[1-(1-X_3)(1-X_4)(1-X_5)]\times[1-(1-X_4)(1-X_5)(1-X_6)]. \end{aligned}$$

(b) Assume that those 6 components operate independently, with reliabilities p_i, $i=1,\dots,6$. **(3.0)**
Obtain the three following quantities: the reliability of the system; the importances of the reliabilities of components 2 and 3.
How do these two importances compare when $p_i = p$, $i=1,\dots,6$?

- **Reliability of the components**
p_i, $i=1,\dots,6$
- **Reliability**
Since $X_i \stackrel{indep}{\sim} \text{Bernoulli}(p_i)$ and $X_i^k \sim X_i$, $k\in\mathbb{N}$, we get, for $\underline{p}=(p_1,\dots,p_6)$:

$$\begin{aligned} r(\underline{p}) &= E\left[\phi(\underline{X})\right] \\ &\stackrel{(a)}{=} E[1-(1-X_1X_2X_4)\times(1-X_1X_2X_5)\times(1-X_1X_3X_6)] \\ &= E\left[1-(1-X_1X_2X_4-X_1X_2X_5+X_1^2X_2^2X_4X_5)\times(1-X_1X_3X_6)\right] \\ &= E[1-(1-X_1X_2X_4-X_1X_2X_5+X_1X_2X_4X_5)\times(1-X_1X_3X_6)] \\ &= E(1-1+X_1X_2X_4+X_1X_2X_5-X_1X_2X_4X_5 \\ &\quad +X_1X_3X_6-X_1^2X_2X_3X_4X_6-X_1^2X_2X_3X_5X_6+X_1^2X_2X_3X_4X_5X_6) \\ &= E(X_1X_2X_4+X_1X_2X_5-X_1X_2X_4X_5 \\ &\quad +X_1X_3X_6-X_1X_2X_3X_4X_6-X_1X_2X_3X_5X_6+X_1X_2X_3X_4X_5X_6) \\ &= p_1p_2p_4+p_1p_2p_5-p_1p_2p_4p_5+p_1p_3p_6-p_1p_2p_3p_4p_6-p_1p_2p_3p_5p_6 \\ &\quad +p_1p_2p_3p_4p_5p_6 \end{aligned}$$

- **Importance of the reliability of components 2 and 3**

$$\begin{aligned} I_r(i) &\stackrel{(1.29)}{=} \frac{\partial r(\underline{p})}{\partial p_i} \\ &= \begin{cases} p_1p_4+p_1p_5-p_1p_4p_5-p_1p_3p_4p_6-p_1p_3p_5p_6+p_1p_3p_4p_5p_6, & i=2 \\ p_1p_6-p_1p_2p_4p_6-p_1p_2p_5p_6+p_1p_2p_4p_5p_6, & i=3 \end{cases} \end{aligned}$$

- **Comparing the importance of the reliability of components 2 and 3 when $p_i = p$**
Given that

$$\begin{aligned} I_r(2)-I_r(3) &\stackrel{p_i=p}{=} (2p^2-p^3-2p^4+p^5)-(p^2-2p^4+p^5) \\ &= p^2-p^3 \\ &= p^2(1-p) \\ &\stackrel{p\in(0,1)}{\geq} 0, \end{aligned}$$

the reliability of component 2 is more important than the reliability of component 3.

(c) Suppose now that $p_i = p = 0.99$, $i=1,\dots,6$. **(3.0)**
Provide two pairs of non trivial bounds for the reliability of the system.
Which bounds are stricter?

- **Reliability of the components**
$p_i = p = 0.99$, $i=1,\dots,6$
- **First pair of bounds for the reliability $r(\underline{p})$**
The components form a coherent system and operate independently, therefore...

Lower bound

$$\begin{aligned} r(\underline{p}) &\stackrel{Th.1.68}{\geq} \prod_{j=1}^{q}\left[1-\prod_{i\in\mathscr{K}_j}(1-p_i)\right] \\ &\stackrel{p_i=p}{=} \prod_{j=1}^{q}\left[1-(1-p)^{\#\mathscr{K}_j}\right] \\ &= [1-(1-p)]\times[1-(1-p)^2]^2\times[1-(1-p)^3]^2 \\ &\stackrel{p=0.99}{=} 0.989800 \end{aligned}$$

Upper bound

$$\begin{aligned} r(\underline{p}) &\stackrel{Th.,1.68}{\leq} 1-\prod_{j=1}^{p^*}\left(1-\prod_{i\in\mathscr{P}_j} p_i\right) \\ &\stackrel{p_i=p}{=} 1-\prod_{j=1}^{p^*}\left(1-p^{\#\mathscr{P}_j}\right) \\ &\stackrel{\#\mathscr{P}_j=2,\forall j}{=} 1-(1-p^3)^3 \end{aligned}$$

$$r(\underline{p}) \overset{p=0.95}{\leq} 0.999974.$$

- **Another pair of bounds for the reliability** $r(\underline{p})$
  Since the components form a coherent system and operate in an independent fashion — hence, in a positively associated manner —, we can also apply Theorem 1.70 (min-max bounds!).

  Another lower bound

$$\begin{aligned}
r(\underline{p}) &\overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i \in \mathscr{P}_j} p_i \\
&\overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathscr{P}_j} \\
&= p^{\min_{j=1,\ldots,p^*} \#\mathscr{P}_j} \\
&\overset{\#\mathscr{P}_j=3,\forall j}{=} p^3 \\
&= 0.970299.
\end{aligned}$$

  Another upper bound

$$\begin{aligned}
r(\underline{p}) &\overset{(1.42)}{\leq} \min_{j=1,\ldots,q} \left[1 - \prod_{i \in \mathscr{K}_j} (1-p_i)\right] \\
&\overset{p_i=p}{=} \min_{j=1,\ldots,q} \left[1-(1-p)^{\#\mathscr{K}_j}\right] \\
&= \left[1-(1-p)^{\min_{j=1,\ldots,q} \#\mathscr{K}_j}\right] \\
&\overset{\#\mathscr{K}_j=1,2,3}{=} 1-(1-p)^1 \\
&= p \\
&= 0.99.
\end{aligned}$$

- **Which bounds are stricter?**
  Since 0.989800 > 0.970299 (resp. 0.999974 > 0.99), the lower (resp. upper) bound given by Theo. 1.68 (resp. 1.70) is stricter than the one obtain by invoking Theo. 1.70 (resp. 1.68).

2. A leaf spring for a truck is assumed to have a time to failure (X, in years) with hazard rate function equal to $\lambda(t) = \left(\frac{t}{4}\right)^3$, $t \geq 0$.

(a) What is the probability that this leaf spring fails after age $E(X)$? **(2.0)**

Obs.: It might be useful to know that $\Gamma(1/4) \simeq 3.625610$.

- **R.v.**
  X = time to failure of a leaf spring for a truck
- **Hazard rate function**
  $\lambda(t) = \left(\frac{t}{4}\right)^3$, $t \geq 0$
- **Survival function**
  Using Prop. 3.3, we get

$$\begin{aligned}
R(t) &= \exp\left[-\int_0^t \lambda(u)\,du\right] \\
&= \exp\left[-\int_0^t \left(\frac{u}{4}\right)^3 du\right] \\
&= \exp\left[-\left(\frac{u}{4}\right)^4\Big|_0^t\right] \\
&= \exp\left[-\left(\frac{t}{4}\right)^4\right], \; t \geq 0.
\end{aligned}$$

- **Important**
  According to Definition 4.21, this survival function coincides with the one of a Weibull distribution with shape parameter (α) and scale parameter (δ) both equal to 4.
- **Expected value**
  Since $X \sim \text{Weibull}(\delta = 4, \alpha = 4)$, we get

$$\begin{aligned}
E(X) &\overset{Exer. 4.22}{=} \delta\,\Gamma\left(\frac{1}{\alpha}+1\right) \\
&= 4 \times \Gamma\left(\frac{1}{4}+1\right) \\
&= 4 \times \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \\
&\simeq 3.625610
\end{aligned}$$

- **Requested probability**

$$\begin{aligned}
R[E(X)] &= \exp\left\{-\left[\frac{E(X)}{4}\right]^4\right\} \\
&= \exp\left(-\frac{3.625610^4}{256}\right) \\
&\simeq 0.509172.
\end{aligned}$$

(b) What would be the expected time to failure if 3 out of 4 leaf springs are crucial from a safety standpoint of the truck? Assume that $T_i \overset{i.i.d.}{\sim} X$, where T_i represent the time to failure of the leaf spring i ($i = 1,\ldots,4$). **(2.0)**

- **Individual times to failure, common reliability function**
  T_i = time to failure of leaf spring i, $i = 1,\ldots,4$
  $T_i \overset{i.i.d.}{\sim} \text{Weibull}(\delta = 4, \alpha = 4)$, $i = 1,\ldots,4$
  $R_i(t) = R(t) = \exp\left[-\left(\frac{t}{4}\right)^4\right]$, $t \geq 0$
- **Duration of the system**
  We are dealing with a $3-out-of-4$ system, thus

$$T = T_{(n-k+1)} = T_{(4-3+1)} = T_{(2)}.$$

- **Reliability function of** T

$$\begin{aligned}
R_T(t) &\overset{(2.8)}{=} F_{binomial(4,1-R(t))}(4-3) \\
&= \sum_{j=0}^{1} \binom{4}{j} [1-R(t)]^j\,[R(t)]^{4-j} \\
&= [R(t)]^4 + 4\,[1-R(t)]\,[R(t)]^3 \\
&= 4\,[R(t)]^3 - 3\,[R(t)]^4, \; t \geq 0
\end{aligned}$$

- **Requested expected value**
  T is a non negative r.v. as a consequence

$$\begin{aligned}
E(T) &\overset{(2.10)}{=} \int_0^{+\infty} R_T(t)\,dt \\
&= \int_0^{+\infty} \{4\,[R(t)]^3 - 3\,[R(t)]^4\}\,dt \\
&= 4\int_0^{+\infty} [R(t)]^3\,dt - 3\int_0^{+\infty} [R(t)]^4\,dt \\
&\overset{(a)}{=} 4\int_0^{+\infty} \exp\left[-\left(\frac{t}{\frac{\delta}{3^{1/\alpha}}}\right)^\alpha\right] dt - 3\int_0^{+\infty} \exp\left[-\left(\frac{t}{\frac{\delta}{4^{1/\alpha}}}\right)^\alpha\right] dt \\
&= 4\int_0^{+\infty} R_{Weibull(\frac{\delta}{3^{1/\alpha}},\alpha)}(t)\,dt - 3\int_0^{+\infty} R_{Weibull(\frac{\delta}{4^{1/\alpha}},\alpha)}(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
E(T) &= 4\times\frac{\delta}{3^{1/\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)-3\times\frac{\delta}{4^{1/\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)\\
&= \left(\frac{4}{3^{1/\alpha}}-\frac{3}{4^{1/\alpha}}\right)\times\delta\Gamma\left(1+\frac{1}{\alpha}\right)\\
&= \left(\frac{4}{3^{1/4}}-\frac{3}{4^{1/4}}\right)\times E(X)\\
&\overset{(a)}{=} 0.918022\times 3.625610\\
&\simeq 3.32839.
\end{aligned}
$$

(c) What can be said about the stochastic ageing character of this system with four leaf springs? (**1.0**)

- **Devising the stochastic ageing character of T**
  Since $\lambda(t)=\left(\frac{t}{4}\right)^3$, $t\geq 0$, is an increasing function in t, we can conclude that

  $$T_i \overset{i.i.d.}{\sim} IHR,\ i=1,\dots,n.$$

  By applying Prop. 3.25, we can add that any order statistic is also IHR, namely

  $$T=T_{(2)}\in IHR.$$

(d) Now, suppose the four leaf springs operate in a positively associated manner. (**1.5**)
Obtain an appropriate lower bound for the expected time to failure of this *new* system.

- **Important**
  We are dealing with positively associated r.v. therefore we can resort to Theorem 2.22 to provide bounds for R_T.
- **Lower bound for $R_T(t)$**
  Note that the minimal path sets associated with a $3-out-of-4$ system are

  $$
  \begin{aligned}
  \mathscr{P}_1 &= \{1,2,3\}\\
  \mathscr{P}_2 &= \{1,2,4\}\\
  \mathscr{P}_3 &= \{1,3,4\}\\
  \mathscr{P}_4 &= \{2,3,4\}\\
  p^* &= 4 \text{ minimal path sets.}
  \end{aligned}
  $$

  Consequently,

  $$
  \begin{aligned}
  R_T(t) &\overset{T2.22}{\geq} \max_{j=1,\dots,p^*}\left[\prod_{i\in\mathscr{P}_j}R_i(t)\right]\\
  &\overset{R_i(t)=R(t)}{=} \max_{j=1,\dots,p^*}[R(t)]^{\#\mathscr{P}_j}\\
  &= [R(t)]^{\min_{j=1,\dots,p^*}\#\mathscr{P}_j}\\
  &\overset{\#\mathscr{P}_j=3,\forall j}{=} [R(t)]^3.
  \end{aligned}
  $$

- **Lower bound for μ**

  $$
  \begin{aligned}
  E(T) &\overset{(2.10)}{=} \int_0^{+\infty}R_T(t)\,dt\\
  &\geq \int_0^{+\infty}[R(t)]^3\,dt\\
  &\overset{(b)}{=} \int_0^{+\infty}R_{Weibull(\frac{\delta}{3^{1/\alpha}},\alpha)}(t)\,dt\\
  &= \frac{1}{3^{1/4}}\times E(X)\\
  &\simeq 2.75486.
  \end{aligned}
  $$

3. Twenty-four vehicles have completed a 74 000 kilometer scheduled run at an automobile company's proving grounds. During the test eight radiator hoses failed and were replaced at kilometers: 2 760, 3 700, 7 100, 17 220, 29 500, 48 400, 52 600, 65 000.

(a) Is the exponential assumption reasonable in light of this data at a 5% significance level? (**2.0**)

- **Life test**
  Since the test had a scheduled end after exactly $t_0=74000$ km and as soon as radiator hose failed it was replaced with a new one, we are dealing with a
  - Type I/item censored testing with replacement.
- **R.v.**
  $T_{(i)}$ = time of the i^{th} failure
  $Z_i=T_{(i)}-T_{(i-1)}$ = time between the i^{th} and $(i-1)^{th}$ failures
  $Z_i \overset{i.i.d.}{\sim} Z,\ i\in\mathbb{N}$
- **Censored data**
  $n=24$
  $r=8$ failures during the life test
  $(t_{(1)},\dots,t_{(r)})=(2760,3700,7100,17220,29500,48400,52600,65000)$
  $(z_1,\dots,z_r)=(2760,940,3400,10120,12280,18900,4200,12400)$
- **Hypotheses**
  $H_0: Z\sim\text{Exponential}(\lambda)$
  $H_1: Z\sim\text{Weibull}(\lambda^{-1},\alpha),\ \alpha\neq 1$
- **Significance level**
  $\alpha_0=5\%$
- **Test statistic** (Bartlett's test)

  $$
  \begin{aligned}
  B_r &\overset{(5.19)}{=} \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r Z_i}{r}\right)-\frac{1}{r}\sum_{i=1}^r\ln(Z_i)\right]\\
  &\overset{a}{\sim}_{H_0} \chi^2_{(r-1)}
  \end{aligned}
  $$

- **Rejection region of H_0**

  $$
  \begin{aligned}
  W &= \left(0,F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right)\cup\left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2),+\infty\right)\\
  &\overset{r=8,\alpha_0=0.05}{=} (0,1.690)\cup(16.01,+\infty)
  \end{aligned}
  $$

- **Decision**
  The observed value of the test statistic is

  $$
  \begin{aligned}
  b_r &= \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r z_i}{r}\right)-\frac{1}{r}\sum_{i=1}^r\ln(z_i)\right]\\
  &\simeq \frac{2\times 8}{1+\frac{8+1}{6\times 8}}\times\left[\ln\left(\frac{65\,000}{8}\right)-\frac{1}{8}\times 69.1536\right]\\
  &\simeq 4.83033\\
  &\notin W=(0,1.690)\cup(16.01,+\infty).
  \end{aligned}
  $$

  Therefore we should not reject H_0 for at significance level $\alpha_0\leq 5\%$ [or at any at significance level smaller than 5%].

(b) Calculate the ML estimate of the expected number of replacements of the original radiator hoses during the 24 000 kilometer warranty period assuming that: two million vehicles have been sold; proving ground kilometers are assumed to be equivalent to customer kilometers; the times to replacement are exponentially distributed. (**2.0**)

- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is in this particular case:
  $$\begin{aligned}\tilde{t} &= n \times t_0 \\ &= 24 \times 74\,000 \\ &= 1\,776\,000\end{aligned}$$
- **ML estimate of λ**
  According to Table 5.13, it is equal to
  $$\begin{aligned}\hat{\lambda} &= \frac{r}{\tilde{t}} \\ &= \frac{8}{1\,776\,000}.\end{aligned}$$
- **Another unknown parameter**
  If two million vehicles are sold, then the r.v. Y, representing number of replacements of the original radiator hoses during the 24 000 kilometer warranty period, has a binomial distribution with parameters $m = 2\,000\,000$ and $p = P(T \le 24\,000) = 1 - e^{-24\,000\times\lambda}$.
  Its expected value is equal to
  $$h(\lambda) = 2\,000\,000 \times \left(1 - e^{-24\,000\times\lambda}\right).$$
- **Requested ML estimate**
  By invoking the invariance property of the ML estimators, we can add that
  $$\begin{aligned}\widehat{h(\lambda)} &= h(\hat{\lambda}) \\ &= 2\,000\,000 \times \left(1 - e^{-24\,000\times\hat{\lambda}}\right) \\ &= 2\,000\,000 \times \left(1 - e^{-24\,000\times\frac{8}{1776000}}\right) \\ &\simeq 204\,938.878.\end{aligned}$$

(c) Compute a 95% confidence interval for the probability that a radiator hose is not replaced during the 24 000 kilometer warranty period. **(2.0)**

- **Unknown parameter**
  $P(T > 24\,000) = e^{-24\,000\times\lambda}$
- **Confidence interval for λ**
  Since we are dealing with a Type I/item censored testing with replacement, we get
  $$\begin{aligned}CI_{(1-\alpha)\times 100\%}(\lambda) &\overset{Table 5.16}{=} [\lambda_L; \lambda_U] \\ &= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2\times\tilde{t}}\right] \\ CI_{95\%}(\lambda) &\overset{(a)}{=} \left[\frac{F^{-1}_{\chi^2_{(16)}}(0.025)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(18)}}(0.975)}{2\times\tilde{t}}\right] \\ &= \left[\frac{6.908}{2\times 1\,776\,000}; \frac{31.53}{2\times 1\,776\,000}\right] \\ &= [0.000001945; 0.000008877].\end{aligned}$$
- **Confidence interval for $R(24\,000) = e^{-24\,000\times\lambda}$**
  $R(24\,000)$ is a decreasing function of $\lambda > 0$, thus
  $$\begin{aligned}CI_{95\%}(R(24\,000)) &= \left[e^{-24\,000\,\lambda_U}; e^{-24\,000\,\lambda_L}\right] \\ &= \left[e^{-24\,000\times 0.000008877}; e^{-24\,000\times 0.000001945}\right] \\ &\simeq [0.808117; 0.954393].\end{aligned}$$

4. Consider the following multiple choice questions. Select and justify the best possible answer.

(a) The target fraction of live births from c-sections is $p_0 = 0.15$. A sample of 125 live births led to 25 c-sections performed in the sample. When plotted on a $p-$chart, the observed value of the control statistic suggests that the process is: (A) out-of-control; (B) in-control. **(0.5)**

- **Best possible answer**
  B. The observed value of the control statistic is equal to $t = \frac{25}{125} = 0.2$. Since it is above the center line $p_0 = 0.14$, we only need to calculate the
  $$UCL = p_0 + 3\sqrt{\frac{p_0(1-p_0)}{n}} = 0.14 + 3\sqrt{\frac{0.15(1-0.15)}{15}} \simeq 0.245812,$$
  and conclude that $t < UCL$, thus this sample suggests that the fraction of c-sections is in-control.

(b) The in-control ARL of the $np-$chart, with $n = 20$ and 3–sigma limits, used to control the previous fraction of c-sections is approximately equal to: (A) 370.4; (B) 169.492; (C) 168.886. **(1.0)**

- **Best possible answer**
  B.
- **Control statistic of the $np-$chart and its distribution**
  Y_N = number of nonconforming items in the N^{th} batch, $N \in \mathbb{N}$
  $Y_N \sim \text{Binomial}(n, p)$
- **3-sigma control limits**
  $$\begin{aligned}LCL &= \left\lceil \max\{0, np_0 - 3\times\sqrt{np_0(1-p_0)}\}\right\rceil \\ &= \left\lceil \max\left\{0, 20\times 0.15 - 3\times\sqrt{20\times 0.15\times(1-0.15)}\right\}\right\rceil \\ &= \left\lceil \max\left\{0, 3 - 3\sqrt{2.55}\right\}\right\rceil \\ &= \lceil \max\{0, -1.791\}\rceil \\ &= 0 \\ UCL &= \left\lfloor np_0 + 3\times\sqrt{np_0(1-p_0)}\right\rfloor \\ &= \left\lfloor 20\times 0.15 - 3\times\sqrt{20\times 0.15\times(1-0.15)}\right\rfloor \\ &= \left\lfloor 3 + 3\sqrt{2.55}\right\rfloor \\ &= \lfloor 7.791\rfloor \\ &= 7.\end{aligned}$$
- **Probability of triggering a signal**
  $$\begin{aligned}\xi(\theta) &= 1\times P(Y_N \notin [LCL, UCL] \mid p = p_0 + \theta) \\ &= 1 - \left[F_{Binomial(n,p_0+\theta)}(UCL) - F_{Binomial(p_0+\theta)}(LCL-1)\right].\end{aligned}$$
- **Requested in-control ARL**
  Since $RL(0) \sim \text{geometric}(\xi(0))$,
  $$\begin{aligned}ARL(0) &= \frac{1}{\xi(0)} \\ &= \frac{1}{1 - \left[F_{Binomial(20,0.15+0)}(7) - F_{Binomial(20,0.15+0)}(0-1)\right]} \\ &\overset{tables}{=} \frac{1}{1 - 0.9941 + 0} \\ &\simeq 169.492.\end{aligned}$$
- **Obs.**
  If we use a calculator instead of the tables then the best possible answer is **C**: $ARL(0) \simeq 168.886$.

5. An engineer is monitoring the proportion p of incompletely filled low voltage liquid crystal display units in a high-yield process. The associated control statistic X is the cumulative count of conforming display units between two nonconforming ones. The p.f. and c.d.f. of X is given by $P_p(X=x)=(1-p)^x\,p$, $x\in\mathbb{N}_0$, and $F_p(x)=1-(1-p)^{1+x}$, $x\in\mathbb{N}_0$, where the target value of p is known and denoted by $p_0\in(0,1)$ and the true value of p is equal to $\rho\times p_0$, with $0<\rho<1/p_0$.

(a) The engineer is planning to use a chart with conventional k−sigma control limits ($k\in\mathbb{R}^+$): $LCL=(1-p_0)/p_0-k\sqrt{1-p_0}/p_0$ and $UCL=(1-p_0)/p_0+k\sqrt{1-p_0}/p_0$. **(1.5)**

Justify these control limits. For which values of k is the LCL positive? Elaborate on the importance of a positive LCL in this particular context.

- **Control statistic**
  X = number of nonconforming items between two consecutive conforming items
  $X\sim \text{Geometric}^\star(p)\equiv \text{BinomialN}^\star(r=1,p)$
- **P.f., c.d.f., expected value and variance of the control statistic**
  $P_p(X=x)=(1-p)^x\,p,\ x\in\mathbb{N}_0$
  $F_p(x)=1-(1-p)^{1+x},\ x\in\mathbb{N}_0$
  $E_p(X)=\frac{1-p}{p}$
  $V_p(X)=\frac{1-p}{p^2}$
- **On the control limits**
  They are of the type
  $$E_{p_0}(X)\pm k\sqrt{V_{p_0}(X)}=\frac{1-p_0}{p_0}\pm k\sqrt{\frac{1-p_0}{p_0^2}},$$
  where k is a positive constant.
- **Requested values of k**
  $$\begin{aligned} k\in\mathbb{R}^+\ :\ & LCL>0\\ & (1-p_0)/p_0-k\sqrt{(1-p_0)/p_0^2}>0\\ & k<\sqrt{1-p_0}.\end{aligned}$$
- **Importance of a positive LCL**
  It is essential that the chart has a positive LCL so the chart is able to detect decreases in p (i.e., to detect a quality improvement) in a fairly quick fashion.
  If $LCL\not>0$, we are bound to deal with an upper one-sided chart, whose out-of-control ARL in the presence of some decreases in p is surely (and unreasonably) larger than the in-control ARL.

(b) A quality control practitioner suggested the use of *exact probability limits* defined as $LCL_\alpha=\frac{\ln(1-\alpha/2)}{\ln(1-p_0)}$ and $UCL_\alpha=\frac{\ln(\alpha/2)}{\ln(1-p_0)}$, where α represents the *acceptable risk of false alarm*. **(2.0)**

Obtain the *exact probability limits* and the probability that this new chart triggers a valid signal when $p_0=10^{-3}$, $\alpha=0.005$ and $\rho=1,1.1$. Comment on these two ARL values.

- **Exact limits**
  $$\begin{aligned} LCL_\alpha &= \frac{\ln(1-\alpha/2)}{\ln(1-p_0)}\\ &= \frac{\ln(1-0.005/2)}{\ln(1-0.001)}\\ &= 2.501878\\ UCL_\alpha &= \frac{\ln(\alpha/2)}{\ln(1-p_0)}\\ &= \frac{\ln(0.005/2)}{\ln(1-0.001)}\\ &\simeq 5988.468315\end{aligned}$$

- **Probability of a signal**
  $$\begin{aligned}\xi_\alpha(\rho) &= P_{\rho\times p_0}(X\not\in[LCL_\alpha,UCL_\alpha])\\ &= 1-\left[F_{\rho\times p_0}(UCL_\alpha)-F_{\rho\times p_0}(LCL_\alpha^-)\right]\\ &= 1-\left[F_{\rho\times p_0}(5988.468315)-F_{\rho\times p_0}(2.501878^-)\right]\\ &= 1-\left[1-(1-\rho\times p_0)^{1+5988}\right]+\left[1-(1-\rho\times p_0)^{1+2}\right]\\ &= 1-(1-\rho\times 0.001)^3+(1-\rho\times 0.001)^{5989}\end{aligned}$$
- **ARL function and requested ARL values**
  Let us remind the reader that, since we are dealing with a Shewhart chart with independent control statistics, the ARL is equal to a reciprocal of the probability of a signal:
  $$ARL_\alpha(\rho) = \frac{1}{\xi_\alpha(\rho)}.$$
  Consequently:
  $$\begin{aligned}ARL_\alpha(1) &= \frac{1}{1-(1-0.001)^3+(1-0.001)^{5989}}\\ &\simeq 181.961;\\ ARL_\alpha(1.1) &= \frac{1}{1-(1-1.1\times 0.001)^3+(1-1.1\times 0.001)^{5989}}\\ &\simeq 214.210.\end{aligned}$$
- **Comments**
  Note that:
  - $ARL_\alpha(1)=181.961<\frac{1}{\alpha}=200$, i.e., the in-control ARL (does not coincide with and) is smaller than the reciprocal of the *acceptable risk of false alarm* — an undesirable property of this chart;
  - regretfully $ARL_\alpha(1.1)=214.210>ARL_\alpha(1)=181.961$, i.e., it takes longer, in average, to trigger a valid signal in the presence of a 10% increase in the fraction of nonconforming units than to trigger a false alarm — another unwelcome property.

(c) A statistician anticipated both downward and upward shifts and decided to adopt an alternative chart with control limits $LCL^\star=4$ and $UCL^\star=7428$. Moreover, it triggers a signal with probability: **(2.0)**

- one if the control statistic X is below $LCL^\star$ or above $UCL^\star$;
- $\gamma_{LCL^\star}=0.415872$ (resp. $\gamma_{UCL^\star}=0.349557$) if the control statistic is equal to $LCL^\star$ (resp. $UCL^\star$).

The in-control ARL of this new chart is equal to $\frac{1}{\alpha}=200$.

Verify that, when p shifts from its target value $p_0=0.001$ to $p=1.1\times p_0$, the out-of-control ARL is 194.950. Comment on this ARL value.

- **Probability of a signal**
  Judging by the description above, when $p=\rho\times p_0$ ($\rho\in(0,1/p_0)$), this alternative chart triggers a signal with probability
  $$\begin{aligned}\xi^\star(\rho) &= 1\times\left(1-\left\{F_{\rho\times p_0}(UCL^\star)-F_{\rho\times p_0}[(LCL^\star)^-]\right\}\right)\\ &\quad+\gamma_{LCL^\star}\times P_{\rho\times p_0}(X=LCL^\star)\\ &\quad+\gamma_{UCL^\star}\times P_{\rho\times p_0}(X=UCL^\star)\\ &= \left[1-(1-\rho\times 0.001)^{1+(4-1)}+(1-\rho\times 0.001)^{1+7428}\right]\\ &\quad+0.415872\times(1-\rho\times 0.001)^4\,\rho\times 0.001\\ &\quad+0.349557\times(1-\rho\times 0.001)^{7428}\,\rho\times 0.001.\end{aligned}$$
  Thus, the probability of a valid signal, when $\rho=1.1$, is given by
  $$\begin{aligned}\xi^\star(1.1) &= \left[1+(1-1.1\times 0.001)^{1+7428}-(1-1.1\times 0.001)^{1+4}\right]\\ &\quad+0.415872\times(1-1.1\times 0.001)^4\,1.1\times 0.001\\ &\quad+0.349557\times(1-1.1\times 0.001)^{7428}\,1.1\times 0.001\\ &\simeq 0.005130.\end{aligned}$$

- **ARL function and requested ARL value**
  Since we are still dealing with a Shewhart chart with independent control statistics, we have

$$ARL^\star(\rho) = \frac{1}{\xi^\star(\rho)}$$
$$ARL^\star(1.1) = \frac{1}{0.005130} \simeq 194.950.$$

- **Comment**
  Unlike the chart with *exact probability limits*, this alternative chart is able to signal an increase in p sooner (in average) than to trigger a false alarm — $ARL^\star(1) > ARL^\star(1.1)$ — a very desirable property.

6. The high-voltage output of a certain power supply used in a copy machine is assumed to have a normal distribution with nominal mean and standard deviation equal to $\mu_0 = 350$ and $\sigma_0 = 2.0$ (V dc at 20 milliamps). Samples of $n = 9$ power supply units have been inspected every half-hour. The process mean and standard deviation have increased and the magnitudes of the associated shifts are $\delta = \sqrt{n}(\mu-\mu_0)/\sigma_0 = 0.25$ and $\theta = \sigma/\sigma_0 = \sqrt{23.574603/21.95}$, respectively.

(a) Find and interpret the 1st. quartile of the out-of-control RL of a $\bar{X}-$chart with $3-$sigma limits. **(2.0)**

- **Quality characteristic**
  X = high-voltage output of a certain power supply used in a copy machine
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistic of the standard $\bar{X}-$chart and its distribution**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, where $\delta \in \mathbb{R}$ (resp. $\theta \geq 1$) represents the magnitude of a shift in μ (resp. an upward shift in σ)
- **Control limits**
  $LCL_\mu = \mu_0 - \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$,
  where $\gamma_\mu = 3$, $\mu_0 = 350$ and $\sigma_0 = 2.0$.
- **Shifts in the process mean and standard deviation**
  $\delta = \sqrt{n}(\mu-\mu_0)/\sigma_0 = 0.25$
  $\theta = \frac{\sigma}{\sigma_0} = \sqrt{\frac{23.574603}{21.95}} \simeq 1.036346$
- **Probability of a signal**

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= P(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta, \theta) \\
&= \ldots \\
&= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{3-0.25}{1.036346}\right) - \Phi\left(\frac{-3-0.25}{1.036346}\right)\right] \\
&\simeq 1 - [\Phi(2.65) - \Phi(-3.14)] \\
&\overset{tables}{=} 1 - [0.9960 - (1-0.999155)] \\
&= 0.004845
\end{aligned}$$

- **Requested** 25% **percentage point of the out-of-control** RL_μ
  Given that $RL_\mu(\delta,\theta) \sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right)$, we have

$$\begin{aligned}
F^{-1}_{RL_\mu(\delta,\theta)}(p) &\overset{Table\ 9.2}{=} \inf\left\{m \in \mathbb{N} : F_{RL_\mu(\delta,\theta)}(m) \geq p\right\} \\
&= 1 - \left[1 - \xi_\mu(\delta,\theta)\right]^m \geq p \\
&= m \times \ln\left[1 - \xi_\mu(\delta,\theta)\right] \leq \ln(1-p) \\
&\overset{\ln[1-\xi_\mu(\delta,\theta)]<0}{=} m \geq \frac{\ln(1-p)}{\ln\left[1-\xi_\mu(\delta,\theta)\right]} \\
&= m \geq \frac{\ln(1-0.25)}{\ln(1-0.004845)} \\
&= m \geq 59.233,
\end{aligned}$$

  thus, $F^{-1}_{RL_\mu(\delta,\theta)}(0.25) = 60$.
- **Interpretation of the** $F^{-1}_{RL_\sigma(\delta,\theta)}(0.25)$
  When there is a shift both in the location and spread of this quality characteristic with magnitude $(\delta,\theta) = (0.25, \sqrt{23.574603/21.95})$, the probability that the $\bar{X}-$chart triggers a valid signal within the first 60 samples is of at least 25%.

(b) An upper one-sided Shewhart chart for σ^2 was set with $UCL = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$, where $\gamma_\sigma = 23.574603$; thus, its in-control ARL is equal to 370.4. **(2.0)**
Compare the first quartile of the out-of-control RL of this chart with the one you obtained in (a).

- **Control statistic of the upper one-sided S^2 chart and its distribution**
  S^2_N = variance of the N^{th} random sample of size n, $N \in I\!N$
  $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, where θ ($\theta \geq 1$) represents the magnitude of the upward shift in the standard deviation σ.
- **Control limits of the upper one-sided S^2- chart**
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$
- **Shift in process standard deviation**
  $\theta = \frac{\sigma}{\sigma_0} = \sqrt{\frac{23.574603}{21.95}}$
- **Probability of triggering a signal**

$$\begin{aligned}
\xi_\sigma(\theta) &= P\left(S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\
&= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] \\
&= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right) \\
&= 1 - F_{\chi^2_{(9-1)}}\left[\frac{23.574603}{\left(\sqrt{23.574603/21.95}\right)^2}\right] \\
&= 1 - F_{\chi^2_{(8)}}(21.95) \\
&\overset{tables}{=} 1 - 0.995 \\
&= 0.005
\end{aligned}$$

- **Requested** 25% **percentage point of the out-of-control** RL_σ
  Since we are dealing once more with a Shewhart chart with independent control statistics,

$$RL_\sigma(\theta) \sim \text{Geometric}(\xi_\sigma(\theta)).$$

  and

$$\begin{aligned}
F^{-1}_{RL_\sigma(\theta)}(p) &\overset{Table\ 9.2}{=} \inf\left\{m \in I\!N : F_{RL_\sigma(\theta)}(m) \geq p\right\} \\
&= 1 - [1 - \xi_\sigma(\theta)]^m \geq p \\
&= m \geq \frac{\ln(1-p)}{\ln[1-\xi_\sigma(\theta)]} \\
&= m \geq \frac{\ln(1-0.25)}{\ln(1-0.005)} \\
&= m \geq 57.392,
\end{aligned}$$

  hence $F^{-1}_{RL_\sigma(\theta)}(0.25) = 58$.

- **Comment**
  Even though the RL of chart for μ depends on both δ and θ and the RL of the chart for σ is only influenced by θ, the out-of-control $RL_\mu(0.25, \sqrt{23.574603/21.95})$ and $RL_\sigma(\sqrt{23.574603/21.95})$ have similar 25% percentage points [and 50%, 75%, 95% percentage points for that matter, suggesting that both chart have similar detection power in the presence of that particular shift in μ and σ].
  [We believe the small magnitudes of both shifts may have played an important role on this result.]

(c) Compute the probability that the first 10 samples are associated with at least two valid signal triggered by the joint scheme. **(1.5)**

- **Probability of a signal**
  According to Exercise 10.38, the joint scheme triggers a signal with probability

$$\begin{aligned}
\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \not\in [LCL_\mu, UCL_\mu] \text{ or } S_N^2 \not\in [LCL_\sigma, UCL_\sigma] \mid \delta,\theta\right) \\
&= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta) \\
&\simeq 0.004845 + 0.005 - 0.004845 \times 0.005 \\
&\simeq 0.00982078.
\end{aligned}$$

  [Recall that: the joint scheme signals as soon as at least one of its constituent charts triggers a signal; and $\bar{X}_N$ and S_N^2 are independent control statistics given δ and θ.]

- **Auxiliary r.v.**
  Let W denote the number of valid signals triggered by the joint scheme in the first 5 samples. Since we are dealing with Shewhart charts [with no runs rules, etc.] then

$$W \sim \text{Binomial}\left(10, \xi_{\mu,\sigma}(0.25, \sqrt{23.574603/21.95} \simeq 0.009823860)\right).$$

- **Requested probability**

$$\begin{aligned}
P(W \geq 2) &= 1 - P(W \leq 1) \\
&\simeq 1 - (1 - 0.009823860)^{10} - 10 \times 0.009823860 \times (1 - 0.009823860)^{10-1} \\
&\simeq 0.004121.
\end{aligned}$$

(d) How would you briefly describe the obtention of the PMS_{III} of the joint EWMA scheme EE^+? **(1.5)**

- **Probability of a misleading signal of Type III**
  Let $RL_{E^+-\mu}(\delta,\theta)$ (resp. $RL_{E^+-\sigma}(\theta)$) denote the RL of the upper one-sided EWMA chart for μ (resp. σ), where $\delta = \sqrt{n}(\mu - \mu_0)/\sigma$ $(\delta \geq 0)$ and $\theta = \sigma/\sigma_0$ $(\theta \geq 1)$ are the magnitudes of the shits in the process mean and standard deviation. Then, according to Theorem 10.42, the probability of a misleading signal of Type III is equal to

$$\begin{aligned}
PMS_{III}(\theta) &= P[RL_{E^+-\sigma}(\theta) > RL_{E^+-\mu}(0,\theta)] \\
&= \sum_{i=1}^{+\infty} P_{RL_{E^+-\mu}(0,\theta)}(i) \times \overline{F}_{RL_{E^+-\sigma}(\theta)}(i) \\
&= \sum_{i=1}^{+\infty} \left[\overline{F}_{RL_{E^+-\mu}(0,\theta)}(i-1) - \overline{F}_{RL_{E^+-\mu}(0,\theta)}(i)\right] \times \overline{F}_{RL_{E^+-\sigma}(\theta)}(i), \quad \theta > 1.
\end{aligned}$$

- **Obtaining $PMS_{III}(\theta)$**
  Following page 100 from the lecture notes, we can obtain an approximation to the $PMS_{III}(\theta)$ by replacing the survival functions in the formula of this probability by the Markovian approximations described thoroughly on pages 91–92.

7. The drained weight after filling of contents of a can of tomatoes is required to be at most 22.8 oz.

(a) Set a single sampling plan by VARIABLES, with KNOWN STANDARD DEVIATION with risk points $(p_1, 1-\alpha) = (1\%, 0.95)$ and $(p_2, \beta) = (5\%, 0.1)$, and confirm that this plan is indeed related to these **(2.0)**

risk points.

- **Single sampling plan by variables with** KNOWN STANDARD DEVIATION
  n_σ (sample size)
  k_σ (acceptance constant)
  σ (known standard deviation)
  U (upper specification limit)

- **Producer's and consumer's risk points**
  $(p_1, 1-\alpha) = (1\%, 0.95)$
  $(p_2, \beta) = (5\%, 0.1)$.

- **Obtaining n_σ and k_σ**
  According to (13.32),

$$(n_\sigma, k_\sigma) \quad : \quad \begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(1-\alpha) - \Phi^{-1}(\beta)}{\Phi^{-1}(p_2) - \Phi^{-1}(p_1)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(p_2)\Phi^{-1}(1-\alpha) - \Phi^{-1}(p_1)\Phi^{-1}(\beta)}{\Phi^{-1}(\beta) - \Phi^{-1}(1-\alpha)}. \end{cases}$$

$$\begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(0.95) - \Phi^{-1}(0.1)}{\Phi^{-1}(0.05) - \Phi^{-1}(0.01)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(0.05)\,\Phi^{-1}(0.95) - \Phi^{-1}(0.01)\,\Phi^{-1}(0.1)}{\Phi^{-1}(0.1) - \Phi^{-1}(0.95)}. \end{cases}$$

$$\begin{cases} n_\sigma \stackrel{table}{=} \left[\frac{1.6449 - (-1.2816)}{(-1.6449) - (-2.3263)}\right]^2 = 18.445601 \\ k_\sigma \stackrel{table}{=} \frac{(-1.6449) \times 1.6449 - (-2.3263) \times (-1.2816)}{(-1.2816) - 1.6449} = 1.943305. \end{cases}$$

  We should take $n_\sigma = \lceil 18.445601 \rceil = 19$ and $k_\sigma = 1.943305$. In fact

$$\begin{aligned}
P_a(p_1) &= \Phi\left(\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_1)\right]\right) \\
&= \Phi\left(\sqrt{19}\left[-1.943305 - (-2.3263)\right]\right) \\
&\simeq \Phi(1.67) \\
&\stackrel{table}{=} 0.9525 \\
&\geq 1 - \alpha = 0.95 \\
P_a(p_2) &= \Phi\left(\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_2)\right]\right) \\
&= \Phi\left(\sqrt{19}\left[-1.943305 - (-1.6449)\right]\right) \\
&\simeq \Phi(-1.30) \\
&\stackrel{table}{=} 1 - 0.9032 \\
&= 0.0968 \\
&\leq \beta = 0.1.
\end{aligned}$$

(b) Consider now a sampling plan by variables with UNKNOWN STANDARD DEVIATION and obtain its (n_s, k_s). **(3.0)**
Verify that the approximate values of $P_a(p_1)$ (resp. $P_a(p_2)$) is larger (resp. smaller) than or equal to $1-\alpha$ (resp. β), when $(n_s, k_s) = (59, 1.937713)$.
Compare the last pair (n_s, k_s) to (n_σ, k_σ).
Note: In case you did not solve (a), consider $(n_\sigma, k_\sigma) = (19, 1.943305)$.

- **Single sampling plan by variables with** UNKNOWN STANDARD DEVIATION
  n_s (sample size)
  k_s (acceptance constant)

- **Obtaining n_s and k_s**
  Capitalizing on (13.38) and on the fact that

$$\begin{aligned}
u &= 3n_\sigma\,(k_\sigma^2 - 2) + 8 \\
&= 3 \times 19 \times (1.943305^2 - 2) + 8 \\
&\simeq 109.256756
\end{aligned}$$

$$
\begin{aligned}
v &= 3n_\sigma^2 k_\sigma^2 \\
&= 3 \times 19^2 \times 1.943305^2 \\
&\simeq 4089.878372
\end{aligned}
$$

we obtain

$$
\begin{aligned}
n_s &= n_\sigma + \frac{u + \sqrt{u^2 + 24v}}{12} \\
&= 15 + \frac{109.256756 + \sqrt{109.256756^2 + 24 \times 4089.878372}}{12} \\
&\simeq 55.755091 \\
k_s &= \sqrt{\frac{3n_s - 3}{3n_s - 4}}\, k_\sigma \\
&\simeq \sqrt{\frac{3 \times 55.755091 - 3}{3 \times 55.755091 - 4}} \times 1.943305 \\
&\simeq 1.937381.
\end{aligned}
$$

- If we consider $n_s = 59$ and $k_s = 1.937713$ then

$$
\begin{aligned}
P_a(p_1) &\overset{(13.39)}{\simeq} \Phi(\theta_{p_1}) \\
&\overset{(13.41)}{\simeq} \Phi\left[\frac{\Phi^{-1}(1-p_1) - k_s\sqrt{\frac{3n_s-4}{3n_s-3}}}{\sqrt{\frac{1+\frac{3n_s k_s^2}{6n_s-8}}{n_s}}}\right] \\
&= \Phi\left[\frac{2.3263 - 1.937713\sqrt{\frac{3\times59-4}{3\times59-3}}}{\sqrt{\frac{1+\frac{3\times59\times1.937713^2}{6\times59-8}}{59}}}\right] \\
&\simeq \Phi(1.77) \\
&\overset{table}{=} 0.9616 \\
&\geq 1-\alpha = 0.95 \\
P_a(p_2) &= \Phi\left[\frac{1.6449 - 1.937713\sqrt{\frac{3\times59-4}{3\times59-3}}}{\sqrt{\frac{1+\frac{3\times59\times1.937713^2}{6\times59-8}}{59}}}\right] \\
&\simeq \Phi(-1.29) \\
&\overset{table}{=} 1 - 0.9015 \\
&= 0.0985 \\
&\leq \beta = 0.1.
\end{aligned}
$$

- **Comment**
  Admitting that the standard deviation is unknown is more realistic:
  - it does not change the acceptance constant significantly (in this exercise $k_s \simeq k_\sigma$ down to the first decimal place);
  - but it requires the collection of a much larger sample (in this case $n_s = 3.1 \times n_\sigma$).

(c) Suppose a sample of size n_s was taken from a lot, and $\bar{x} = 22.876942$, $s = 1.116276$. **(1.0)**
Should the lot be accepted or rejected?

- **Checking whether or not the lot should be accepted**
  The lot should be accepted iff

$$
Q = \frac{U - \bar{x}}{s} \geq k_s,
$$

where Q is the quality index, U is the upper specification limit, $\bar{x}$ and s represent the mean of a sample with size n_s, and k_s the acceptance constant. For this sample, we have

$$
\begin{aligned}
Q &= \frac{22.8 - 22.876942}{1.116276} \\
&= -0.068927 \\
&\ngeq 1.937713,
\end{aligned}
$$

therefore we should reject the lot.