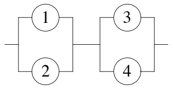
Department of Mathematics, IST — Probability and Statistics Unit

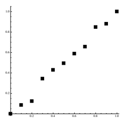
Reliability and Quality Control

1st. Test ("Recurso") 1st. Semester — **2016/17**
Duration: **1h30m** **2017/02/01 — 11:30AM**, Room **C13**

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. A detection system for the CO level in a science lab depends on 4 sensors arranged according to the system block diagram on the left :





Admit sensor i ($i = 1,\dots,4$) never triggers false alarms and it emits a signal with probability p_i when the CO level exceeds a critical threshold.

(a) Obtain the structure function of this system. (DO NOT simplify it!) **(1.0)**
Calculate its reliability when the 4 sensors operate in an independent manner and with reliability $p_i = p = 0.9$.

- **Structure function**
  According to the system block diagram above, we have

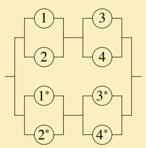
$$\begin{aligned}\phi(\underline{X}) &= \min\{\max\{X_1, X_2\}, \max\{X_3, X_4\}\} \\ &\overset{(1.4),(1.5)}{=} [1-(1-X_1)(1-X_2)] \times [1-(1-X_3)(1-X_4)] \\ &= (X_1+X_2-X_1X_2) \times (X_3+X_4-X_3X_4).\end{aligned}$$

- **Reliability of the components**
  $p_i = p = 0.9,\ i = 1,\dots,4$
- **System reliability**

$$\begin{aligned}r(\underline{p}) &= E\left[\phi(\underline{X})\right] \\ &= E\left[(X_1+X_2-X_1X_2)\times(X_3+X_4-X_3X_4)\right] \\ &\overset{X_i \overset{indep}{\sim} Ber(p_i)}{=} (p_1+p_2-p_1p_2)\times(p_3+p_4-p_3p_4) \\ &\overset{p_i=p}{=} (2p-p^2)^2 \\ &\overset{p=0.9}{=} 0.9801.\end{aligned}$$

(b) Consider two alternative systems resulting from a replication of the original detection system and a replication of the original sensors. **(3.0)**
Draw the associated block diagrams and obtain their reliabilities.
Which of these two new arrangements should be adopted in order that the probability of detecting a critical CO level is maximized?

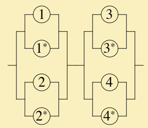
- **Replication at the system level (RSL)**
  The replication of the original detection system is associated with the following block diagram:



- **Reliability — RSL**

$$\begin{aligned}r_{RSL} &\overset{(1.26)}{=} 1-[1-r(\underline{p})]\times[1-r(\underline{p}^*)] \\ &\overset{\underline{p}=\underline{p}^*}{=} 1-[1-r(\underline{p})]^2 \\ &\overset{(a)}{=} 1-(1-0.9801)^2 \\ &\simeq 0.999604.\end{aligned}$$

- **Replication at the component level (RCL)**
  The replication of the original sensors is associated with:



- **Reliability — RCL**
  Note that

$$\begin{aligned}r_{RCL} &\overset{(1.27)}{=} r[\underline{1}-(\underline{1}-\underline{p})\bullet(\underline{1}-\underline{p}^*)] \\ &= r[(1,1,1,1)-(1-p_1,1-p_2,1-p_3,1-p_4)\bullet(1-p_1^*,1-p_2^*,1-p_3^*,1-p_4^*)] \\ &= r[1-(1-p_1)(1-p_1^*),1-(1-p_2)(1-p_2^*),1-(1-p_3)(1-p_3^*), \\ &\qquad 1-(1-p_4)(1-p_4^*)] \\ &\overset{p_i=p_i^*=p=0.9}{=} r(0.99,0.99,0.99,0.99) \\ &\overset{(a)}{=} (2\times 0.9-0.9^2)^2 \\ &= 0.999800.\end{aligned}$$

- **Comment**
  Expectedly, replicating sensors leads to a higher reliability than replicating the detection system:

$$r_{RCL} = 0.9998 \geq 0.999604 = r_{RSL}.$$

  Thus, the replication of the original sensors is the preferred arrangement if the probability of detecting a critical CO level is to be maximized.

(c) Admit that the 4 sensors of the original detection system operate in a positively associated fashion. **(3.0)**
Provide two pairs of lower and upper bounds for the reliability of the original detection device.
Comment on the strictest bounds.

- **Reliability of the components**
  $p_i = p = 0.9,\ i = 1,\dots,4$
- **First pair of bounds for the reliability** $r(\underline{p})$
  Since the 4 sensors form a coherent system and operate in a positively associated manner, we can apply Theorem 1.65 and obtain a...
  First lower bound

$$\begin{aligned}r(\underline{p}) &\geq \prod_{i=1}^{n} p_i \\ &\overset{p_i=p}{=} p^4 \\ &\overset{p=0.9}{=} 0.6561;\end{aligned}$$

First upper bound

$$\begin{aligned} r(\underline{p}) &\leq 1 - \prod_{i=1}^{n}(1-p_i) \\ &\stackrel{p_i=p}{=} 1-(1-p)^4 \\ &\stackrel{p=0.9}{=} 0.9999. \end{aligned}$$

- **Second pair of bounds for the reliability** $r(\underline{p})$
For the same reasons we have pointed out, we can also apply Theorem 1.70. However, before we proceed we ought to identify the...

Minimal path sets

$$\begin{aligned} \mathscr{P}_1 &= \{1,3\} \\ \mathscr{P}_2 &= \{1,4\} \\ \mathscr{P}_3 &= \{2,3\} \\ \mathscr{P}_4 &= \{2,4\} \\ p^* &= 4 \text{ minimal path sets} \end{aligned}$$

Minimal cut sets

$$\begin{aligned} \mathscr{K}_1 &= \{1,2\} \\ \mathscr{K}_2 &= \{3,4\} \\ q &= 2 \text{ minimal cut sets} \end{aligned}$$

Second lower bound

$$\begin{aligned} r(\underline{p}) &\stackrel{Th.1.70}{\geq} \max_{j=1,\dots,p^*}\left\{\prod_{i\in\mathscr{P}_j} p_i\right\} \\ &\stackrel{p_i=p}{=} \max_{j=1,\dots,p^*} p^{\#\mathscr{P}_j} \\ &\stackrel{p\in(0,1)}{=} p^{\min_{j=1,\dots,p^*}\#\mathscr{P}_j} \\ &= p^2 \\ &\stackrel{p=0.9}{=} 0.81 \end{aligned}$$

Second upper bound

$$\begin{aligned} r(\underline{p}) &\stackrel{Th.,1.70}{\leq} \min_{j=1,\dots,q}\left[1-\prod_{i\in\mathscr{K}_j}(1-p_i)\right] \\ &\stackrel{p_i=p}{=} \min_{j=1,\dots,q}\left[1-(1-p)^{\#\mathscr{K}_j}\right] \\ &\stackrel{p\in(0,1)}{=} 1-(1-p)^{\min_{j=1,\dots,q}\#\mathscr{K}_j} \\ &= 1-(1-p)^2 \\ &\stackrel{p=0.9}{=} 0.99. \end{aligned}$$

- **Comment on the strictest bounds**
Since $0.81 > 0.6561$ and $0.99 < 0.877521$, the strictest lower and upper bounds are the ones given by Theorem 1.70 because this result capitalizes on the topology of the system (unlike Theorem 1.65).

2. Assume the weather radar system on a commercial aircraft has a time to failure with a Weibull distribution, with shape parameter $\alpha = 0.5$ and mean of 1140 hours, and answer the following questions.

(a) Compute the maximum length of flight such that the reliability will not be less than 0.9? What is the probability of failure of the weather radar system during a 4 hour flight? **(2.0)**

- **Relevante r.v.**
T_i = time to failure of the weather radar system i

- **Distribution**
$T_i \sim \text{Weibull}(\delta,\alpha)$, where δ and $\alpha = 0.5$ denote the scale and shape parameters (resp.).
Note that we can invoke Exerc. 4.22 and write

$$\begin{aligned} \delta \;:\; & E(T_i) = 1140 \\ & \delta \times \Gamma\left(1+\frac{1}{\alpha}\right) = 1140 \\ & \delta = \frac{1140}{\Gamma\left(1+\frac{1}{0.5}\right)} \\ & \delta = \frac{1140}{2!} \\ & \delta = 570. \end{aligned}$$

- **Requested maximum length of flight**
By invoking result (4.22), we get

$$\begin{aligned} t \;:\; & P(T_i > t) \geq 0.9 \\ & R(t) \geq 0.9 \\ & \exp\left[-\left(\frac{t}{\delta}\right)^{\alpha}\right] \geq 0.9 \\ & \left(\frac{t}{\delta}\right)^{\alpha} \leq -\ln(0.9) \\ & t \leq \delta \times [-\ln(0.9)]^{\frac{1}{\alpha}} \\ & t \leq 570 \times [-\ln(0.9)]^{\frac{1}{0.5}} \end{aligned}$$

Thus, the maximum length of flight such that the reliability will not be less than 0.9 is

$$570 \times [-\ln(0.9)]^2 \simeq 6.327478 \text{ (hours)}.$$

- **Requested probability**

$$\begin{aligned} P(T_i \leq 4) &= 1 - R(4) \\ &\stackrel{(4.22)}{=} 1 - \exp\left[-\left(\frac{4}{570}\right)^{0.5}\right] \\ &\simeq 0.080358. \end{aligned}$$

(b) Admit 2 weather radar systems are set in parallel and operate in a positively associated manner. Obtain bounds for the probability of failure of this parallel system during a 4 hour flight. Calculate an appropriate lower bound for the associated expected time to failure. **(3.0)**

- **Individual times to failure and common survival function**
T_i = time to failure of the weather radar system i, $i = 1,\dots,n$
T_i, $i = 1,\dots,n$, are positively associated r.v. with common survival function

$$R(t) = \exp\left[-\left(\frac{t}{\delta}\right)^{\alpha}\right],\ t \geq 0.$$

- **Duration of the parallel system**
$T = \max\{T_1, T_2\}$

- **Important**
We are dealing with positively associated r.v. therefore we can resort to Theorem 2.22 (or alternatively to Theorem 2.19) to provide bounds for $1 - R_T(4)$.

- **Bounds for** $1 - R_T(4)$ **(Theorem 2.22)**

Minimal path sets

$$\begin{aligned} \mathscr{P}_1 &= \{1\} \\ \mathscr{P}_2 &= \{2\} \\ p^* &= 2 \text{ minimal path sets} \end{aligned}$$

Minimal cut set

$$\begin{aligned}\mathscr{K}_1 &= \{1,2\}\\ q^* &= 1 \text{ minimal cut set}\end{aligned}$$

Lower bound for the reliability function $R_T(t)$

$$\begin{aligned}R_T(t) &\overset{T2.22}{\geq} \max_{j=1,\dots,p^*}\left[\prod_{i\in\mathscr{P}_j} R_i(t)\right]\\ &\overset{R_i(t)=R(t)}{=} \max_{j=1,\dots,p^*}[R(t)]^{\#\mathscr{P}_j}\\ &= [R(t)]^{\min_{j=1,\dots,p^*}\#\mathscr{P}_j}\\ &= R(4)\\ &\overset{(a)}{\simeq} 1-0.080358.\end{aligned}$$

- **Upper bound for the reliability function** $R_T(t)$

$$\begin{aligned}R_T(t) &\overset{T2.22}{\leq} \min_{j=1,\dots,q}\left\{1-\prod_{i\in\mathscr{K}_j}[1-R_i(t)]\right\}\\ &\overset{R_i(t)=R(t)}{=} \min_{j=1,\dots,q}\left\{1-[1-R(t)]^{\#\mathscr{K}_j}\right\}\\ &= 1-[1-R(t)]^{\min_{j=1,\dots,q}\#\mathscr{K}_j}\\ &= 1-[1-R(t)]^2\\ &\overset{t=4,(a)}{=} 1-0.080358^2\\ &\overset{t=5.32}{=} 1-0.006457.\end{aligned}$$

Conclusion

$$\begin{aligned}1-(1-0.006457) &\leq 1-R(4)\leq 1-(1-0.080358)\\ 0.006457 &\leq 1-R(4)\leq 0.080358.\end{aligned}$$

- [**Bounds for** $1-R_T(4)$ **(Theorem 2.19)**
In this case, we have

$$\begin{aligned}\prod_{i=1}^n R_i(t) &\leq R_T(t)\leq 1-\prod_{i=1}^n[1-R_i(t)]\\ \prod_{i=1}^n[1-R_i(t)] &\leq 1-R_T(t)\leq 1-\prod_{i=1}^n R_i(t).\end{aligned}$$

Since $n=2$, $t=4$ and the r.v. are identically distributed, we get the requested bounds:

$$\begin{aligned}[1-R(4)]^2 &\leq 1-R_T(4)\leq 1-[R(4)]^2\\ 0.080358^2 &\leq 1-R_T(4)\leq 1-(1-0.080358)^2\\ 0.006457 &\leq 1-R_T(4)\leq 0.154259.\end{aligned}$$

Unsurprisingly, the upper bound is not as strict as the one obtained by resorting to Theorem 2.19...]

- **Lower bound for** $\mu = E(T)$
Let us remind the reader that T_i has a Weibull dist. with scale and shape parameters δ and $\alpha = 0.5 < 1$ (resp.). Thus, according to subsection 4.3.4 (see table in page 100), $T_i \sim DHR$. Furthermore,

$$T_i \sim DHR \overset{Prop.3.36}{\Rightarrow} T_i \in DHRA.$$

Consequently,

$$\begin{aligned}\mu &\overset{Th.3.64}{\geq} \int_0^{+\infty}\left[1-\prod_{i=1}^n\left(1-e^{-t/\mu_i}\right)\right]dt\\ &\overset{n=2,\mu_i=E(T_i)=\mu^*}{=} \int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^2\right]dt\\ &= \int_0^{+\infty}\left(2e^{-t/\mu^*}-e^{-2t/\mu^*}\right)dt\end{aligned}$$

$$\begin{aligned}\mu &\geq \left(-2\mu^*\,e^{-t/\mu^*}+\frac{\mu^*}{2}\,e^{-2t/\mu^*}\right)\Bigg|_0^{+\infty}\\ &= \frac{3\mu^*}{2}\\ &= \frac{3\times 1140}{2}\\ &= 1710.\end{aligned}$$

(c) What would be the expected time to failure if those 2 weather radar systems in parallel operated independently? (**2.0**)

- **Individual times to failure and common distribution**
T_i = time to failure of the weather radar system i, $i=1,2$
$T_i \overset{i.i.d.}{\sim} \text{Weibull}(\delta,\alpha)$, $i=1,2$
- **Duration of the parallel system**
$T_{(2)} = \max\{T_1,T_2\}$
- **Survival function of** $T_{(2)}$

$$\begin{aligned}R_{T_{(2)}} &\overset{(2.5)}{=} 1-[1-R(t)]^2\\ &= 2R(t)-[R(t)]^2.\end{aligned}$$

- **Requested expected value**
$T_{(2)}$ is a non negative r.v. as a consequence

$$\begin{aligned}E(T_{(2)}) &\overset{(2.10)}{=} \int_0^{+\infty} R_{T_{(2)}}(t)\,dt\\ &= \int_0^{+\infty}\{2R(t)-[R(t)]^2\}\,dt\\ &\overset{(b)}{=} 2\int_0^{+\infty}R(t)\,dt-\int_0^{+\infty}\exp\left[-2\left(\frac{t}{\delta}\right)^\alpha\right]dt\\ &= 2\int_0^{+\infty}R(t)\,dt-\int_0^{+\infty}\exp\left[-\left(\frac{t}{\frac{\delta}{2^{1/\alpha}}}\right)^\alpha\right]dt\\ &= 2\int_0^{+\infty}R_{Weibull(\delta,\alpha)}(t)\,dt-\int_0^{+\infty}R_{Weibull(\frac{\delta}{2^{1/\alpha}},\alpha)}(t)\,dt\\ &= 2\delta\times\Gamma\left(1+\frac{1}{\alpha}\right)-\frac{\delta}{2^{1/\alpha}}\times\Gamma\left(1+\frac{1}{\alpha}\right)\\ &= \left(2-\frac{1}{2^{1/\alpha}}\right)\times\delta\times\Gamma\left(1+\frac{1}{\alpha}\right)\\ &= \left(2-\frac{1}{2^{1/0.5}}\right)\times 1140\\ &\overset{(a)}{=} \frac{7}{4}\times 1140\\ &= 1995.\end{aligned}$$

3. The time to failure of a brand of rescue vehicles is under consideration.

(a) A statistician collected the distances run (in km) until failure of 10 of those vehicles, $(84.4, 126.4, 399.1, 521.2, 628.2, 814.8, 983.8, 1619.8, 1776.3, 2970.4)$, and obtained the TTT plot shown above (picture on the right). (**2.0**)
Exemplify the obtention of the first 4 points of such plot.

- **Failure times**
T_i = distance run (in km) to failure of rescue vehicle i, $i=1,\dots,10$
- **Ordered sample**
$(84.4, 126.4, 399.1, 521.2, 628.2, 814.8, 983.8, 1619.8, 1776.3, 2970.4)$

- **Total time on test up to time $t_{(i)}$**

$$\begin{aligned}
\tau\left(t_{(0)}\right) &= 0 \\
\tau\left(t_{(i)}\right) &= \sum_{j=1}^{i}(n-j+1)\left[t_{(j)}-t_{(j-1)}\right],\ i=1,\ldots,n \\
\tau\left(t_{(10)}\right) &= (10-1+1)\times(84.4-0)+(10-2+1)\times(126.4-84.4)+(10-3+1)\times(399.1-126.4) \\
&\quad +\cdots+(10-10+1)\times(2970.4-1619.8) \\
&= 9924.4
\end{aligned}$$

- **Abcissae of the TTT plot**
  $\frac{i}{n}$, $i=0,1,\ldots,n$
- **Ordinates of the TTT plot**
  $\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$, $i=1,\ldots,n$.
- **First 4 points of the TTT plot**

i	$\frac{i}{n}$	$\tau\left(t_{(i)}\right)=\sum_{j=1}^{i}(n-j+1)\left[t_{(j)}-t_{(j-1)}\right]$	$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$
0	0	0	0
1	$\frac{1}{10}=0.1$	$(10-1+1)\times(84.4-0)=844$	$\frac{844}{9924.4}\simeq 0.085043$
2	$\frac{2}{10}=0.2$	$844+(10-2+1)\times(126.4-84.4)=1222$	$\frac{1222}{9924.4}\simeq 0.123131$
3	$\frac{3}{10}=0.3$	$1222+(10-3+1)\times(399.1-126.4)=3403.6$	$\frac{3403.6}{9924.4}\simeq 0.342953$

(b) She used the R software to perform the Atkinson test for exponentiality[1] and obtained a $p-$value of 0.6631. **(1.5)**

Comment on this $p-$value.

Does the result of the Atkinson test for exponentiality agree with the TTT plot?

Why isn't the chi-square goodness-of-fit test reasonable in light of this sample?

- **Comment on the $p-$value**
  Recall that the $p-$value is the largest significance level leading to the non rejection of the null hypothesis. Thus, for these particular data set and null hypothesis $H_0: T \sim \text{Exponential}(\lambda)$, $\lambda > 0$:
  - we should not reject H_0 for any significance levels $\alpha_0 \leq p-\text{value} = 0.6631$, namely the usual significance levels (1%, 5%, 10%);
  - we should reject H_0 for any significance levels $\alpha_0 > p-\text{value} = 0.6631$.

  The exponential model seems to be very reasonable in light of the data set.
- **Atkinson test for exponentiality and the TTT plot**
  The result of the Atkinson test for exponentiality is consistent with the TTT plot.
  In fact, the points of the TTT plot are roughly around a 45^o line and, according to Note 5.5, this suggests that the data should be modelled by an Exponential distribution[, that is, a memoryless distribution therefore with constant hazard rate (CHR)].
- **Additional comment**
  The chi-square goodness-of-fit test should NOT be performed in this particular case because we are dealing with a small sample.
  [Let us remind the reader that this test is based on a statistic whose distribution under H_0 is only known asymptotically.]

(c) The supplier of the rescue vehicles was requested to perform a life test with replacement over a 6 000 km test track and recorded a total 8 failures. **(2.5)**

Compute the UMVU estimate and a 90% confidence interval for the probability that the rescue vehicle reaches a target positioned 160 km away and returns to the central depot.

[1] For more details about this test the reader is referred to: Mimoto, N. and Zitikis, R. (2008). The Atkinson index, the Moran statistic, and testing exponentiality. *Journal of the Japan Statistical Society* **38**, 187–205.

- **Life test**
  Since the test is scheduled to end after exactly $t_0 = 6\,000$ km (the length of the test track) and a rescue vehicle is replaced as soon as it fails, we are dealing with a
  - Type I/item censored testing with replacement.
- **Censored data**
  $n=1$ (a single rescue vehicle is put to test at a time)
  $r=8$ failures during the life test
- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}
\tilde t &= n\times t_0 \\
&= 1\times 6000 \\
&= 6000\,\text{km}
\end{aligned}$$

- **Relevant r.v.**
  T_i = distance run by rescue vehicle i
- **Distribution assumption**
  $T_i \overset{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$.
  This is fairly reasonable since we did not reject $H_0: T\sim\text{Exponential}(\lambda)$, $\lambda>0$ in (b).
- **Unknown parameter**
  $P(T > 2\times 160) = R(320) = e^{-320\lambda}$
- **UMVU estimate of $e^{-320\lambda}$**
  According to Table 5.14, the UMVUE of $R_T(t)$ is, for $t = 320 < \tilde t = 6000$ and $r>0$, equal to

$$\begin{aligned}
\tilde R(t) &= \left(1-\tilde t^{-1}\times t\right)^r \\
&= \left(1-\frac{320}{6000}\right)^8 \\
&\simeq 0.645025.
\end{aligned}$$

- **Confidence interval for λ**

$$\begin{aligned}
CI_{(1-\alpha)\times 100\%}(\lambda) &\overset{Table\,5.16}{=} [\lambda_L;\lambda_U] \\
&= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde t};\frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2\times\tilde t}\right] \\
CI_{90\%}(\lambda) &\overset{(a)}{=} \left[\frac{F^{-1}_{\chi^2_{(16)}}(0.05)}{2\times 6000};\frac{F^{-1}_{\chi^2_{(18)}}(0.95)}{2\times 6000}\right] \\
&= \left[\frac{7.962}{12\,000};\frac{28.87}{12\,000}\right] \\
&\simeq [0.000664; 0.002406].
\end{aligned}$$

- **Confidence interval for $R(320)=e^{-320\lambda}$**
  Since $R(320)$ is a decreasing function of $\lambda>0$, we get

$$\begin{aligned}
CI_{90\%}(R(320)) &= \left[e^{-320\lambda_U}; e^{-320\lambda_L}\right] \\
&\simeq \left[e^{-320\times 0.002406}; e^{-320\times 0.000664}\right] \\
&\simeq [0.463050; 0.808576].
\end{aligned}$$