Department of Mathematics, IST — Probability and Statistics Unit

# Reliability and Quality Control

**2nd. Test** ("Recurso") 1st. Semester — **2016/17**
Duration: **1h30m** **2017/02/01** — **1:00PM**, Room **C13**

- Please justify your answers.
- This test has **two pages** and **four questions**. The total of points is **20.0**.

**1.** Consider the following multiple choice questions. Select and justify the best possible answer.

(a) A $(\bar{X}, S^2)$ joint scheme, with $n = 4$ and control limits $LCL_\mu = 348.5$, $UCL_\mu = 351.5$, $LCL_\sigma = 0$ and $UCL_\sigma = 4.28$, is used to monitor the output voltage (of a high-voltage power supply), which is assumed to be normally distributed with nominal mean and variance $\mu_0 = 350$ and $\sigma_0^2 = 1$. **(0.5)**

The new sample $(355, 351, 356, 349)$ from the process suggests that:

(A) the process mean has decreased; (B) the process mean has increased;
(C) the process variance has increased; (D) both (B) and (C).

- **Best possible answer**
  **D**. The mean and variance of the sample $\underline{x} = (355, 351, 356, 349)$ are:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 352.75 \quad > \quad UCL_\mu = 351.5$$

$$s^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\,\bar{x}^2\right) = 10.91(6) \quad > \quad UCL_\sigma = 4.71881.$$

As a consequence, the new sample suggests that the process mean and variance have increased.

(b) The in-control ARL of the $S^2$−chart equals: (A) 500; (B) 370.4; (C) 200. **(1.0)**

- **Best possible answer**
  **C.**
- **Quality characteristic**
  $X$ = voltage output
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistic**
  $S_N^2$ = variance of the $N^{th}$ random sample of size $n$
- **Distribution**
  $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$
  where $\theta = \frac{\sigma}{\sigma_0} \geq 1$ represents the magnitude of an upward shift in $\sigma$.
- **Control limits of the $S^2$−chart**
  $LCL_\sigma = 0$
  $UCL_\sigma = 4.28$
- **Probability of a false alarm**

$$\begin{aligned}
\xi_\sigma(1) &= P\left(S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta = 1\right) \\
&= P\left[\frac{(n-1)\,S_N^2}{\sigma_0^2} > \frac{(n-1)\,UCL_\sigma}{\sigma_0^2} \mid \theta = 1\right] \\
&= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma_0^2}\right] \\
&= 1 - F_{\chi^2_{(3)}}(12.84) \\
&\overset{table}{=} 0.005.
\end{aligned}$$

- **Requested in-control ARL**
  Since $RL_\sigma(1) \sim \text{geometric}(\xi_\sigma(1))$, $ARL_\sigma(1) = \frac{1}{\xi_\sigma(1)} = 200$.



**2.** A statistician has been monitoring a process for injection moulding of a part of an automobile instrument panel. A part is considered nonconforming if it has even one occurrence of any of the four types of defects: flash; splay; voids; short shots.

(a) Samples of $n$ panels are examined every 45 minutes and the number of nonconforming panels is added to a $np$−chart with 3–sigma control limits. **(1.5)**

What is the minimum sample size so that this chart has a positive LCL when the target fraction of nonconforming is $p_0 = 0.05$?

Elaborate on this minimum sample size and on the importance of a positive LCL in such a context.

- **Requested minimum sample size**

$$\begin{aligned}
n \; : \; & LCL > 0 \\
& n\,p_0 - 3 \times \sqrt{n\,p_0(1-p_0)} > 0 \\
& \sqrt{n} > 3 \times \sqrt{\frac{1-p_0}{p_0}} \\
& n > 9 \times \frac{1-p_0}{p_0} \\
& n > 171.
\end{aligned}$$

Therefore the requested minimum sample size is $n^\star = 172$.

- **Comment on the minimum sample size**
  Requiring a positive LCL can lead to impractical sample sizes, in case when we are dealing with small values of $p_0$, such as the one we have just obtained.
- **Importance of a positive LCL**
  It is essential that $np$−chart has a positive LCL in order to be able detect a decrease in the expected number of nonconforming items (i.e., to detect a quality improvement) in a fairly quick fashion.
  If $LCL \not> 0$, we are bound to deal with an upper one-sided $np$−chart, whose out-of-control ARL in the presence of some decreases in $p$ is surely (and unreasonably) larger than the in-control ARL.

(b) The statistician anticipated both downward and upward shifts and decided to adopt an alternative $np$−chart. It requires $n = 20$ and the use of control limits $LCL^\star = 0$ and $UCL^\star = 5$. Moreover, it triggers a signal with: **(2.5)**

- probability one if the sample number of nonconforming items is below $LCL^\star$ or above $UCL^\star$;
- probabilities $\gamma_{LCL^\star} = 0.0062$ and $\gamma_{UCL^\star} = 0.0612$ if the sample number of nonconforming items is equal to $LCL^\star$ and $UCL^\star$, respectively.

Verify that this alternative chart triggers a signal with probability approximately equal to:

- 0.0027, in the absence of assignable causes;
- 0.0029 (resp. 0.0030), when the fraction of nonconforming items shifts from its target value $p_0 = 0.05$ to 0.04 (resp. 0.06).

Comment on these probabilities.

- **Control statistic of the $np$−chart and its distribution**
  $Y_N$ = number of nonconforming items in the $N^{th}$ batch, $N \in \mathbb{N}$
  $Y_N \sim \text{Binomial}(n, p)$
- **Probability of triggering a signal**
  Judging by the description above, when $p = \lambda_0 + \theta$ ($\theta \in (-p_0, 1-p_0)$), this alternative $np$−chart triggers a signal with probability

$$\begin{aligned}
\xi^\star(\theta) &= 1 \times P(Y_N \notin [LCL^\star, UCL^\star] \mid p = p_0 + \theta) \\
&\quad + \gamma_{LCL^\star} \times P(Y_N = LCL^\star \mid \lambda = p_0 + \theta) \\
&\quad + \gamma_{UCL^\star} \times P(Y_N = UCL^\star \mid \lambda = p_0 + \theta) \\
&= 1 - \left[F_{Binomial(n,p_0+\theta)}(UCL^\star) - F_{Binomial(p_0+\theta)}(LCL^\star - 1)\right] \\
&\quad + \gamma_{LCL^\star} \times \left[F_{Binomial(p_0+\theta)}(LCL^\star) - F_{Binomial(p_0+\theta)}(LCL^\star - 1)\right] \\
&\quad + \gamma_{UCL^\star} \times \left[F_{Binomial(p_0+\theta)}(UCL^\star) - F_{Binomial(p_0+\theta)}(UCL^\star - 1)\right].
\end{aligned}$$

- **Probability of a false alarm**

$$\begin{aligned}\xi^\star(0) &= 1-\left[F_{Binomial(20,0.05+0)}(5)-F_{Binomial(20,0.05+0)}(0-1)\right]\\ &\quad +0.006230\times\left[F_{Binomial(20,0.05+0)}(0)-F_{Binomial(20,0.05+0)}(0-1)\right]\\ &\quad +0.061232\times\left[F_{Binomial(20,0.05+0)}(5)-F_{Binomial(20,0.05+0)}(5-1)\right]\\ &\overset{tables}{=} 1-(0.9997-0)+0.0062\times(0.3585-0)+0.0612\times(0.9997-0.9974)\\ &\simeq 0.0027\end{aligned}$$

- **Probability of a valid signal when** $p=0.05+\theta=0.04,0.06$

$$\begin{aligned}\xi^\star(-0.01) &= 1-\left[F_{Binomial(20,0.05-0.01)}(5)-F_{Binomial(20,0.05-0.01)}(0-1)\right]\\ &\quad +0.006230\times\left[F_{Binomial(20,0.05-0.01)}(0)-F_{Binomial(20,0.05-0.01)}(0-1)\right]\\ &\quad +0.061232\times\left[F_{Binomial(20,0.05-0.01)}(5)-F_{Binomial(20,0.05-0.01)}(5-1)\right]\\ &\overset{tables}{=} 1-(0.9999-0)+0.0062\times(0.4420-0)+0.0612\times(0.9999-0.9990)\\ &\simeq 0.0029\\ \xi^\star(0.01) &= 1-\left[F_{Binomial(20,0.05+0.01)}(5)-F_{Binomial(20,0.05+0.01)}(0-1)\right]\\ &\quad +0.006230\times\left[F_{Binomial(20,0.05+0.01)}(0)-F_{Binomial(20,0.05+0.01)}(0-1)\right]\\ &\quad +0.061232\times\left[F_{Binomial(20,0.05+0.01)}(5)-F_{Binomial(20,0.05+0.01)}(5-1)\right]\\ &\overset{tables}{=} 1-(0.9991-0)+0.0062\times(0.2901-0)+0.0612\times(0.9991-0.9944)\\ &\simeq 0.0030.\end{aligned}$$

- **Comment**

In this case, we trigger a signal in the presence of a 20% decrease or increase in $p$ with a probability larger than the probability of emitting a false alarm — a very desirable property. We have indeed:

$$\xi^\star(-0.01),\ \xi^\star(-0.01)>\xi^\star(0).$$

Unlike the $np-$chart with 3–sigma limits and $n=20$, this alternative $np-$chart is able to signal a decrease in $p$ (from the target value $p_0=0.05$ to 0.04) sooner (in average) than to trigger a false alarm.

(c) In the table below you can find the number of nonconforming items ($y_N$) and some observed values ($z_N$) of the control statistic of an UPPER ONE-SIDED CUSUM chart for binomial output with $n=20$, $UCL_C=17$, reference value $k=1$, and no head start. This chart is used with the purpose of speeding up the detection of small-to-moderate upward shifts in $p$. **(1.5)**

| $N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_N$ | 3 | 5 | 3 | 2 | 2 | 4 | 2 | 2 | 3 | 2 |
| $z_N$ | | 6 | 8 | 9 | 10 | 13 | 14 | 15 | | |

Fill this table and check whether the upper one-sided CUSUM chart is responsible for a signal.

- **Upper one-sided CUSUM chart for binomial output**
  - Control limits
    $LCL_C=0$
    $UCL_C=x=17$
  - Reference value
    $k=1$
  - Initial value of the control statistic
    $u=0$ (no head-start)
  - Control statistic

$$Z_N=\begin{cases}u=0, & N=0\\ \max\{0,Z_{N-1}+(Y_N-k)\}, & N\in\mathbb{N}.\end{cases}$$

[Note that $Y_N$ represents the number of nonconforming items in the $N^{th}$ sample, $N\in I\!N$.]



- **Missing observed values**

The first and the two last observed values of the control statistic are:

$$\begin{aligned}z_1 &= \max\{0,0+(3-1)\}\\ &= 2;\\ z_9 &= \max\{0,15+(3-1)\}\\ &= 17;\\ z_{10} &= \max\{0,17+(2-1)\}\\ &= 18.\end{aligned}$$

- **Comment**

Since $z_{10}\not\in[LCL_C,UCL_C=[0,17]$ the upper one-sided CUSUM chart for binomial output triggers a signal at the $10^{th}$ sample and the process is deemed out-of-control.

(d) How could you obtain the out-of-control ARL of this chart when $p=0.06$? **(2.0)**
How would this out-of-control ARL compare with the corresponding ARL of the chart in (b)?

- **Out-of-control RL**

It is represented by $RL^0(\theta)$ and has a phase-type distribution with parameters $(\underline{e}_0,\mathbf{Q}(\theta))$, where $\underline{e}_0=(1,0,\ldots,0)$ is the first vector of the orthonormal basis of $\mathbb{R}^{x+1}$ and

$$\begin{aligned}\mathbf{Q}(\theta) &\overset{(10.8),(10.10)}{=} [p_{ij}(\theta)]_{i,j=0}^{x}\\ &= \begin{bmatrix} F_\theta(k) & P_\theta(k+1) & P_\theta(k+2) & \cdots & P_\theta(k+x)\\ F_\theta(k-1) & P_\theta(k) & P_\theta(k+1) & \cdots & P_\theta(k+x-1)\\ F_\theta(k-2) & P_\theta(k-1) & P_\theta(k) & \cdots & P_\theta(k+x-2)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ F_\theta(k-x) & P_\theta(k-x+1) & P_\theta(k-x+2) & \cdots & P_\theta(k)\end{bmatrix},\end{aligned}$$

where $F_\theta$ and $P_\theta$ represent the c.d.f. and the p.f. of a r.v. with a Binomial$(n,p=p_0+\theta)$ distribution. Note that $n=20$, $p_0=0.05$, $k=1$, $x=17$ and $\theta\in(-p_0,0)\cup(0,1-p_0)$.

- **Out-of-control ARL**

Let $\mathbf{I}$ and $\underline{\mathbf{1}}$ be the $(x+1)\times(x+1)$ identity matrix and a vector of $(x+1)$ ones, then according to Table 10.3, $ARL^0(\theta)$ is given by

$$ARL^0(\theta)=\underline{\mathbf{e}}_0^\top\left[\mathbf{I}-\mathbf{Q}(\theta)\right]^{-1}\underline{\mathbf{1}}.$$

- **Comment**

The out-of-control ARL, when $p$ shifts from $p_0=0.05$ to $p_0+0.01=0.06$, is bound to be smaller than $ARL^\star(0.01)\simeq 1/0.0030\simeq 337.969$. [Additional calculations led to $ARL^0(0.01)\simeq 81.629$.] This is basically due to the fact that we are dealing with a UPPER ONE-SIDED CUSUM chart [instead of a variant of the TWO-SIDED $np-$chart].

**3.** The manufacturer of exercise weights makes a full range of dumbbells. Recently, a new set of mould has been developed for the casing of a 6-pound iron dumbbell. This quality characteristic is assumed to have a normal distribution with nominal mean and variance equal to $\mu_0=6$ and $\sigma_0^2=0.1^2$ (respectively) and a sample of $n=4$ dumbbells is collected every 2 hours. Furthermore, the control limits of the individual:

- $\bar{X}-$chart are equal to $LCL_\mu=\mu_0-\gamma_\mu\times\frac{\sigma_0}{\sqrt{n}}$ and $UCL_\mu=\mu_0+\gamma_\mu\times\frac{\sigma_0}{\sqrt{n}}$, where $\gamma_\mu=2.81$;
- $S^2-$chart are given by $LCL_\sigma=0$ and $UCL_\sigma=\frac{\sigma_0^2}{n-1}\times\gamma_\sigma$, where $\gamma_\sigma=12.84$.

(a) Admit that: the process mean $\mu$ on-target and the variance has increased; the magnitude of the upward shift in the process standard deviation is equal to $\theta=\sigma/\sigma_0=\sqrt{12.84/11.34}$. **(2.5)**
Compute the probability that the first 5 samples are associated with exactly one valid signal triggered by the joint scheme.

- **Quality characteristic**

$X$ = outside diameter of a motor shaft
$X\sim$ Normal$(\mu,\sigma^2)$, where $\mu$ and $\sigma^2$ represent the process mean and variance, respectively.

- **Control statistics**
  $\bar{X}_N$ = mean of the $N^{th}$ random sample of size $n$
  $S_N^2$ = variance of the $N^{th}$ random sample of size $n$
- **Distributions**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$,
  where $\delta = \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}} \le 0$ (resp. $\theta = \frac{\sigma}{\sigma_0} \ge 1$) represents the magnitude of a shift in $\mu$ (resp. an upward shift in $\sigma$).
  $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$
- **Control limits of the individual charts**
  $LCL_\mu = \mu_0 - \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$, where $\gamma_\mu = 2.81$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$, where $\gamma_\sigma = 12.84$.
- **Probabilities of triggering a signal**
  Taking into account the distribution of the control statistics, the STANDARD $\bar{X}-$chart and the UPPER ONE-SIDED $S^2-$chart trigger a signal with probabilities:

$$\begin{aligned}\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta,\theta\right)\\ &= \ldots\\ &= 1-\left[\Phi\left(\frac{\gamma_\mu-\delta}{\theta}\right)-\Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\right], \delta\in\mathbb{R}, \theta\ge 1;\\ \xi_\sigma(\theta) &= P\left(S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right)\\ &= \ldots\\ &= 1-F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right), \theta \ge 1.\end{aligned}$$

  Furthermore, according to Exercise 10.38, the joint scheme triggers a signal with probability

$$\begin{aligned}\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \delta,\theta\right)\\ &= \xi_\mu(\delta,\theta)+\xi_\sigma(\theta)-\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta), \delta\in\mathbb{R}, \theta\ge 1.\end{aligned}$$

  [Recall that: the joint scheme signals as soon as at least one of its constituent charts triggers a signal; and $\bar{X}_N$ and $S_N^2$ are independent control statistics given $\delta$ and $\theta$.]
  Since the process mean $\mu$ on-target and the magnitude of the upward shift in the process standard deviation is equal to $\theta = \sigma/\sigma_0 = \sqrt{12.84/11.34}$, we get the following probabilities of a valid signal:

$$\begin{aligned}\xi_\mu\left(0,\sqrt{12.84/11.34}\right) &\overset{tables}{=} 1-\left[\Phi\left(\frac{2.81-0}{\sqrt{12.84/11.34}}\right)-\Phi\left(\frac{-2.81-0}{\sqrt{12.84/11.34}}\right)\right]\\ &\simeq 1-[\Phi(2.64)-\Phi(-2.64)]\\ &\simeq 1-\{\Phi(2.64)-(1-\Phi(-2.64))\}\\ &\overset{tables}{=} 1-[0.9959-(1-0.9959)]\\ &= 0.0082;\\ \xi_\sigma\left(0,\sqrt{12.84/11.34}\right) &= 1-F_{\chi^2_{(4-1)}}\left(\frac{12.84}{\left(\sqrt{12.84/11.34}\right)^2}\right)\\ &= 1-F_{\chi^2_{(3)}}(11.34)\\ &\overset{tables}{=} 1-0.990\\ &= 0.010;\\ \xi_{\mu,\sigma}\left(0,\sqrt{12.84/11.34}\right) &\simeq 0.0082+0.010-0.0082\times 0.010\\ &= 0.018118.\end{aligned}$$

- **Requested probability**
  Let $W$ denote the number of valid signals triggered by the joint scheme in the first 5 samples. Since we are dealing with Shewhart charts [with no runs rules, etc.] then



$$W \sim \text{Binomial}\left(5, \xi_{\mu,\sigma}\left(0,\sqrt{12.84/11.34}\right)\simeq 0.018118\right).$$

Moreover, the probability that the first 5 samples are associated with exactly one valid signal triggered by the joint scheme is given by

$$\begin{aligned}P(W=1) &\simeq \binom{5}{1}\times 0.018118^1\times(1-0.018118)^{5-1}\\ &\simeq 0.084201.\end{aligned}$$

(b) Find the value of the probability of a misleading signal of Type III when $\theta = \sqrt{12.84/11.34}$. **(1.0)**
**Hint**: You may find useful to know that $PMS_{III}(\theta) = \frac{\xi_\mu(0,\theta)\times[1-\xi_\sigma(\theta)]}{\xi_\mu(0,\theta)+\xi_\sigma(\theta)-\xi_\mu(0,\theta)\times\xi_\sigma(\theta)}$.

- **Probability of a misleading signal of Type III**

$$PMS_{III}(\theta) \overset{Table\,10.12}{=} \frac{1-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]}{[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)]^{-1}-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]}$$

or

$$PMS_{III}(\theta) = \frac{\xi_\mu(0,\theta)\times[1-\xi_\sigma(\theta)]}{\xi_\mu(0,\theta)+\xi_\sigma(\theta)-\xi_\mu(0,\theta)\times s\xi_\sigma(\theta)}$$

- **Requested PMS of Type III**

$$\begin{aligned}PMS_{III}(\sqrt{12.84/11.34}) &\overset{(a)}{\simeq} \frac{0.0082}{0.990^{-1}-(1-0.0082)}\\ &\simeq 0.448063\end{aligned}$$

or

$$\begin{aligned}PMS_{III}(\sqrt{12.84/11.34}) &\overset{(a)}{\simeq} \frac{0.0082\times(1-0.010)}{0.0082+0.010-0.0082\times 0.010}\\ &\simeq 0.448063.\end{aligned}$$

(c) Prove that, for any $\theta > 1$: $PMS_{III}(\theta) \le 0.5 \Leftrightarrow ARL_\mu(0,\theta)+1 \ge ARL_\sigma(\theta)$. **(1.5)**
Comment the value of the PMS of Type III you obtained in (b) in light of this equivalence.

- **Requested proof**
  Capitalizing on the hint in (b) and on the fact that the ARL function of a Shewhart-type chart is the reciprocal of the probability of a signal, we get:

$$\begin{aligned}PMS_{III}(\theta)\le 0.5 \;\Leftrightarrow\; & \frac{\xi_\mu(0,\theta)\times[1-\xi_\sigma(\theta)]}{\xi_\mu(0,\theta)+\xi_\sigma(\theta)-\xi_\mu(0,\theta)\times\xi_\sigma(\theta)}\le 0.5\\ & 2\xi_\mu(0,\theta)-2\xi_\mu(0,\theta)\times\xi_\mu(0,\theta)\le\xi_\mu(0,\theta)+\xi_\sigma(\theta)-\xi_\mu(0,\theta)\times\xi_\sigma(\theta)\\ & \xi_\mu(0,\theta)-\xi_\mu(0,\theta)\times\xi_\sigma(\theta)\le\xi_\sigma(\theta)\\ & \frac{1}{\xi_\sigma(\theta)}-1\le\frac{1}{\xi_\mu(0,\theta)}\\ & ARL_\mu(0,\theta)+1\ge ARL_\sigma(\theta).\end{aligned}$$

- **Comment**
  The probability of misidentifying a shift in $\sigma$ with magnitude $\theta = \sqrt{12.84/11.34}$ by a shift in $\mu$, $PMS_{III}(\theta)$, we obtained in (b) is not larger than 0.5 because we have indeed

$$\begin{aligned}ARL_\mu(0,\theta)+1 &\simeq \frac{1}{0.0082}+1\\ &\simeq 122.951\\ &\ge ARL_\sigma(\theta)\simeq\frac{1}{0.01}=100.\end{aligned}$$

  [We are dealing with two Shewhart-type individual charts therefore the associated RL, $RL_\mu(\delta,\theta)$ and $RL_\sigma(\theta)$, are geometrically distributed r.v. Furthermore, the ARL are equal to $ARL_\mu(\delta,\theta) = 1/\xi_\mu(\delta,\theta)$ and $ARL_\sigma(\theta) = 1/\xi_\sigma(\theta)$.]

**4.** An engineer is assisting in the quality evaluations of a major food processing plant and suggests a single sampling plan for ATTRIBUTES to inspect of batches of a canned beans before their expedition. Assume the batch size ($N$) is much larger than the sample size $n = 20$ and that the engineer considered the acceptance number $c = 1$.

(a) What is the probability of acceptance of a batch containing 1% (resp. 10%) rejectable cans? (**1.5**)
Does this sampling plan comply with the producer's and consumer's risk points ($p_1 = AQL = 1\%, 1-\alpha = 0.95$) and ($p_2 = LTPD = 10\%, \beta = 0.10$)?

- **Single sampling plan for attributes)**
  $n = 20$ (sample size)
  $c = 1$ (acceptance number)
- **Auxiliary r.v. and its approximate distribution**
  $$D = \text{number of rejectable cans in the sample} \overset{a}{\sim} \text{Binomial}(20, p)$$
- **Requested probabilities of acceptance**
  $$\begin{aligned} P_a(p) &= P(D \le c) \\ &\simeq F_{Binomial(20,p)}(1) \\ &\overset{tables}{=} \begin{cases} 0.9831, & p = 1\% \\ 0.3917, & p = 10\%. \end{cases} \end{aligned}$$
- **Comment**
  The sampling plan DOES COMPLY with the producer's risk point, ($p_1 = AQL = 1\%, 1-\alpha = 0.95$), because $P_a(AQL) = 0.9831 > 0.95$.
  However, it DOES NOT COMPLY with the consumer's risk point, ($p_2 = LTPD = 10\%, \beta = 0.10$), for $P_a(LTPD) = 0.3917 > 0.10$.

(b) The quality engineer has objected to the use of $n = 20$ and $c = 1$ and recommended another single sampling plan for attributes derived using the Wetherill and Brown procedure and the adoption of RECTIFYING INSPECTION. Define such sampling plan. (**2.5**)
How should the assistant engineer procede if a sample is collected and two rejectable cans are to be found?

- **Producer's and consumer's risk points**
  $(p_1 = AQL, 1-\alpha) = (1\%, 0.95)$
  $(p_2 = LTPD, \beta) = (10\%, 0.10)$
- **Obtaining the acceptance number and sample size**
  According to Wetherill and Brown (1991, p. 257) and page 129 of the lecture notes (in particular, formulae (13.11), (13.10) and (13.12)), the acceptance number $c$ and sample size $n$ of a sampling plan for attributes, associated to risk points ($p_1 = AQL, 1-\alpha$) and ($p_2 = LTPD, \beta$), can be approximately obtained:
  - $c$ should be taken as the smallest integer satisfying
    $$r(c) \le \frac{p_2}{p_1},$$
    where $r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$;
  - $n$ should be taken as the smallest integer satisfying
    $$\frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{2p_2} \le n \le \frac{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}{2p_1},$$
    most likely the ceiling of the lower bound above.

  Using the tables to determine $F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta = 0.90)$ and $F^{-1}_{\chi^2_{(2(c+1))}}(\alpha = 0.05)$, we get



| $c$ | $r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$ | Is $r(c) \le \frac{p_2}{p_1} = \frac{0.1}{0.01} = 10$? |
|---|---|---|
| 0 | $\frac{F^{-1}_{\chi^2_{(2(0+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(0+1))}}(0.05)} = \frac{4.605}{0.103} \simeq 44.709$ | NO! |
| 1 | $\frac{F^{-1}_{\chi^2_{(2(1+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(1+1))}}(0.05)} = \frac{7.779}{0.711} \simeq 10.941$ | NO! |
| 2 | $\frac{F^{-1}_{\chi^2_{(2(2+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(2+1))}}(0.05)} = \frac{10.64}{1.635} \simeq 6.508$ | **YES!** |

Consequently, $c = 2$. Moreover,
$$\begin{aligned} n &= \left\lceil \frac{F^{-1}_{\chi^2_{(2(1+1))}}(1-0.1)}{2 \times 0.1} \right\rceil \\ &\overset{table}{=} \left\lceil \frac{10.64}{2 \times 0.1} \right\rceil \\ &= \lceil 53.2 \rceil \\ &= 54. \end{aligned}$$

- **Comment**
  The lot should be accepted iff the number of rejectable cans, in a sample of $n = 54$, does not exceed $c = 2$, and that is the case.
  Besides accepting the batch, the engineer should replace the 2 rejectable cans with conforming ones, after all rectifying inspection was adopted.

(c) Calculate the average outgoing quality (AOQ) of the sampling plan obtained in (b), when batches contain 4.1427% of rejectable cans. (**2.0**)
Verify that the average outgoing quality limit (AOQL) of the single sampling plan in (b) is achieved in the vicinity of $p = p^\star \simeq 0.0414271$.

- **Alternative single sampling plan for attributes**
  $N$ (batch size much larger than the sample size)
  $n = 54$ (sample size)
  $c = 2$ (acceptance number)
- **Auxiliary r.v. and its approximate distribution**
  $D =$ number of defective units in the sample $\overset{a}{\sim}$ Binomial$(n, p)$
- **Requested probability of acceptance**
  $$\begin{aligned} P_a(p) &= P(D \le c) \\ &\simeq F_{Binomial(n,p)}(c) \\ &= \sum_{i=0}^{c} \binom{n}{i} \times p^i \times (1-p)^{n-i} \\ P_a(p^\star) &= (1-0.041427)^{54} + 54 \times 0.041427 \times (1-0.041427)^{53} \\ &\quad + \frac{54 \times 53}{2} \times 0.041427^2 \times (1-0.041427)^{52} \\ &\simeq 0.611477 \end{aligned}$$
- **Requested average outgoing quality of a single sampling plan with rectifying inspection**
  $$\begin{aligned} AOQ(p) &\overset{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\ &\overset{n/N\ very\ small}{\simeq} p\,P_a(p) \\ AOQ(p^\star) &\simeq 0.041427 \times 0.611477 \\ &\simeq 0.025332. \end{aligned}$$

- **Verifying that** $AOQ(p^\star = 0.041427) \simeq AOQL$
  Let us remind the reader that $AOQL = \max_{p\in[0,1]} AOQ(p)$. Moreover, if we invoke the identity found in the list of probability formulae

$$F_{Beta(\alpha,\beta)}(p) = 1 - F_{Binomial(\alpha+\beta-1,p)}(\alpha-1),$$

for $\alpha, \beta \in \mathbb{N}$, then

$$\begin{aligned} AOQ(p) &\simeq p \times F_{Binomial(3+52-1,p)}(3-1) \\ &= p \times \left[1 - F_{Beta(3,52)}(p)\right]. \end{aligned}$$

Finally,

$$\begin{aligned} \left.\frac{d\,AOQ(p)}{dp}\right|_{p=p^\star} &\simeq \left.\left\{\left[1 - F_{Beta(3,52)}(p)\right] - p \times f_{Beta(3,52)}(p)\right\}\right|_{p=p^\star} \\ &= P_a(p^\star) - p^\star \times \frac{\Gamma(3+52)}{\Gamma(3)\,\Gamma(52)}\,(p^\star)^{3-1}\,(1-p^\star)^{52-1} \\ &\simeq 0.611477 - 0.041427 \times \frac{54!}{2!\,51!} \times 0.041427^2 \times (1-0.041427)^{51} \\ &\simeq 0.000003, \end{aligned}$$

thus, the AOQL of the single sampling plan with rectifying inspection is achieved in the vicinity of $p = p^\star \simeq 0.0414271$.
[Alternatively,

$$\begin{aligned} \frac{d\,AOQ(p)}{dp} &\simeq P_a(p) + p \times \frac{d\,P_a(p)}{dp} \\ &= P_a(p) + p \times \Big[-54 \times (1-p)^{53} + 54 \times (1-p)^{53} - 54 \times 53 \times p \times (1-p)^{52} \\ &\qquad +54 \times 53 \times p \times (1-p)^{52} - \frac{54 \times 53 \times 52}{2} \times p^2 \times (1-p)^{51}\Big] \\ &\overset{p=p^\star}{\simeq} 0.611477 + 0.041427 \times (-14.7603) \\ &\simeq 0.000002, \end{aligned}$$

thus, the AOQL...]