Department of Mathematics, IST — Probability and Statistics Unit

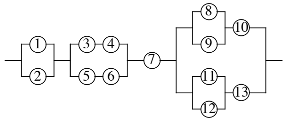
Reliability and Quality Control

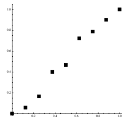
1st. Test — 1st. Semester — **2016/17**

Duration: **1h30m** — **2016/11/19 — 8AM**, Room **P8**

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. A skid protection system to be used on a two-wheel vehicle is shown below (picture on the left).[1]





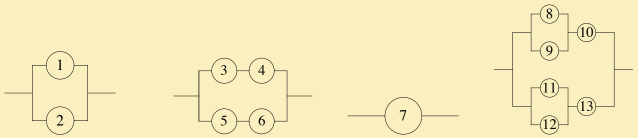
(a) Provide an expression for the structure function in terms of minima (min) and maxima (max). (DO NOT write it in terms of minimal path sets or minimal cut sets! DO NOT simplify it!) **(1.0)**

- **Structure function**
  Given the system block diagram above, we get

$$\begin{aligned}\phi(\underline{X}) \;=\; & \min\{ \max\{X_1, X_2\}, \\ & \max\{ \min\{X_3, X_4\}, \min\{X_5, X_6\} \}, \\ & X_7, \\ & \max\{ \min\{\max\{X_8, X_9\}, X_{10}\}, \min\{\max\{X_{11}, X_{12}\}, X_{13}\} \} \}.\end{aligned}$$

(b) Obtain the reliability of the skid protection system when the 13 components operate independently with reliabilities $p_i = p = 0.99$, $i = 1, 2, \ldots, 13$. **(2.5)**

- **Reliability of the components**
  $p_i = p = 0.99$, $i = 1, 2, \ldots, 13$
- **Reliability**
  Taking into account that we are dealing with a series system consisting of the four subsystems below



and that $X_i \overset{iid}{\sim} \text{Bernoulli}(p)$, $i = 1, \ldots, 13$, we can add that the reliability of the skid protection system is the product of the following reliabilities:

$$\begin{aligned}E[\max\{X_1, X_2\}] &\overset{(1.21)}{=} E[1 - (1 - X_1) \times (1 - X_2)] \\ &= 1 - (1 - p)^2;\end{aligned}$$

$$\begin{aligned}E[\max\{ \min\{X_3, X_4\}, \min\{X_5, X_6\} \}] &\overset{(1.21),(1.20)}{=} E[1 - (1 - X_3X_4) \times (1 - X_5X_6)] \\ &= 1 - (1 - p^2)^2;\end{aligned}$$

$$E(X_7) = p;$$

[1] Components 1 to 13 represent: 1-battery; 2-generator; 3-sensor (wheel 1); 4-logic unit (wheel 1); 5-sensor (wheel 2); 6-logic unit (wheel 2); 7-command unit; 8-vacuum solenoid (wheel 1); 9-electric solenoid (wheel 1); 10-actuator (wheel 1); 11-vacuum solenoid (wheel 2); 12-electric solenoid (wheel 2); 13-actuator (wheel 2).

$$\begin{aligned}& E[\max\{ \min\{\max\{X_8, X_9\}, X_{10}\}, \\ & \quad \min\{\max\{X_{11}, X_{12}\}, X_{13}\} \}] \overset{(1.21),(1.20)}{=} E(1 - \{1 - [1 - (1 - X_8)(1 - X_9)] \times X_{10}\} \times \\ & \qquad \{1 - [1 - (1 - X_{11})(1 - X_{12})] \times X_{13}\}) \\ & \qquad = 1 - \left\{1 - \left[1 - (1 - p)^2\right] \times p\right\}^2.\end{aligned}$$

We finally get the requested reliability

$$\begin{aligned}r(\underline{p}) &= E\left[\phi(\underline{X})\right] \\ &\overset{p=0.99}{\simeq} 0.9999 \times 0.999604 \times 0.99 \times 0.999898 \\ &\simeq 0.989408.\end{aligned}$$

(c) Provide three lower bounds for the reliability of this system. Comment on the strictest bound. **(3.0)**

- **Three lower bounds**
  Since the 5 components form a coherent system and operate independently — thus, positively associated —, we can apply theorems 1.65, 1.68 and 1.70.
  - By capitalizing on Theorem 1.65 we get

$$\begin{aligned}r(\underline{p}) &\geq \prod_{i=1}^{n} p_i \\ &\overset{p_i=p}{=} p^{13} \\ &\overset{p=0.99}{\simeq} 0.877521\end{aligned}$$

  - Since the minimal cut sets are

$$\begin{aligned}\mathcal{K}_1 &= \{7\} \\ \mathcal{K}_2 &= \{1, 2\} \\ \mathcal{K}_3 &= \{3, 5\} \\ \mathcal{K}_4 &= \{3, 6\} \\ \mathcal{K}_5 &= \{4, 5\} \\ \mathcal{K}_6 &= \{4, 6\} \\ \mathcal{K}_7 &= \{10, 13\} \\ \mathcal{K}_8 &= \{8, 9, 13\} \\ \mathcal{K}_9 &= \{10, 11, 12\} \\ \mathcal{K}_{10} &= \{8, 9, 11, 12\} \\ q &= 10 \text{ (minimal cut sets)},\end{aligned}$$

  Theorem 1.68 leads to

$$\begin{aligned}r(\underline{p}) &\geq \prod_{j=1}^{q} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - p_i)\right] \\ &\overset{p_i=p}{=} \prod_{j=1}^{q} \left[1 - (1 - p)^{\#\mathcal{K}_j}\right] \\ &= [1 - (1 - p)] \times \left[1 - (1 - p)^2\right]^6 \times \left[1 - (1 - p)^3\right]^2 \times \left[1 - (1 - p)^4\right] \\ &\overset{p=0.99}{\simeq} 0.989404\end{aligned}$$

  - A closer look at the system block diagram prompted us to conclude that all the p^* minimal path sets have 6 components. For example: $\{1, 3, 4, 7, 8, 10\}$, $\{1, 3, 4, 7, 9, 10\}$ or $\{1, 3, 4, 7, 11, 13\}$. This result and Theorem 1.70 lead to

$$r(\underline{p}) \geq \max_{j=1,\ldots,p^*} \prod_{i \in \mathcal{P}_j} p_i$$

$$\begin{aligned} r(\underline{p}) &\overset{p_i=p}{\geq} \max_{j=1,\ldots,p^*} p^{\#\mathscr{P}_j} \\ &= p^{\min_{j=1,\ldots,p^*} \#\mathscr{P}_j} \\ &\overset{\#\mathscr{P}_j=6,\forall j}{=} p^6 \\ &\overset{p=0.99}{\simeq} 0.941480. \end{aligned}$$

- **Comment on the strictest lower bound**
  Since $0.989404 > 0.941480 > 0.877521$, the strictest lower bound is the one given by Theorem 1.68.
  This result is somewhat expected because Theorem 1.68 capitalizes not only on the topology of the system (unlike Theorem 1.65), but also on the fact that we are dealing with independent components (unlike Theorem 1.70 meant for positively associated components).

2. Assume that four wheel bolts are adequate from a design/safety standpoint, however the wheel attachment under consideration has five bolts.

Let T_i be the time (in 10^3 days) until losing the wheel bolt i $(i = 1,\ldots,5)$ and admit that T_i, $i = 1,\ldots,5$, are independent r.v. with common failure function $\lambda(t) = 2t$, for $t \geq 0$.

(a) Derive the reliability function of the wheel attachment under consideration, $R_T(t)$, and prove that the associated expected value, $E(T)$, is equal to $\frac{5}{4}\sqrt{\pi} - 2\sqrt{\pi/5} \simeq 0.630236$ (in 10^3 days). **(2.5)**

Hint: Recall that $\int_0^{+\infty} \frac{1}{\sqrt{2\pi\times\sigma^2}} e^{-\frac{t^2}{2\times\sigma^2}}\, dt = \int_0^{+\infty} f_{N(0,\sigma^2)}(t)\, dt = 0.5$.

- **Individual times and common hazard function**
  T_i = time until losing the wheel bolt i, $i = 1,\ldots,5$
  T_i are i.i.d. r.v. with common hazard function $\lambda(t) = 2t$, $t \geq 0$
- **Common reliability function**
  Prop. 3.3 leads to

$$\begin{aligned} R(t) &= \exp\left[-\int_0^t \lambda(u)\, du\right] \\ &= \exp\left[-\int_0^t 2u\, du\right] \\ &= e^{-t^2},\ t \geq 0. \end{aligned}$$

- **Operation time of the wheel attachment**
  T = duration of a $4-\text{out}-\text{of}-5$ system
  $T \overset{Ex.2.7}{=} T_{(n-k+1)} \overset{n=5,k=4}{=} T_{(2)}$
- **Reliability function of T**

$$\begin{aligned} R_T(t) &\overset{(2.8)}{=} F_{Binomial(n,F(t))}(n-k) \\ &= F_{Binomial(5,1-e^{-t^2})}(5-4) \\ &= \sum_{m=0}^{1} \binom{5}{m} \left(1-e^{-t^2}\right)^m \left(e^{-t^2}\right)^{5-m} \\ &= e^{-5t^2} + 5\left(1-e^{-t^2}\right) e^{-4t^2} \\ &= 5e^{-4t^2} - 4e^{-5t^2},\ t \geq 0. \end{aligned}$$

- **Associated expected value**

$$\begin{aligned} E(T) &\overset{(2.10)}{=} \int_0^{+\infty} R_T(t)\, dt \\ &= \int_0^{+\infty} \left(5e^{-4t^2} - 4e^{-5t^2}\right) dt \end{aligned}$$

$$\begin{aligned} E(T) &= 5\sqrt{2\pi\times 1/8}\int_0^{+\infty} \frac{1}{\sqrt{2\pi\times 1/8}} e^{-\frac{t^2}{2\times 1/8}}\, dt - 4\sqrt{2\pi\times 1/10}\int_0^{+\infty} \frac{1}{\sqrt{2\pi\times 1/10}} e^{-\frac{t^2}{2\times 1/10}}\, dt \\ &= 5\sqrt{2\pi\times 1/8}\int_0^{+\infty} f_{N(0,1/8)}(t)\, dt - 4\sqrt{2\pi\times 1/10}\int_0^{+\infty} f_{N(0,1/10)}(t)\, dt \\ &= \frac{5}{2}\sqrt{\pi}\times 0.5 - 4\sqrt{\frac{\pi}{5}}\times 0.5 \\ &= \frac{5}{4}\sqrt{\pi} - 2\sqrt{\frac{\pi}{5}} \\ &\simeq 0.630236. \end{aligned}$$

QED

(b) What can be said about the stochastic ageing character of wheel attachment under consideration? **(1.0)**

- **Devising the stochastic ageing character of T**
  Since $\lambda(t) = 2t$ is an increasing function in t, we can conclude that

$$T_i \overset{i.i.d.}{\sim} IHR,\ i = 1,\ldots,n.$$

  By applying Prop. 3.25, we can add that any order statistic is also IHR, namely

$$T = T_{(2)} \in IHR.$$

(c) Capitalize on the stochastic ageing character of T to calculate bounds for $E(T)$ given that:

(i) the median is equal to $\xi_{0.5} = F_T^{-1}(0.5) \simeq 0.613678$; **(1.5)**

- **Bounds for $E(T)$**
  Recall that $T \in IHR$ and $p = 0.5 \leq 1 - e^{-1} \simeq 0.632121$. Consequently, we can apply Theorem 3.52 and state that

$$\begin{aligned} -\frac{p\xi_p}{\ln(1-p)} &\leq \mu = E(T) \leq -\frac{\xi_p}{\ln(1-p)} \\ -\frac{0.5\times 0.613678}{\ln(1-0.5)} &\leq E(T) \leq -\frac{0.613678}{\ln(1-0.5)} \\ 0.442675 &\leq E(T) \leq 0.885351. \end{aligned}$$

(ii) the common expected time until losing the wheel bolt i is equal to $\mu^* = E(T_i) = \frac{\sqrt{\pi}}{2} \simeq 0.886227$, for $i = 1,\ldots,5$. **(2.5)**

- **Preliminaries**
  We are dealing with a coherent system characterized as follows:
  - $T_i \overset{i.i.d.}{\sim} IHR,\ i = 1,\ldots,5 \overset{Prop.3.36}{\Rightarrow} T_i \overset{i.i.d.}{\sim} IHRA,\ i = 1,\ldots,5$;
  - $\mu^* = E(T_i) = \mu_i = \frac{\sqrt{\pi}}{2} \simeq 0.886227$, $i = 1,\ldots,5$;
  - since we are dealing with a $4-out-of-5$ system, there are in total $p^* = \binom{5}{4} = 5$ minimal path sets, all with cardinal $\#\mathscr{P}_j = 4$, $j = 1,\ldots,5$;
    [$\mathscr{P}_1 = \{1,2,3,4\}, \mathscr{P}_2 = \{1,2,3,5\}, \mathscr{P}_3 = \{1,2,4,5\}, \mathscr{P}_4 = \{1,3,4,5\}, \mathscr{P}_5 = \{2,3,4,5\}$;]
  - since $4-out-of-5$ system functions with at most one inoperative component, there are in total $q = \binom{5}{2} = 10$ minimal cut sets, all with cardinal $\mathscr{K}_j = 2$, $j = 1,\ldots,10$.
    [$\mathscr{K}_1 = \{1,2\}, \mathscr{K}_2 = \{1,3\}, \ldots, \mathscr{K}_{10} = \{4,5\}$.]

  Thus, we can apply Theorem 3.69 and obtain bounds for $E(T)$...

- **Lower bound for $E(T)$**

$$\begin{aligned}
\mu &= E(T)\\
&\geq \max_{j=1,\ldots,p^*}\left\{\left(\sum_{i\in\mathcal{P}_j}\mu_i^{-1}\right)^{-1}\right\}\\
&\overset{\mu_i=\mu^*}{=} \max_{j=1,\ldots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\}\\
&= \frac{\mu^*}{\min_{j=1,\ldots,p}\{\#\mathcal{P}_j\}}\\
&= \frac{\mu^*}{4}\\
&\simeq 0.221557.
\end{aligned}$$

- **Upper bound for $E(T)$**

$$\begin{aligned}
\mu &= E(T)\\
&\leq \min_{j=1,\ldots,q}\int_0^{+\infty}\left[1-\prod_{i\in\mathcal{K}_j}\left(1-e^{-t/\mu_i}\right)\right]dt\\
&\overset{\mu_i=\mu^*}{=} \min_{j=1,\ldots,q}\int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^{\#\mathcal{K}_j}\right]dt\\
&= \int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^{\min_{j=1,\ldots,q}\#\mathcal{K}_j}\right]dt\\
&= \int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^2\right]dt\\
&= \int_0^{+\infty}\left(2e^{-t/\mu^*}-e^{-2t/\mu^*}\right)dt\\
&= \left(-2\mu^*\,e^{-t/\mu^*}+\frac{\mu^*}{2}\,e^{-t/\mu^*}\right)\Bigg|_0^{+\infty}\\
&= 2\mu^*-\frac{\mu^*}{2}\\
&= \frac{3\mu^*}{2}\\
&\simeq 1.329340.
\end{aligned}$$

- **[Comment**
  Knowing
  - the IHR ageing character and the median of T leads to stricter bounds for $E(T)$

  than knowing
  - the expected value of the duration of each component $E(T_i)$, their IHR/IHRA ageing character and the topology of the system;

  not to mention that Theorem 3.52 is applied to a stricter class of distribution (IHR distributions) than Theorem 3.69 (IHRA distributions).]

3. In an electrical distribution system, electronically operated circuit breakers can be activated to interrupt the current.[2]

(a) After having collected a COMPLETE sample of failure times (in 10^3 days) of circuit breakers $(0.07, 0.22, 0.6, 0.73, 1.35, 1.56, 2.11, 3.08)$, a statistician obtained the TTT plot shown above (picture on the right). **(2.5)**

Should the data be described by an exponential model at a 10% significance level ? Comment on the result of the goodness of fit test in light of the TTT plot above.

[2] If the current exceeds 105% of the rated line current it is required that the circuit breakers open, thereby disconnecting the supply.

- **Life test / data**
  COMPLETE data (no censoring!): $r = n = 8$
  Ordered failure times: $(t_{(1)},\ldots,t_{(n)}) = (0.07, 0.22, 0.6, 0.73, 1.35, 1.56, 2.11, 3.08)$

- **Hypotheses**
  $H_0: T \sim \text{Exponential}(\lambda)$
  $H_1: T \sim \text{Weibull}(\lambda^{-1},\alpha),\ \alpha \neq 1$

- **Significance level**
  $\alpha_0 = 10\%$

- **Test statistic** (Bartlett's test)

$$\begin{aligned}
B_r &\overset{(5.17)}{=} \frac{2r}{1+\frac{r+1}{6r}}\left(\ln\left(\frac{\sum_{i=1}^r T_{(i)}}{r}\right)-\frac{1}{r}\sum_{i=1}^r \ln\left[T_{(i)}\right]\right)\\
&\overset{a}{\sim}_{H_0} \chi^2_{(r-1)}
\end{aligned}$$

- **Rejection region of H_0**

$$\begin{aligned}
W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right)\cup\left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2),+\infty\right)\\
&\overset{r=8,\,\alpha_0=0.1}{=} (0, 2.167)\cup(14.07,+\infty)
\end{aligned}$$

- **Decision**
  The observed value of the test statistic is

$$\begin{aligned}
b_r &= \frac{2r}{1+\frac{r+1}{6r}}\left(\ln\left(\frac{\sum_{i=1}^r t_{(i)}}{r}\right)-\frac{1}{r}\sum_{i=1}^r \ln\left[t_{(i)}\right]\right)\\
&= \frac{2\times 8}{1+\frac{8+1}{6\times 8}}\left(\ln\left(\frac{0.07+\cdots+3.08}{8}\right)-\frac{1}{8}\{\ln(0.07)+\cdots+\ln(3.08)\}\right)\\
&= 13.473684\times\left(\ln\left(\frac{9.72}{8}\right)-\frac{-2.382516}{8}\right)\\
&= 6.636579\\
&\not\in W = (0, 2.167)\cup(14.07,+\infty),
\end{aligned}$$

  therefore we should not reject H_0 for any significance level $\alpha \leq 10\%$.

- **Comment**
  This decision is consistent with the TTT plot above. In fact, its points are roughly around a 45^o line and, according to Note 5.5, this suggests that the data should be modelled by an Exponential distribution[, that is, a memoryless distribution therefore with constant hazard rate (CHR)].

(b) Determine the ML estimate of the median failure time of a circuit breaker and the UMVU estimate of the probability that such failure time exceeds 10^3 days. **(1.5)**

- **Preliminary comment**
  In light of (a), it is fairly reasonable to admit that the failure times of circuit breakers are exponentially distributed r.v.

- **ML estimate of $F_T^{-1}(0.5)$**
  According to Table 5.10 and result (5.20), the ML estimate of $F_T^{-1}(0.5) = -\frac{1}{\lambda}\ln(1-0.5)$ is given by

$$\begin{aligned}
\hat{F}_T^{-1}(0.5) &= -\frac{1}{\hat{\lambda}}\ln(1-0.5)\\
&= -\bar{t}\ln(1-0.5)\\
&= -1.215\times\ln(0.5)\\
&= 0.842174.
\end{aligned}$$

- **UMVU estimate of $P(T > 10^3$ days$)$**
  Since $t = 1 < n\bar{t} = 8\times 1.215 = 9.72$, we can capitalize on result (5.21) to obtain the requested UMVU estimate:

$$
\begin{aligned}
UMVUE[R_T(t)] &= \left(1-\frac{\hat{\lambda}\,t}{n}\right)^{n-1}\\
&= \left(1-\frac{t}{n\bar{t}}\right)^{n-1}\\
&\overset{t=1,etc.}{=} \left(1-\frac{1}{8\times 1.125}\right)^{8-1}\\
&\simeq 0.467683.
\end{aligned}
$$

(c) Obtain a 95% confidence interval for the expected duration of the life test if the statistician decides to use 8 circuit breakers and end the life test at the 4^{th} failure epoch. (**2.0**)

- **Life test / distribution assumption**
  The description above suggests a Type II/item censored testing without replacement, with $r = 4$ and $n = 8$.
  The assumption $T_i \overset{i.i.d.}{\sim}$ Exponential(λ), $i = 1,\ldots,n$, is fairly reasonable since we did not reject H_0 for all the usual significance levels $(1\%, 5\%, 10\%)$.
- **Expected duration of the life test**
  According to (4.12), the expectation is given by
  $$E(T_{r:n}) = \sum_{i=1}^{r}\frac{1}{(n-i+1)\lambda} \overset{r=4,n=8}{\simeq} 0.634524\times\frac{1}{\lambda},$$
- **Unknown parameters**
  λ
  $E(T_{4:n}) \simeq 0.634524 \times \frac{1}{\lambda}$
- **Confidence interval for λ — Type II/item censored testing without replacement**
  According to Table 5.16,
  $$
  \begin{aligned}
  CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L;\lambda_U]\\
  &= \left[\frac{F_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}};\frac{F_{\chi^2_{(2r)}}(1-\alpha/2)}{2\times\tilde{t}}\right].
  \end{aligned}
  $$
  If we take into account that
  - $n = 8$, $r = 4$,
  - the censored data is $(t_{(1)},\ldots,t_{(r)}) = (0.07, 0.22, 0.6, 0.73)$,
  - the cumulative total time in a Type II/item censored test without replacement is given by
  $$
  \begin{aligned}
  \tilde{t} &\overset{Def.5.17}{=} \sum_{i=1}^{r} t_{(i)} + (n-r)\times t_{(r)}\\
  &= 1.62 + (8-4)\times 0.73\\
  &= 4.54,
  \end{aligned}
  $$

  we get
  $$
  \begin{aligned}
  CI_{95\%}(\lambda) &= \left[\frac{F_{\chi^2_{(8)}}(0.025)}{2\times 4.54};\frac{F_{\chi^2_{(8)}}(0.975)}{2\times 4.54}\right]\\
  &\overset{Tables}{=} \left[\frac{2.180}{9.08};\frac{17.53}{9.08}\right]\\
  &\simeq [0.240088; 1.930617]
  \end{aligned}
  $$
- **Confidence interval for $E(T_{4:n})$ — Type II/item censored testing without replacement**
  Since $E(T_{4:n}) \simeq 0.634524 \times \frac{1}{\lambda}$ is a decreasing function of $\lambda > 0$, we obtain
  $$
  \begin{aligned}
  CI_{95\%}(E(T_{4:n})) &= \left[0.634524\times\frac{1}{\lambda_U}, 0.634524\times\frac{1}{\lambda_L}\right]\\
  &\simeq [0.328664; 2.642881] \quad (\text{in } 10^3 \text{ days}).
  \end{aligned}
  $$

- [**Confidence interval for λ and $E(T_{4:n})$ — assuming complete data**
  - **Pivotal quantity for λ**
    According to p. 141 of the lecture notes,
    $$Z = 2\lambda\sum_{i=1}^{n} T_i \sim \chi^2_{(2n)}.$$
  - **Percentage points (balanced ones)**
    Considering the confidence level $(1-\alpha)\times 100\% = 95\%$ and $n = 8$, we obtain
    $$
    \begin{aligned}
    a_\alpha &= F^{-1}_{\chi^2_{(2n)}}(1-\alpha/2)\\
    &= 6.908\\
    b_\alpha &= F^{-1}_{\chi^2_{(2n)}}(1-\alpha/2)\\
    &= 28.85.
    \end{aligned}
    $$
  - **Inverting the double inequality $a_\alpha \le Z \le b_\alpha$**
    $$
    \begin{aligned}
    P(a_\alpha \le Z \le b_\alpha) &= 1-\alpha\\
    P\left(a_\alpha \le 2\lambda\sum_{i=1}^{n} T_i \le b_\alpha\right) &= 1-\alpha\\
    P\left(\frac{a_\alpha}{2\sum_{i=1}^{n} T_i} \le \lambda \le \frac{b_\alpha}{2\sum_{i=1}^{n} T_i}\right) &= 1-\alpha
    \end{aligned}
    $$
  - **Confidence interval for λ**
    $$
    \begin{aligned}
    CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L, \lambda_U]\\
    &= \left[\frac{a_\alpha}{2\sum_{i=1}^{n} t_i}, \frac{b_\alpha}{2\sum_{i=1}^{n} t_i}\right]\\
    CI_{95\%}(\lambda) &= \left[\frac{6.908}{2\times 9.72}, \frac{28.85}{2\times 9.72}\right]\\
    &\simeq [0.355350, 1.484054]
    \end{aligned}
    $$
  - **Confidence interval for $E(T_{r:n})$**
    Since $E(T_{4:n}) \simeq 0.634524 \times \frac{1}{\lambda}$ is a decreasing function of $\lambda > 0$, we obtain
    $$
    \begin{aligned}
    CI_{95\%}(E(T_{4:n})) &= \left[0.634524\times\frac{1}{\lambda_U}, 0.634524\times\frac{1}{\lambda_L}\right]\\
    &\simeq [0.427561, 1.785629] \quad (\text{in } 10^3 \text{ days}).]
    \end{aligned}
    $$