Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

2nd. Test (“Recurso") 1st. Semester — **2014/15**

Duration: **1h30m** **2014/01/31** — **9:45AM**, Room **V1.27**

- Please justify your answers.
- This test has **two pages** and **four questions**. The total of points is **20.0**.

1. Consider the following multiple choice questions, and select and justify the best possible answer.

   (i) If the control limits based on three standard deviations of the control statistic are replaced with those based on two standard deviations: (**1.0**)

   A. the resulting control chart is more sensitive to the real demands of the process;

   B. adjustments to the process based on the two sigma limits may increase process variation.

   - **Best possible answer**
     B. Using two sigma limits in place of three sigma limits results in responding to random variation as if it were due to assignable causes and leads to unnecessary adjustments and tampering with the process; this in turn leads to an increase in process variation.

   (ii) When an $\bar{X}-$chart is used to control the process mean using a sample size of 12: (**0.5**)

   A. the S^2-chart should be used to control the process variance;

   B. the $R-$chart (range chart) should be used to control the process standard deviation.

   - **Best possible answer**
     A. R loses its efficiency as an estimator of σ, as the sample size increases, therefore the range chart should not be used when the sample size is as large as 12. [Keller (2011, p. 276)[1] mentions that the range chart should not be used for subgroups larger than 10.]

2. A maintenance group improves the effectiveness of its repair work by monitoring the fraction p of maintenance requests that require a second call to complete the repair. Ten weeks of data led to:

i	1	2	3	4	5	6	7	8	9	10
Total number of requests (n_i)	200	250	250	250	250	200	200	150	150	150
Requests requiring a 2nd. call (y_i)	6	8	9	7	3	4	2	10	0	2

   (a) An approach to dealing with variable sample size (n_i) is to use a standardized control chart, where the points are plotted in standard deviation units. Such a control chart has the center line at zero, and upper and lower control limits of $+3$ and -3, respectively. (**1.5**)

   Identify its control statistic when you are trying to monitor p and the maintenance group considers a target value $p_0 = 0.03$. Does the process appear to be in statistical control?

   - **Relevant r.v. and its distribution**
     $Y_i =$ number of requests requiring a 2nd. call in the i^{th} sample of size n_i, $i \in \mathbb{N}$
     $Y_i \sim \text{Binomial}(n_i, p)$
     $p = P(\text{request requiring a 2nd. call})$

[1] Keller, P. (2011). *Statistical Process Control DeMystified* (Hard stuff made easy). New York: MacGraw Hill.

   - **Control statistic of the standardized chart for p**
     Judging by the description above and the fact that estimator of p is $\frac{Y_i}{n_i}$ at sample i and that in control

$$E(Y_i/n_i) = p_0$$

$$V(Y_i/n_i) = \frac{p_0\,(1-p_0)}{n_i},$$

     the control statistic of the standardized chart for p should be

$$Z_i = \frac{\frac{Y_i}{n_i} - p_0}{\sqrt{\frac{p_0\,(1-p_0)}{n_i}}}.$$

   - **Checking whether the process is in statistical control**
     For instance, taking the largest value of y_i — coincidentally associated with the smallest sample size — we obtain

$$\begin{aligned} z_8 &= \frac{\frac{y_8}{n_8} - p_0}{\sqrt{\frac{p_0\,(1-p_0)}{n_8}}} \\ &= \frac{\frac{10}{150} - 0.03}{\sqrt{\frac{0.03\times(1-0.03)}{150}}} \\ &= 3.589791 \\ &\notin [LCL, UCL] = [-3, 3]. \end{aligned}$$

     Hence we can add that the process does not appear to be in statistical control.

   (b) Now, admit a much smaller sample size fixed and equal to $n = 15$. Obtain values of the in-control ARL of a $p-$chart with 3–sigma limits and also its out-of-control ARL when the fraction of non-conforming items suddenly shifts from its target value $p_0 = 0.03$ to 0.04 (resp. 0.02). Comment. (**2.0**)

   - **Control statistic and distribution**
     $X_N = \frac{Y_N}{n}$, $N \in I\!N$
     $n \times X_N = Y_N \sim \text{Binomial}(n, p = p_0 + \delta)$, where $n = 15$, $p_0 = 0.03$ and δ $(0 < \delta < 1 - p_0)$ represents the magnitude of the shift in p.

   - **Control limits of the p chart**

$$\begin{aligned} LCL &= \max\left\{0, p_0 - \gamma\sqrt{\frac{p_0(1-p_0)}{n}}\right\} \\ &= \max\left\{0, 0.03 - 3\times\sqrt{\frac{0.03\times(1-0.03)}{15}}\right\} \\ &= \max\{0, -0.102136\} \\ &= 0 \\ UCL &= p_0 + \gamma\sqrt{\frac{p_0(1-p_0)}{n}} \\ &= 0.03 + 3\times\sqrt{\frac{0.03\times(1-0.03)}{15}} \\ &= 0.162136 \end{aligned}$$

   - **Probabilities of triggering a signal**
     Since $X_N \geq 0$ and $LCL = 0$, we get:

$$\begin{aligned}\xi(\delta) &= P(X_N \notin [LCL, UCL] \mid \delta)\\ &= P(n \times X_N > n \times UCL \mid \delta)\\ &= 1 - F_{Binomial(n,p=p_0+\delta)}(n \times UCL)\\ &= 1 - F_{Binomial(15,p=0.03+\delta)}(2.432044)\\ &= 1 - F_{Binomial(15,p=0.03+\delta)}(2)\\ &\overset{table}{=} \begin{cases} 1-0.9906 = 0.0094, & p = p_0 = 0.03\ (\delta = 0)\\ 1-0.9797 = 0.0203, & p = 0.04\ (\delta = 0.01)\\ 1-0.9970 = 0.0030, & p = 0.02\ (\delta = -0.01).\end{cases}\end{aligned}$$

- **Run length and requested ARL**
  We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:
  $RL(\delta) \sim \text{Geometric}\,(\xi(\delta))$.

  Thus,
  $$\begin{aligned}ARL(\delta) &= \frac{1}{\xi(\delta)}\\ &= \begin{cases} 0.0094^{-1} \simeq 106.383, & p = p_0 = 0.03\ (\delta = 0, \text{in-control})\\ 0.0203^{-1} \simeq 49.261, & p = 0.04\ (\delta = 0.01, \text{out-of-control, upward shift})\\ 0.0030^{-1} \simeq 333.333, & p = 0.02\ (\delta = -0.01, \text{out-of-control, downward shift}).\end{cases}\end{aligned}$$

- **Comment**
  In this case we have $ARL(-0.01) > ARL(0)$, that is, we deal with a[n upper one-sided] $p-$chart, whose ARL in the presence of a downward shift is unreasonably larger than the in-control ARL.

(c) What is the minimum sample size that guarantees a positive lower control limit? List one consequence of such a limit in the performance of the $p-$chart. **(1.5)**

- **Obtaining a positive lower control limit**
  Given that $p_0 = 0.025$,
  $$\begin{aligned} n\ :\ & LCL > 0\\ & p_0 - 3\sqrt{\frac{p_0(1-p_0)}{n}} > 0\\ & n > \frac{(1-p_0)\times 3^2}{p_0}\\ & n > \frac{(1-0.02)\times 3^2}{0.02}\\ & n > 291,\end{aligned}$$
  and the minimum sample size that guarantees a positive lower control limit is $n = 292$.

- **Importance of a positive lower control limit**
  When dealing with a $p-$chart it is essential to have a positive lower control limit in order to detect in a fairly quick fashion a decrease in the fraction of nonconforming items (i.e., quality improvement), namely with $ARL(\delta < 0) < ARL(0)$.

3. In 1879, A.A. Michelson measured the speed of light in air using a modification of a method proposed by the French physicist J.B.L. Foucault.

(a) Ten of these individual measurements, reported in kilometers per second and with 299 000 km/s subtracted from it, are: **(1.0)**

Measurement	850	1000	740	980	900	930	1070	650	920	760

A statistician performed a Anderson–Darling goodness-of-fit test using *Mathematica* and got a $p-$value of 0.787343. Is there any evidence that the measurements above are normally distributed?

- **Result of the goodness-of-fit test**
  Recall that the $p-$ value is the largest significance level leading to the non rejection of the null hypothesis. Thus, for these particular data set and null hypothesis:
  - we should not reject $H_0 : T \sim \text{Normal}(\mu, \sigma^2)$, $\mu, \sigma^2 > 0$ for any significance levels $\alpha_0 \leq p -$ value $= 0.787343$, namely the usual significance levels $(1\%, 5\%, 10\%)$;
  - we should reject H_0 for any significance levels $\alpha_0 > p -$ value $= 0.787343$.

  The FAMILY of normal distributions, $\{\text{Normal}(\mu, \sigma^2) : \mu, \sigma^2 > 0\}$, seems to be very reasonable in light of the data set.[2]

(b) Suppose you decide to collect samples of $n = 2$ speed measurements and to operate a STANDARD $\bar{X}-$chart and an UPPER one-sided S^2-chart to detect shifts from (μ_0, σ_0) to $(\mu = \mu_0 + \delta \times \sigma_0/\sqrt{n}, \sigma = \theta \times \sigma_0)$, $\delta \in \mathbb{R}$ and $\theta \geq 1$.

(i) Determine the probability that a signal is triggered by the STANDARD $\bar{X}-$chart with $ARL_\mu(0,1) = 100$ samples, when $\theta = \sqrt{\frac{6.635}{5.024}}$. Recalculate this probability for the UPPER one-sided S^2-chart with the same in-control ARL. Comment. **(3.0)**

- **Quality characteristic**
  $X =$ speed measurement
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistics**
  $\bar{X}_N =$ mean of the N^{th} random sample of size n
  $\bar{S}^2_N =$ variance of the N^{th} random sample of size n
- **Distributions**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, where $\delta = \frac{\mu - \mu_0}{\sigma_0/\sqrt{n}} \leq 0$ (resp. $\theta = \frac{\sigma}{\sigma_0} \geq 1$) represents the magnitude of a shift in μ (resp. an upward shift in σ).
  $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$.
- **Control limits**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

[2] The Anderson-Darling test is a statistical test of whether a given sample of data is drawn from a given probability distribution; when applied to testing if a normal distribution adequately describes a set of data, it is one of the most powerful statistical tools for detecting most departures from normality (http://en.wikipedia.org/wiki/Anderson-Darling_test).

- **Probability of triggering a signal**

  Taking into account the distribution of the control statistics, the $\bar{X}-$chart triggers a signal with probability

$$\begin{aligned}\xi_\mu(\delta,\theta) &= 1 - P\left(LCL_\mu \le \bar{X}_N \le UCL_\mu \mid \delta,\theta\right)\\ &= \ldots\\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu-\delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\right],\ \delta\in\mathbb{R},\ \theta\ge 1,\end{aligned}$$

  and the upper one-sided S^2-chart with probability

$$\begin{aligned}\xi_\sigma(\theta) &= P\left(\bar{S}^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right)\\ &= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right]\\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right),\ \theta\ge 1.\end{aligned}$$

- **Run lengths and ARL**

  We are dealing with Shewhart charts therefore the number of samples collected until the $\bar{X}-$ and S^2-charts trigger a signal, given δ and θ, are such that:

$$\begin{aligned}RL_\mu(\delta,\theta) &\sim \text{Geometric}(\xi_\mu(\delta,\theta));\\ ARL_\mu(\delta,\theta) &= \frac{1}{\xi_\mu(\delta,\theta)};\\ RL_\sigma(\theta) &\sim \text{Geometric}\left(\xi_\sigma(\theta)\right);\\ ARL_\sigma(\theta) &= \frac{1}{\xi_\sigma(\theta)}.\end{aligned}$$

- **Obtaining γ_μ and γ_σ**

  The constant γ_μ is such that the in-control ARL, $ARL_\mu(0,1)$, is equal to 100 samples, that is,

$$\begin{aligned}\gamma_\mu\ :\ & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1)\\ & 1 - [\Phi(\gamma_\mu) - \Phi(-\gamma_\mu)] = \frac{1}{ARL_\mu(0,1)}\\ & \gamma_\mu = \Phi^{-1}\left[1 - \frac{1}{2\times ARL_\mu(0,1)}\right]\\ & \gamma_\mu = \Phi^{-1}\left[1 - \frac{1}{ARL_\mu(0,1)}\right]\\ & \gamma_\mu = \Phi^{-1}(0.995)\\ & \gamma_\mu \overset{table}{=} 2.5758.\end{aligned}$$

  Similarly,

$$\begin{aligned}\gamma_\sigma\ :\ & \frac{1}{\xi_\sigma(1)} = ARL_\sigma(1)\\ & 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = \frac{1}{ARL_\sigma(1)}\\ & \gamma_\sigma = F_{\chi^2_{(n-1)}}\left[1 - \frac{1}{ARL_\sigma(1)}\right]\\ & \gamma_\sigma = F_{\chi^2_{(2-1)}}\left(1 - \frac{1}{100}\right)\\ & \gamma_\sigma = F_{\chi^2_{(1)}}(0.99)\\ & \gamma_\sigma \overset{table}{=} 6.635.\end{aligned}$$

- **Probabilities of triggering a valid signal when $\delta = 0$ and $\theta = \sqrt{\frac{6.635}{5.024}}$**

$$\begin{aligned}\xi_\mu(\delta,\theta) &= 1 - \left[\Phi\left(\frac{\gamma_\mu-\delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\right]\\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu}{\theta}\right)\right]\\ &= 2\times\left[1 - \Phi\left(\frac{\gamma_\mu}{\theta}\right)\right]\\ &= 2\times\left[1 - \Phi\left(\frac{2.5758}{\sqrt{\frac{6.635}{5.024}}}\right)\right]\\ &\simeq 2\times[1 - \Phi(2.24)]\\ &\overset{table}{=} 2\times(1 - 0.9875)\\ &= 0.025\\ \xi_\sigma(\theta) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right)\\ &= 1 - F_{\chi^2_{(2-1)}}\left[\frac{6.635}{\left(\sqrt{\frac{6.635}{5.024}}\right)^2}\right]\\ &\simeq 1 - F_{\chi^2_{(1)}}(5.024)\\ &\overset{table}{=} 1 - 0.975\\ &= 0.025.\end{aligned}$$

- **Comment**

  For $\delta = 0$ and $\theta = \sqrt{\frac{6.635}{5.024}}$, the probabilities coincide, that is, the RL of the $\bar{X}-$ and S^2-charts have the same geometric distribution with parameter 0.025.

  [Let us remind the reader that the distribution of $\bar{X}-$chart also depends on θ, therefore this chart is also sensitive to shifts in spread. Moreover, our numerical investigations lead us to conclude that, <u>for $n = 2$</u>, $RL_\mu(0,\theta) \sim RL_\sigma(0,\theta)$, $\theta \ge 1$. Investigate!]

(ii) Compute the out-of-control ARL and SDRL of your joint scheme for μ and σ in the presence of the shift mentioned in (b)(i). **(1.5)**

- **RL of the joint scheme**

  According to Section 10.6 of the lecture notes:

$$\begin{aligned}RL_{\mu,\sigma}(\delta,\theta) &\overset{st}{=} \min\{RL_\mu(\delta,\theta), RL_\sigma(\theta)\}\\ &\sim \text{Geometric}\left(\xi_{\mu,\sigma}(\delta,\theta)\right)\\ \xi_{\mu,\sigma}(\delta,\theta) &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta)\times\xi_\sigma(\theta).\end{aligned}$$

- **Requested out-of-control ARL and SDRL**

  For $\delta = 0$ and $\theta = \sqrt{\frac{6.635}{5.024}}$,

$$\begin{aligned}\xi_{\mu,\sigma}(\delta,\theta) &\overset{(b)(i)}{=} 0.025 + 0.025 - 0.025\times 0.025\\ &= 0.049375\\ ARL_{\mu,\sigma}(\delta,\theta) &= \frac{1}{\xi_{\mu,\sigma}(\delta,\theta)}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{0.049375} \\
&\simeq 20.253 \\
SDRL_{\mu,\sigma}(\delta,\theta) &\stackrel{\text{Table 9.2}}{=} \frac{\sqrt{1-\xi_{\mu,\sigma}(\delta,\theta)}}{\xi_{\mu,\sigma}(\delta,\theta)} \\
&= \frac{\sqrt{1-0.049375}}{0.049375} \\
&\simeq 19.747.
\end{aligned}$$

(c) Assume a shift from (μ_0, σ_0) to $(\mu = \mu_0 + \delta \times \sigma_0/\sqrt{n}, \sigma = \theta \times \sigma_0)$ occurred. In this case the probability that an arbitrary $\bar{X}-$chart (resp. S^2-chart) detects such a shift is represented by $\xi_\mu(\delta,\theta)$ (resp. $\xi_\sigma(\theta)$). Prove that $P[RL_\mu(\delta,\theta) = RL_\sigma(\theta)] = \frac{\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta)}{\xi_\mu(\delta,\theta)+\xi_\sigma(\theta)-\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta)}$.[3] Interpret this result. **(2.0)**

- **Run lengths**
  [For $\delta \in \mathbb{R}$ and $\theta \geq 1$,]
  $$\begin{aligned} RL_\mu(\delta,\theta) &\sim \text{Geometric}(\xi_\mu(\delta,\theta)) \\ RL_\sigma(\theta) &\sim \text{Geometric}(\xi_\sigma(\theta)). \end{aligned}$$

- **To prove**
  $P[RL_\mu(\delta,\theta) = RL_\sigma(\theta)] = \frac{\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta)}{\xi_\mu(\delta,\theta)+\xi_\sigma(\theta)-\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta)}$

- **Proof**
  If we apply the total probability law and capitalize on the independence between $\bar{X}$ and S^2, thus, of those two RL, we successively get
  $$\begin{aligned}
  P[RL_\mu(\delta,\theta) = RL_\sigma(\theta)] &= \sum_{i=1}^{+\infty} P[RL_\mu(\delta,\theta) = RL_\sigma(\theta) \mid RL_\sigma(\theta) = i] \times P[RL_\sigma(\theta) = i] \\
  &= \sum_{i=1}^{+\infty} P[RL_\mu(\delta,\theta) = i] \times P[RL_\sigma(\theta) = i] \\
  &= \sum_{i=1}^{+\infty} [1-\xi_\mu(\delta,\theta)]^{i-1}\,\xi_\mu(\delta,\theta) \times [1-\xi_\sigma(\theta)]^{i-1}\,\xi_\sigma(\theta) \\
  &= \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta) \sum_{i=0}^{+\infty} \{[1-\xi_\mu(\delta,\theta)] \times [1-\xi_\sigma(\theta)]\}^i \\
  &= \frac{\xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)}{1-[1-\xi_\mu(\delta,\theta)] \times [1-\xi_\sigma(\theta)]} \\
  &= \frac{\xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)}{\xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)}.
  \end{aligned}$$
  QED

- **Interpretation**
  Since the joint scheme for μ and σ triggers a signal with probability
  $$\xi_{\mu,\sigma}(\delta,\theta) = \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta),$$
  the result
  $$P[RL_\mu(\delta,\theta) = RL_\sigma(\theta)] = \frac{\xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)}{\xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)},$$

[3] This is termed the probability of a simultaneous signal.

can be obviously interpreted as a conditional probability — it corresponds to the probability that the chart for μ and σ both trigger a signal, given that the joint scheme was responsible for an alarm.

4. The density of a plastic part used in a cellular telephone is required to be at most $0.70 g/cm^3$.

(a) Admit a sampling plan by variables is adopted with KNOWN STANDARD DEVIATION. Set such a plan with risk points $(p_1, 1-\alpha) = (2\%, 0.9)$ and $(p_2, \beta) = (10\%, 0.05)$. **(2.0)**

- **Sampling plan by variables with KNOWN STANDARD DEVIATION**
  n_σ (sample size)
  k_σ (acceptance constant)
  σ (known standard deviation)
  U (upper specification limit)

- **Producer's and consumer's risk points**
  $(p_1, 1-\alpha) = (2\%, 0.9)$
  $(p_2, \beta) = (10\%, 0.05)$

- **Obtaining n_σ and k_σ**
  According to (13.32),
  $$\begin{aligned}
  (n_\sigma, k_\sigma) \ : \ &\begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(1-\alpha)-\Phi^{-1}(\beta)}{\Phi^{-1}(p_2)-\Phi^{-1}(p_1)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(p_2)\Phi^{-1}(1-\alpha)-\Phi^{-1}(p_1)\Phi^{-1}(\beta)}{\Phi^{-1}(\beta)-\Phi^{-1}(1-\alpha)}. \end{cases} \\
  &\begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(0.9)-\Phi^{-1}(0.05)}{\Phi^{-1}(0.1)-\Phi^{-1}(0.02)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(0.1)\,\Phi^{-1}(0.9)-\Phi^{-1}(0.02)\,\Phi^{-1}(0.05)}{\Phi^{-1}(0.05)-\Phi^{-1}(0.9)}. \end{cases} \\
  &\begin{cases} n_\sigma \stackrel{table}{=} \left[\frac{1.2816-(-1.6449)}{(-1.2816)-(-2.0537)}\right]^2 = 14.366466 \\ k_\sigma \stackrel{table}{=} \frac{(-1.2816)\times 1.2816-(-2.0537)\times(-1.6449)}{(-1.6449)-1.2816} = 1.715575. \end{cases}
  \end{aligned}$$
  We should take $n_\sigma = \lceil 14.366466 \rceil = 15$ and $k_\sigma = 1.715575$. In fact
  $$\begin{aligned}
  P_a(p_1) &= \Phi\left(\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_1)\right]\right) \\
  &= \Phi\left(\sqrt{15}\,[-1.715575 - (-1.2816)]\right) \\
  &\simeq \Phi(1.31) \\
  &\stackrel{table}{=} 0.9049 \\
  &\geq 1-\alpha = 0.9 \\
  P_a(p_2) &= \Phi\left(\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_2)\right]\right) \\
  &= \Phi\left(\sqrt{10}\,[-1.715575 - (-1.2816)]\right) \\
  &\simeq \Phi(-1.68) \\
  &\stackrel{table}{=} 1 - 0.9535 \\
  &= 0.0465 \\
  &\leq \beta = 0.05.
  \end{aligned}$$

(b) Admit a statistician suggested the adoption of a sampling plan by variables with UNKNOWN STANDARD DEVIATION. Use the appropriate formulae to obtain n_s and k_s and verify that when **(3.0)**

$n_s = 39$ and $k_s = 1.708034$ the approximate values of $P_a(p_1)$ (resp. $P_a(p_2)$) is larger (resp. smaller) than or equal to $1-\alpha$ (resp. β). Comment on these values of n_s and k_s in light of (a).

Note: In case you did not solve (a), consider $n_\sigma = 15$ and $k_\sigma = 1.715575$.

- **Single sampling plan by variables with** UNKNOWN STANDARD DEVIATION
  n_s (sample size)
  k_s (acceptance constant)

- **Obtaining n_s and k_s**
  Capitalizing on (13.38) and on the fact that

$$\begin{aligned} u &= 3n_\sigma\,(k_\sigma^2 - 2) + 8 \\ &= 3 \times 15 \times (1.715575^2 - 2) + 8 \\ &\simeq 50.443891 \\ v &= 3n_\sigma^2\,k_\sigma^2 \\ &= 3 \times 15^2 \times 1.715575^2 \\ &\simeq 1986.658367 \end{aligned}$$

  we obtain

$$\begin{aligned} n_s &= n_\sigma + \frac{u + \sqrt{u^2 + 24v}}{12} \\ &= 15 + \frac{50.443891 + \sqrt{50.443891^2 + 24 \times 1986.658367}}{12} \\ &\simeq 37.879323 \\ k_s &= \sqrt{\frac{3n_s - 3}{3n_s - 4}}\,k_\sigma \\ &\simeq \sqrt{\frac{3 \times 37.879323 - 3}{3 \times 37.879323 - 4}} \times 1.715575 \\ &\simeq 1.707830. \end{aligned}$$

- If we consider $n_s = 39$ and $k_s = 1.708034$ then

$$\begin{aligned} P_a(p_1) &\overset{(13.39)}{\simeq} \Phi(\theta_{p_1}) \\ &\overset{(13.41)}{\simeq} \Phi\left[\frac{\Phi^{-1}(1-p_1) - k_s\sqrt{\frac{3n_s-4}{3n_s-3}}}{\sqrt{\frac{1+\frac{3n_s k_s^2}{6n_s-8}}{n_s}}}\right] \\ &= \Phi\left[\frac{2.0537 - 1.708034\sqrt{\frac{3\times39-4}{3\times39-3}}}{\sqrt{\frac{1+\frac{3\times39\times1.708034^2}{6\times39-8}}{39}}}\right] \\ &\simeq \Phi(1.39) \\ &\overset{table}{=} 0.9177 \\ &\geq 1 - \alpha = 0.9 \\ P_a(p_2) &= \Phi\left[\frac{1.2816 - 1.708034\sqrt{\frac{3\times39-4}{3\times39-3}}}{\sqrt{\frac{1+\frac{3\times39\times1.708034^2}{6\times39-8}}{39}}}\right] \\ &\simeq \Phi(-1.65) \end{aligned}$$

$$\begin{aligned} &\overset{table}{=} 1 - 0.9505 \\ &= 0.0495 \\ &\leq \beta = 0.05. \end{aligned}$$

- **Comment**
  Admitting that the standard deviation is unknown is more realistic:
  - it does not change the acceptance constant significantly (in this exercise $k_s \simeq k_\sigma$ down to the first decimal place);
  - but it requires the collection of a much larger sample (in this case $n_s = 2.6 \times n_\sigma$).

(c) Suppose that a sample of the appropriate size was taken from a lot, and $\bar{x} = 0.73$ and $s = 1.05 \times 10^{-2}$. Should the lot be accepted or rejected? (**1.0**)

- **Checking whether or not the lot should be accepted**
  The lot should be accepted iff

$$Q = \frac{U - \bar{x}}{s} \geq k_s,$$

  where Q is the quality index, U is the upper specification limit, $\bar{x}$ and s represent the mean of a sample with size n_s, and k_s the acceptance constant. For this sample, we have

$$\begin{aligned} Q &= \frac{0.70 - 0.73}{1.05 \times 10^{-2}} \\ &= -2.85714 \\ &\not\geq 1.70803, \end{aligned}$$

  therefore we should reject the lot.