Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

2nd. Test 1st. Semester — **2014/15**

Duration: **1h30m** **2014/01/08** — **8AM**, Room **P1**

- Please justify your answers.
- This test has **two pages** and **four questions**. The total of points is **20.0**.

1. Consider the following multiple choice questions, and select and justify the best possible answer.

   (i) An $\bar{X}-$chart will generally detect: (**0.5**)

   A. all shifts in the process mean, but not necessarily immediately after they occur;

   B. some process shifts, and is designed to rarely indicate that the process mean has shifted when it has not.

   - **Best possible answer**
     B. There is no guarantee that all process shifts are detected by an $\bar{X}-$chart (this fact rules out answer A); this chart is designed to rarely trigger false alarms.

   (ii) When customer specifications are used in place of statistical control limits: (**1.0**)

   A. the control chart is more sensitive to detecting meaningful shifts in the process;

   B. unnecessary adjustments are often made to the process.

   - **Best possible answer**
     B. Using customer specifications in place of statistical control limits results in responding to random variation as if it were due to assignable causes and leads to unnecessary adjustments and tampering with the process; this in turn leads to an increase in process variation.

2. Surface defects have been counted on 10 rectangular steel plates, and the data are shown below:

Plate numbers	1	2	3	4	5	6	7	8	9	10
Number of defects	2	3	4	3	1	2	5	0	7	7

   (a) Set up an adequate control chart with estimated 3–sigma limits. Does the process producing the plates appear to be in statistical control? (**1.5**)

   - **Control statistic of the $c-$chart and its distribution**
     Y_N = number of defects in the N^{th} sample of size n $(n = 1)$, $N \in \mathbb{N}$
     $Y_N \sim \text{Poisson}(\lambda)$
   - **ML estimate of the target mean λ_0**
     It is well known that $\hat{\lambda} = \bar{x}$, for the Poisson model. Thus, considering the 10 single item samples as a larger sample of size 10, we get

$$\begin{aligned}\hat{\lambda}_0 &= \frac{1}{10} \times (2 + 3 + \ldots + 7 + 7) \\ &= 3.4.\end{aligned}$$

   - **Estimated 3-sigma control limits**

$$\begin{aligned}LCL &= \max\{0,\, \hat{\lambda}_0 - 3 \times \sqrt{\hat{\lambda}_0}\} \\ &= \max\left\{0,\, 3.4 - 3 \times \sqrt{3.4}\right\} \\ &= \max\{0, -2.131727\} \\ &= 0 \\ UCL &= \hat{\lambda}_0 + 3 \times \sqrt{\hat{\lambda}_0} \\ &= 3.4 + 3 \times \sqrt{3.4} \\ &= 8.931727\end{aligned}$$

     [The control statistic only takes values in $\mathbb{N}_0$, thus, UCL could have been set to 8.]

   - **Checking whether the process is in statistical control**
     Since $y_N \in [LCL, UCL]$, $N = 1, \ldots, 10$, therefore we can say that the process producing the plates appears to be in statistical control.

   (b) Estimate not only the probability of a false alarm, but also the probability that, if the expected number of defects per steel plate should suddenly shift to 4 (resp. 3), your control chart would detect this shift on the next sample? Comment. (**2.0**)

   - **Estimate of the probability of triggering a signal (on the next sample)**
     Y_N is a r.v. taking values in $\mathbb{N}_0$, then the $c-$chart triggers a signal when $Y_N < LCL = 0$ or $Y_N > UCL = 8.931727$, that is, when $Y_N > 8$. Thus, when $\lambda = \lambda_0 + \theta$ $(\theta \in (-\lambda_0, +\infty))$, the estimate of the probability of triggering a signal (on the next sample) is given by

$$\begin{aligned}\hat{P}[RL(\theta) = 1] &\overset{Table\, 9.2}{=} \hat{\xi}(\theta) \\ &= P(Y_N \notin [LCL, UCL] \mid \lambda = \hat{\lambda}_0 + \theta) \\ &= P(Y_N > 8 \mid \lambda = \hat{\lambda}_0 + \theta) \\ &= 1 - F_{Poisson(\hat{\lambda}_0 + \theta)}(8),\end{aligned}$$

     after all, $RL(\theta) \sim \text{Geometric}\,(\xi(\theta))$.

   - **Requested estimates**

$$\begin{aligned}1 - F_{Poisson(3.4)}(8) &= 1 - 0.9917 \\ &= 0.0083, \quad \lambda = \hat{\lambda}_0 = 3.4 \text{ (estimate of the prob. false alarm)} \\ 1 - F_{Poisson(4)}(8) &= 1 - 0.9786 \\ &= 0.0214, \quad \lambda = 4 \\ 1 - F_{Poisson(3)}(8) &= 1 - 0.9962 \\ &= 0.0038, \quad \lambda = 3\end{aligned}$$

   - **Comment**
     Since

$$\hat{\xi}(3 - 3.4) = 0.0038 < 0.0083 = \hat{\xi}(3.4 - 3.4),$$

     this $c-$chart will take longer (in average) to detect a decrease in λ (from 3.4 to 3) than to trigger a false alarm — an extremely undesirable property. [Therefore this chart should not be used if we anticipate downward shifts; after all it is an upper one-sided chart.]

3. Titanium buttons are produced, samples of four are drawn from the production process every 15 minutes, and measurements of hardness made on each button.

(a) Suppose you operate a LOWER one-sided $\bar{X}-$chart, with nominal values of the process mean and standard deviation equal to $\mu_0 = 127.0$ and $\sigma_0 = 3.0$, and in-control ARL equal to 500 samples.
Find the control limit of this chart and compute its out-of-control ARL and SDRL in the presence of a downward shift from the target mean to 125.5. (**2.0**)

- **Quality characteristic**
$X =$ hardness of a titanium button
$X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistic**
$\bar{X}_N =$ mean of the N^{th} random sample of size n

- **Distribution**
$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, where $\delta = \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}} \leq 0$ (resp. $\theta = \frac{\sigma}{\sigma_0} > 0$) represents the magnitude of a downward shift in μ (resp. a shift in σ).

- **Lower control limit of the chart**
$LCL = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$

- **Probability of triggering a signal**
Taking into account the distribution of the control statistic, the lower one-sided chart for μ triggers a signal with probability equal to:

$$\begin{aligned}\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N < LCL \mid \delta,\theta\right)\\ &= \ldots\\ &= \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right), \ \delta \leq 0,\ \theta > 0.\end{aligned}$$

- **Run length**
We are dealing with a Shewhart chart therefore the number of samples collected until the chart triggers a signal given θ, $RL(\theta)$, is such that:

$$\begin{aligned}RL_\mu(\delta,\theta) &\sim \text{Geometric}(\xi_\mu(\delta,\theta));\\ ARL_\mu(\delta,\theta) &= \frac{1}{\xi_\mu(\delta,\theta)}.\end{aligned}$$

- **Obtaining γ_μ**
The constant γ_μ is such that the in-control ARL, $ARL_\mu(0,1)$, is equal to 500 samples, that is,

$$\begin{aligned}\gamma_\mu \ :\ & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1)\\ & \Phi(-\gamma_\mu) = \frac{1}{ARL_\mu(0,1)}\\ & \gamma_\mu = \Phi^{-1}\left[1 - \frac{1}{ARL_\mu(0,1)}\right]\\ & \gamma_\mu = \Phi^{-1}(0.998)\\ & \gamma_\mu \stackrel{table}{=} 2.8782.\end{aligned}$$

Thus, $LCL_\mu \simeq 122.683$.

- **Magnitude of the shift**
The downward shift in μ has magnitude

$$\begin{aligned}\delta &= \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}}\\ &= \frac{125.5-127}{3/\sqrt{4}}\\ &= -1.\end{aligned}$$

We shall assume that the variance is on target, i.e., $\theta = 1$.

- **Probability of detecting the shift**

$$\begin{aligned}\xi_\mu(\delta,\theta) &= \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\\ &\stackrel{(\delta,\theta)=(-1,1)}{=} \Phi[-2.8782-(-1)]\\ &\simeq \Phi(-1.88)\\ &= 1-\Phi(1.88)\\ &\stackrel{tables}{=} 1-0.9699\\ &= 0.0301\end{aligned}$$

- **Request out-of-control ARL and SDRL**

$$\begin{aligned}ARL_\mu(-1,1) &\stackrel{\text{Table 9.2}}{=} \frac{1}{\xi_\mu(-1,1)}\\ &\simeq \frac{1}{0.0301}\\ &\simeq 33.223\\ SDRL_\mu(-1,1) &\stackrel{\text{Table 9.2}}{=} \frac{\sqrt{1-\xi_\mu(-1,1)}}{\xi_\mu(-1,1)}\\ &\simeq \frac{\sqrt{1-0.0301}}{0.0301}\\ &\simeq 32.719.\end{aligned}$$

(b) What should be the minimum sample size n^* if we require that the expected time — until the shift mentioned in (a), $\frac{\mu-\mu_0}{\sigma_0/\sqrt{n*}} = \frac{125.5-127}{3/\sqrt{n^*}}$, is detected — does not exceed 2.5 hours? (**2.0**)

- **Obtaining the minimum requested sample size**
Since the samples are drawn from the production process every 15 minutes, $\gamma_\mu = 2.8782$, $\mu_0 = 127$, $\mu = 125.5$, $\mu - \mu_0 < 0$, $\sigma_0 = 3$ and $\theta = 1$, the sample size should satisfy

$$\begin{aligned}n \ :\ & ARL_\mu\left(\frac{\mu-\mu_0}{\sigma_0/\sqrt{n}}, 1\right) \leq 2.5 \times 4\\ & \frac{1}{\Phi\left(\frac{-\gamma_\mu - \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}}}{\theta}\right)} \leq 10\\ & \frac{-\gamma_\mu - \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}}}{\theta} \geq \Phi^{-1}\left(\frac{1}{10}\right)\\ & \gamma_\mu + \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}} \leq \Phi^{-1}\left(1-\frac{1}{10}\right)\\ & \sqrt{n} \geq \frac{\sigma_0}{|\mu-\mu_0|} \times \left[\Phi^{-1}\left(1-\frac{1}{10}\right) - \gamma_\mu\right]\end{aligned}$$

$$n \geq \left\{ \frac{\sigma_0}{|\mu - \mu_0|} \times \left[\Phi^{-1}(0.90) - \gamma_\mu\right] \right\}^2$$
$$n \geq \left[\frac{3}{|125.5 - 127|} \times (1.2816 - 2.8782) \right]^2$$
$$n \geq 10.197.$$

Hence, the minimum requested sample size is $n^* = 11$.

(c) Admit that you are collecting samples of n buttons, using a LOWER one-sided $\bar{X}-$chart with control limit $LCL = \mu_0 - \gamma \frac{\sigma_0}{\sqrt{n}}$ ($\gamma > 0$) and there is a shift in the process mean of $(-\gamma)$ standard error units, i.e., $\delta = \frac{\mu - \mu_0}{\sigma_0/\sqrt{n}} = -\gamma$. **(2.0)**

Derive the probability of detecting a shift in the process standard deviation from σ_0 to $\sigma = \theta \times \sigma_0$ ($\theta > 0$) not later than the third sample after this shift occurs? Comment.

- **Probability of triggering a signal when $\delta = -\gamma$**

This lower one-sided chart for μ is similar to the one in (a) which triggers a signal with probability equal to:

$$\begin{aligned}\xi_\mu(\delta, \theta) &= P\left(\bar{X}_N < LCL \mid \delta, \theta\right) \\ &= \Phi\left(\frac{-\gamma - \delta}{\theta}\right), \ \delta \leq 0, \ \theta > 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\xi_\mu(-\gamma, \theta) &= \Phi(0) \\ &= 0.5, \ \theta > 0.\end{aligned}$$

- **Requested probability**

$$\begin{aligned}P[RL_\mu(-\gamma, \theta) \leq 3] &= 1 - \overline{F}_{RL_\mu(-\gamma,\theta)}(3) \\ &\overset{\text{Table 9.2}}{=} 1 - [1 - \xi_\mu(-\gamma, \theta)]^3 \\ &= 1 - (1 - 0.5)^3 \\ &= 0.875, \ \theta > 0.\end{aligned}$$

- **Comment**

When $\delta = -\gamma$, the RL of the lower one-sided $\bar{X}-$chart has a geometric distribution with parameter equal to 0.5, regardless of the magnitude of the shift in the process standard deviation, that is, $RL_\mu(-\gamma_\mu, \theta) \sim \text{Geometric}(0.5)$, $\theta > 0$.

The insensitivity of this chart to shifts in the process standard deviation (when $\delta = -\gamma$) should reads as follows: the lower one-sided $\bar{X}-$chart for μ is not designed to detect changes in the process standard deviation.

(d) Suppose you decide to replace the chart described in (a) by a LOWER one-sided EWMA chart for μ, with an asymptotic limit, no head-start, $n = 4$, $\lambda = 0.05$, $\gamma_{\text{EWMA}} = 2.5846$, in-control ARL approximately equal to 500 samples.

(i) Identify the control statistic and limit of this chart. **(1.0)**

- **Control statistic and asymptotic limit**

Judging by Table 10.10[1] and (10.17), and taking into account that the lower one-sided EWMA chart for μ has no head-start, its control statistic and lower asymptotic limit are equal to:

$$W_N = \begin{cases} 0, & N = 0 \\ \min\left\{0, \ (1-\lambda) \times W_{N-1} + \lambda \times \frac{\bar{X}_N - \mu_0}{\sigma_0/\sqrt{n}}\right\} & \\ \quad = \min\left\{0, \ 0.95 \times W_{N-1} + 0.05 \times \frac{\bar{X}_N - 127}{3/2}\right\}, & N \in \mathbb{N}; \end{cases}$$

$$\begin{aligned}LCL_{\text{EWMA}} &= -\gamma_{\text{EWMA}} \times \sqrt{\frac{\lambda}{2-\lambda}} \\ &= -2.5846 \times \sqrt{\frac{0.05}{2-0.05}} \\ &\simeq -0.413867.\end{aligned}$$

[$UCL_{\text{EWMA}} = +\infty$.]

- **[Alternative control statistic and asymptotic limit**

Another possibility would be to use the sample mean, $\bar{X}_N$, instead of the (in-control) reduced sample mean, $\frac{\bar{X}_N - \mu_0}{\sigma_0/\sqrt{n}}$ (see (10.16) and (10.22)):

$$W_N = \begin{cases} 0, & N = 0 \\ \min\left\{0, \ (1-\lambda) \times W_{N-1} + \lambda \times \bar{X}_N\right\} & \\ \quad = \min\left\{0, \ 0.95 \times W_{N-1} + 0.05 \times \bar{X}_N\right\}, & N \in \mathbb{N}; \end{cases}$$

$$\begin{aligned}LCL_{\text{EWMA}} &= \mu_0 - \gamma_{\text{EWMA}} \times \sqrt{\frac{\lambda \sigma_0^2}{(2-\lambda)\, n}} \\ &= 127 - 2.5846 \times \sqrt{\frac{0.05 \times 3^2}{(2-0.05) \times 4}} \\ &\simeq 126.379.]\end{aligned}$$

(ii) In the presence of a downward shift from the target mean to 125.5, the out-of-control SDRL and CVRL of this chart are approximately equal to 4.110 samples and 0.376, respectively. **(1.5)**

How does this chart compare to the one described in (a) in terms of the ability to detect such a shift?

- **Out-of-control ARL**

Assuming once again that the process standard deviation is on target and recalling that $CVRL = \frac{SDRL}{ARL}$, we get:

$$\begin{aligned}ARL_{EWMA}(1,1) &= \frac{SDRL_{EWMA}(1,1)}{CVRL_{EWMA}(1,1)} \\ &\simeq \frac{4.110}{0.376} \\ &\simeq 10.931.\end{aligned}$$

- **Comment**

Since

$$ARL_\mu(1,1) \overset{(a)}{=} 33.223 > ARL_{EWMA}(1,1) = 10.931,$$

we can add that the $\bar{X}$ chart tends to be, in average and expectedly, slower than the EWMA chart in the detection of a small (or medium) size shift like $\delta = 1$.

Moreover,

$$SDRL_\mu(1,1) \overset{(a)}{=} 32.719 > SDRL_{EWMA}(1,1) = 4.110;$$

quality control practitioners should be reminded of Chebyshev's inequality and that

[1] With the control statistic and asymptotic limit of an upper one-sided EWMA chart for μ.

considerable benefit it is to be gained by adopting a chart with a smaller out-of-control SDRL — which conveniently means smaller probabilities of large deviations from the small out-of-control ARL, thus diminishing the possibility of having observations beyond the lower control limit much sooner or much later than expected.

4. An manufacturer receives components from suppliers A and B and accepts lots based on the following sampling plan for attributes: a random sample equal to 10% of the lot size; a lot is rejected if at least one defective components are found.

(a) Suppliers A and B submit components in lots of 250 and 1000, respectively. **(1.5)**

Determine for each supplier the approximate lot fractions defective for which 10% of the lots will be accepted. Do you consider the manufacturer's sampling plan fair if the consumer's risk point is $(LTPD, \beta) = (5\%, 0.1)$?

- **Single sampling plan (for attributes)**
  $N = 250, 1000$ (lot sizes)
  $n = 0.1 \times N$ (sample sizes)
  $c = 0$ (acceptance number)

- **Auxiliary r.v. and its approximate distribution**

$$\begin{aligned} D_N &= \text{number of nonconforming components in the sample of size } 0.1N \\ &\overset{a}{\sim} \text{Binomial}(0.1N, p),\ N = 250, 1000 \end{aligned}$$

- **Obtaining the requested lot fraction defective**

$$\begin{aligned} p_N \ :\ & P(D_N \leq c) = 0.1 \\ & P(D_N = 0) = 0.1 \\ & \binom{0.1N}{0} p_N^0 \, (1-p_N)^{0.1N-0} = 0.1 \\ & (1-p_N)^{0.1N} = 0.1 \\ & p_N = 1 - 0.1^{\frac{1}{0.1N}} \\ & p_N = \begin{cases} 1 - 0.1^{\frac{1}{25}} \simeq 0.087989, & N = 250 \\ 1 - 0.1^{\frac{1}{100}} \simeq 0.022763, & N = 1000. \end{cases} \end{aligned}$$

[Another possibility: use the fact that $D_N \overset{0.1N>20,\, p<0.1}{\sim}$ Poisson$(0.1N\,p)$, for $N = 250, 1000$. In this case $e^{-0.1N \times p_N} = 0.1 \Leftrightarrow p_N = -\frac{\ln(0.1)}{0.1N}$, i.e., $p_N \simeq 0.092103, 0.023026$, for $N = 250, 1000$.]

- **Comment**
  The plan is definitely unfair since it allows supplier A, who ships in smaller lots, a probability of acceptance of 10% when the quality is poor, namely when $p = p_{250} \simeq 0.087989 > LTPD = 0.05$. [Supplier B, who ships in larger lots, is penalized even when the quality is acceptable $p = p_{1000} \simeq 0.022763 < LTPD = 0.05$.]

(b) An experienced statistician has objected to the use of the manufacturer's sampling plan and recommended that the lots from supplier B should be submitted to a sampling plan for attributes with risk points $(AQL, 1-\alpha) = (0.05\%, 0.95)$ and $(LTPD, \beta) = (2\%, 0.1)$ and rectifying inspection. Define an appropriate single sampling plan for attributes. **(2.0)**

- **Producer's and consumer's risk points**
  $(AQL, 1-\alpha) = (0.05\%, 0.98)$
  $(LTPD, \beta) = (2\%, 0.1)$

- **Obtaining the acceptance number and sample size**
  According to Wetherill and Brown (1991, p. 257), the acceptance number c and sample size n of a sampling plan for attributes, associated to risk points $(AQL = p_1, 1-\alpha)$ and $(LTPD = p_2, \beta)$, can be approximately obtained:[2]
  - c should be taken as the smallest integer satisfying
  $$r(c) \leq \frac{p_2}{p_1},$$
  where $r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$;
  - n should be taken as the smallest integer satisfying
  $$\frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{2p_2} \leq n \leq \frac{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}{2p_1}, \quad (1)$$
  most likely the ceiling of the lower bound above,
  $$\left\lceil \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{2p_2} \right\rceil.$$

Using the tables to determine $F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta = 0.90)$ and Mathematica to obtain $F^{-1}_{\chi^2_{(2(c+1))}}(\alpha = 0.05)$, we get

c	$r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$	Is $r(c) \leq \frac{p_2}{p_1} = \frac{0.02}{0.0005} = 40$?
0	$\frac{F^{-1}_{\chi^2_{(2(0+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(0+1))}}(0.05)} = \frac{4.605}{0.103} \simeq 44.709$	NO!
1	$\frac{F^{-1}_{\chi^2_{(2(1+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(1+1))}}(0.05)} = \frac{7.779}{0.711} \simeq 10.941$	**YES!**

Consequently, $c = 1$. Moreover,

$$\begin{aligned} n &= \left\lceil \frac{F^{-1}_{\chi^2_{(2(1+1))}}(1-0.1)}{2 \times 0.02} \right\rceil \\ &\overset{table}{=} \left\lceil \frac{7.779}{2 \times 0.02} \right\rceil \\ &= \lceil 194.475 \rceil \\ &= 195. \end{aligned}$$

- **[Obs. 1**: With $c = 1$, we get $n = 195 \leq \frac{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}{2p_1} = \frac{0.711}{2 \times 0.0005} = 711$.]
- **[Obs. 2**: The lot should be accepted iff the number of defective items, in a sample of $n = 195$, does not exceed $c = 1$.]

[2] See page 129 of the lecture notes, in particular, formulae (13.11), (13.10) and (13.12).

(c) Admit incoming lots from supplier B contain 1% nonconforming items. **(1.5)**

Calculate the average outgoing quality (AOQ) of the single sampling plan in (b). Comment.

Note: In case you have not solved (b), consider from now on $n = 195$ and $c = 1$.

- **Single sampling plan (for attributes)**
  $N = 1000$ (lot size)
  $n = 195$ (sample size)
  $c = 1$ (acceptance number)

- **Auxiliary r.v. and its approximate distribution**
  $D =$ number of defective units in the sample $\overset{a}{\sim}$ Binomial(n, p)

- **Probability of accepting the lot**

$$\begin{aligned} P_a(p) &= P(D \leq c) \\ &\simeq F_{Binomial(n,p)}(c) \\ &= F_{Binomial(195,0.01)}(1) \\ &= \sum_{i=0}^{1} \binom{195}{i} \times 0.01^i \times (1-0.01)^{195-i} \\ &= (1-0.01)^{195} + 195 \times 0.01 \times (1-0.01)^{194} \\ &\simeq 0.418384 \end{aligned}$$

- **Average outgoing quality of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQ(p) &\overset{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\ &\simeq \frac{0.01 \times (1000-195) \times 0.418384}{1000} \\ &\simeq 0.003368. \end{aligned}$$

- **Relative reduction in the percentage defective and comment**
  It is given by

$$\begin{aligned} \left[1 - \frac{AOQ(p)}{p}\right] \times 100\% &\simeq \left(1 - \frac{0.003368}{0.01}\right) \times 100\% \\ &\simeq 66.32\%, \end{aligned}$$

  a considerable reduction.
  [Alternatively, $AOQ(p) \simeq p\,P_a(p) \simeq 0.004184$ and the relative reduction is approximately 58.16%.] Hence, rectifying inspection is worth doing when $p = 0.01$.

(d) Find the average outgoing quality limit (AOQL) of this single sampling plan with rectifying inspection. **(1.5)**

- **Average outgoing quality (AOQ) of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQ(p) &\overset{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\ &\simeq p \times P_a(p) \\ &\overset{(c)}{=} p \times \left[(1-p)^{195} + 195\,p\,(1-p)^{194}\right] \\ &= (1-p)^{194} \times (p + 194p^2) \end{aligned}$$

- **Average outgoing quality limit (AOQL) of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQL &= \max_{p \in [0,1]} AOQ(p) \\ &= AOQ(p^*), \end{aligned}$$

where

$$p^* \in [0,1] \ : \ \begin{cases} \left.\frac{d\,AOQ(p)}{dp}\right|_{p=p^*} = 0 \\ \left.\frac{d^2\,AOQ(p)}{dp^2}\right|_{p=p^*} < 0 \end{cases}$$

$$\begin{cases} -194(1-p^*)^{193} \times [p^* + 194(p^*)^2] + (1-p^*)^{194} \times (1+388p^*) = 0 \\ --- \end{cases}$$

$$\ldots$$

$$\begin{cases} 38024(p^*)^2 - 193p^* - 1 = 0 \\ --- \end{cases}$$

$$\begin{cases} p^* = \frac{193 \pm \sqrt{193^2 - 4 \times 38024 \times (-1)}}{2 \times 38024} \\ --- \end{cases}$$

As a consequence

$$\begin{aligned} p^* &\simeq 0.008260 \\ AOQL &= AOQ(p^*) \\ &\simeq p^* \times (1-p^*)^{194} \times (1+194p^*) \\ &\simeq 0.004301. \end{aligned}$$