Department of Mathematics, IST — Probability and Statistics Unit

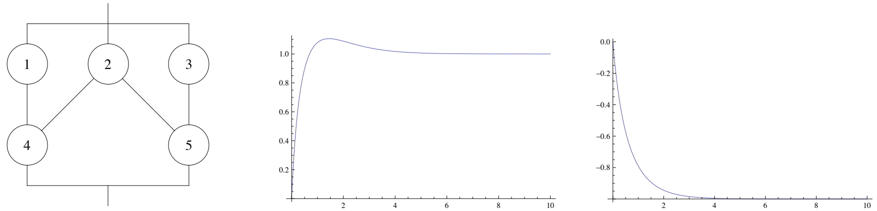
Reliability and Quality Control

1st. Test ("Recurso") 1st. Semester — **2014/15**

Duration: **1h30m** **2015/01/31** — **8AM**, Room **V1.27**

- Please justify your answers.
- This test has **two pages** and **three questions**. The total of points is **20.0**.

1. Assume that a crucial part of a smoke detector constitutes a system with 5 components, set as below (picture on the left).



Part of a smoke detector (left, question 1); plots of $\frac{\frac{dF(t)}{dt}}{1-F(t)}$ and $\frac{\ln[1-F(t)]}{t}$ (center and right, question 2).

(a) Find the minimal path sets and minimal cut sets of this smoke detector part, and provide an expression (do not simplify it!) for its structure function. **(1.5)**

- **Minimal path sets**

$$\begin{aligned} \mathcal{P}_1 &= \{1,4\} \\ \mathcal{P}_2 &= \{2,4\} \\ \mathcal{P}_3 &= \{2,5\} \\ \mathcal{P}_4 &= \{3,5\} \\ p^* &= 4 \text{ minimal path sets} \end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned} \mathcal{K}_1 &= \{4,5\} \\ \mathcal{K}_2 &= \{1,2,3\} \\ \mathcal{K}_3 &= \{1,2,5\} \\ \mathcal{K}_4 &= \{2,3,4\} \\ q &= 4 \text{ minimal cut sets} \end{aligned}$$

- **Structure function**

$$\begin{aligned} \phi(\underline{X}) &\overset{Th. 1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) \\ &= 1 - (1 - X_1X_4) \times (1 - X_2X_4) \times (1 - X_2X_5) \times (1 - X_3X_5). \end{aligned}$$

Obs. — Equivalently,

$$\begin{aligned} \phi(\underline{X}) &\overset{Th. 1.30}{=} \prod_{j=1}^{q}\left[1 - \prod_{i \in \mathcal{K}_j}(1 - X_i)\right] \\ &= [1 - (1 - X_4)(1 - X_5)] \times [1 - (1 - X_1)(1 - X_2)(1 - X_3)] \\ &\quad \times [1 - (1 - X_1)(1 - X_2)(1 - X_5)] \times [1 - (1 - X_2)(1 - X_3)(1 - X_4)]. \end{aligned}$$

(b) Admit that those 5 components operate independently, with reliabilities p_i, $i = 1, \dots, 5$. Compute the three following quantities: the reliability of that part; the importances of the reliabilities of components 1 and 2. **(3.0)**

How do these two importances compare when $p_i = p$, $i = 1, \dots, 5$?

- **Reliability of the components**

p_i, $i = 1, \dots, 5$

- **Reliability**

Taking into account that $X_i \overset{indep}{\sim}$ Bernoulli(p_i) and $X_i^k \sim X_i$, $k \in \mathbb{N}$, we obtain, for $\underline{p} = (p_1, \dots, p_5)$:

$$\begin{aligned} r(\underline{p}) &= E[\phi(\underline{X})] \\ &\overset{(a)}{=} E[1 - (1 - X_1X_4) \times (1 - X_2X_4) \times (1 - X_2X_5) \times (1 - X_3X_5)] \\ &= E\left[1 - (1 - X_1X_4 - X_2X_4 + X_1X_2X_4^2) \times (1 - X_2X_5 - X_3X_5 + X_2X_3X_5^2)\right] \\ &= E(1 - 1 + X_1X_4 + X_2X_4 - X_1X_2X_4^2 \\ &\quad + X_2X_5 - X_1X_2X_4X_5 - X_2^2X_4X_5 + X_1X_2^2X_4^2X_5 \\ &\quad + X_3X_5 - X_1X_3X_4X_5 - X_2X_3X_4X_5 + X_1X_2X_3X_4^2X_5 \\ &\quad - X_2X_3X_5^2 + X_1X_2X_3X_4X_5^2 + X_2^2X_3X_4^2X_5 - X_1X_2^2X_3X_4^2X_5^2) \\ &= E(X_1X_4 + X_2X_4 - X_1X_2X_4 \\ &\quad + X_2X_5 - X_1X_2X_4X_5 - X_2X_4X_5 + X_1X_2X_4X_5 \\ &\quad + X_3X_5 - X_1X_3X_4X_5 - X_2X_3X_4X_5 + X_1X_2X_3X_4X_5 \\ &\quad - X_2X_3X_5 + X_1X_2X_3X_4X_5 + X_2X_3X_4X_5 - X_1X_2X_3X_4X_5) \\ &= p_1p_4 + p_2p_4 - p_1p_2p_4 \\ &\quad + p_2p_5 - p_1p_2p_4p_5 - p_2p_4p_5 + p_1p_2p_4p_5 \\ &\quad + p_3p_5 - p_1p_3p_4p_5 - p_2p_3p_4p_5 + p_1p_2p_3p_4p_5 \\ &\quad - p_2p_3p_5 + p_1p_2p_3p_4p_5 + p_2p_3p_4p_5 - p_1p_2p_3p_4p_5 \\ &= p_1p_4 + p_2p_4 + p_2p_5 + p_3p_5 - p_1p_2p_4 - p_2p_4p_5 - p_2p_3p_5 - p_1p_3p_4p_5 + p_1p_2p_3p_4p_5. \end{aligned}$$

- **Importance of the reliability of components 1 and 2**

$$\begin{aligned} I_r(i) &\overset{(1.29)}{=} \frac{\partial r(\underline{p})}{\partial p_i} \\ &= \begin{cases} p_4 - p_2p_4 - p_3p_4p_5 + p_2p_3p_4p_5, & i = 1 \\ p_4 + p_5 - p_1p_4 - p_4p_5 - p_3p_5 + p_1p_3p_4p_5, & i = 2 \end{cases} \end{aligned}$$

- **Comparing the importance of the reliability of components 1 and 2 when $p_i = p$**

Given that

$$\begin{aligned} I_r(1) - I_r(2) &\overset{p_i = p}{=} (p - p^2 - p^3 + p^4) - (p + p - p^2 - p^2 - p^2 + p^4) \\ &= -p + 2p^2 - p^3 \end{aligned}$$

$$
\begin{aligned}
&= -p(1-2p+p^2)\\
&= -p(1-p)^2\\
&\overset{p\in(0,1)}{\leq} 0,
\end{aligned}
$$

we can add that the reliability of component 2 is more crucial than the one of component 1.

(c) Suppose now that $p_i = p = 0.95$, $i = 1, \ldots, 5$. Provide two pairs of non trivial bounds for the reliability of the smoke detector part. Which bounds are stricter? **(3.0)**

- **Reliability of the components**
  $p_i = p = 0.975$, $i = 1, \ldots, 5$
- **First pair of bounds**
  Since the 5 components form a coherent system and operate independently, we can apply Theorem 1.68.
- **Lower bound for the reliability** $r(\underline{p})$

$$
\begin{aligned}
r(\underline{p}) &\overset{Th.\,1.68}{\geq} \prod_{j=1}^{q}\left[1-\prod_{i\in\mathcal{K}_j}(1-p_i)\right]\\
&\overset{p_i=p}{=} \prod_{j=1}^{q}\left[1-(1-p)^{\#\mathcal{K}_j}\right]\\
&= [1-(1-p)^2]\times[1-(1-p)^3]^3\\
&\overset{p=0.95}{=} 0.997126
\end{aligned}
$$

- **Upper bound for the reliability** $r(\underline{p})$

$$
\begin{aligned}
r(\underline{p}) &\overset{Th.,\,1.68}{\leq} 1-\prod_{j=1,\ldots,p^*}\left(1-\prod_{i\in\mathcal{P}_j}p_i\right)\\
&\overset{p_i=p}{=} 1-\prod_{j=1,\ldots,p^*}\left(1-p^{\#\mathcal{P}_j}\right)\\
&\overset{\#\mathcal{P}_j=2,\forall j}{=} 1-(1-p^2)^4\\
&\overset{p=0.95}{=} 0.999910.
\end{aligned}
$$

- **Another pair of bounds**
  Since the 5 components form a coherent system and operate in an independent fashion — hence, in a positively associated manner —, we can also apply Theorem 1.70 (min-max bounds!).
- **Lower bound for the reliability** $r(\underline{p})$

$$
\begin{aligned}
r(\underline{p}) &\overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*}\prod_{i\in\mathcal{P}_j}p_i\\
&\overset{p_i=p}{=} \max_{j=1,\ldots,p^*}p^{\#\mathcal{P}_j}\\
&= p^{\min_{j=1,\ldots,p^*}\#\mathcal{P}_j}\\
&\overset{\#\mathcal{P}_j=2,\forall j}{=} p^2\\
&= 0.95^2\\
&= 0.9025.
\end{aligned}
$$

- **Upper bound for the reliability**

$$
\begin{aligned}
r(\underline{p}) &\overset{(1.42)}{\leq} \min_{j=1,\ldots,q}\left[1-\prod_{i\in\mathcal{K}_j}(1-p_i)\right]\\
&\overset{p_i=p}{=} \min_{j=1,\ldots,q}\left[1-(1-p)^{\#\mathcal{K}_j}\right]\\
&= \left[1-(1-p)^{\min_{j=1,\ldots,q}\#\mathcal{K}_j}\right]\\
&\overset{\#\mathcal{K}_j=2,3}{=} 1-(1-p)^2\\
&= 1-(1-0.95)^2\\
&= 0.9975.
\end{aligned}
$$

- **Which bounds are stricter?**
  Since $0.997126 > 0.9025$ (resp. $0.999910 > 0.9975$), the lower (resp. upper) bound given by Theo. 1.68 (resp. 1.70) is stricter than the one obtain by applying Theo. 1.70 (resp. 1.68).

2. (a) The lung cancer rate of a t−year−old male smoker is given by $\lambda(t) = 0.027 + 0.025 \times \left(\frac{t-40}{10}\right)^4$, $t \geq 40$.[1]

(i) Assuming that a 40−year−old male smoker survives all other hazards, what is the probability that he survives to age 50 without contracting lung cancer? **(1.5)**

- **R.v.**
  T = lifetime of a male smoker...
- **Hazard rate function**
  $\lambda(t) = 0.027 + 0.025 \times \left(\frac{t-40}{10}\right)^4$, $t \geq 40$
- **Survival function**
  Using Prop. 3.3, we get

$$
\begin{aligned}
R(t) &= \exp\left[-\int_0^t \lambda(u)\,du\right]\\
&= \exp\left\{-\int_{\mathbf{40}}^t\left[0.027+0.025\times\left(\frac{u-40}{10}\right)^4\right]du\right\}\\
&= \exp\left\{\left[-0.027\times u-\frac{0.025\times(u-40)^5}{5\times10^4}\right]\Big|_{\mathbf{40}}^t\right\}\\
&= \exp\left[-0.027\times(t-40)+\frac{0.025\times(t-40)^5}{5\times10^4}\right],\ t\geq 40.
\end{aligned}
$$

- **Requested probability**

$$
\begin{aligned}
R(50) &= \exp\left[-0.027\times(50-40)+\frac{0.025\times(50-40)^5}{5\times10^4}\right]\\
&\simeq 0.726149.
\end{aligned}
$$

(ii) Confirm that $me \simeq 54.360873$ is his median lifetime. **(0.5)**

[1] In the foregoing we are assuming that he remains a smoker throughout his life.

- **Requested confirmation**

$$R(me) = \exp\left[-0.027 \times (54.360873-40) + \frac{0.025 \times (54.360873-40)^5}{5\times 10^4}\right] \simeq 0.5.$$

(b) An OptoEMU sensor is used to monitor the energy a facility uses in real time. Admit its operation time (in 10^3 days) has c.d.f. $F(t) = (1-e^{-t}) \times (1-e^{-2t}) \times I_{(0,+\infty)}(t)$.

(i) A statistician plotted of $\frac{\frac{dF(t)}{dt}}{1-F(t)}$ and $\frac{\ln[1-F(t)]}{t}$ (above, center and right). What do these plots suggest about the stochastic ageing character of the operation time of this sensor? (**1.0**)

- **R.v., c.d.f. and reliability function**
  $T =$ operation time of the OptoEMU sensor
  $F(t) = (1-e^{-t}) \times (1-e^{-2t}) \times I_{(0,+\infty)}(t)$
  $R(t) = 1 - F(t)$

- **Comment on the plot of $\lambda(t)$**
  The plot above (center) suggests that the hazard rate function $\lambda(t)$ is not monotone in t, hence $T \notin IHR, DHR$ by Definition 3.14.

- **Comment on the plot of $\frac{\ln[1-F(t)]}{t}$**
  The other plot above (right) suggests that $\frac{\ln[1-F(t)]}{t} = \ln\left[R^{1/t}(t)\right]$ decreases with t, i.e., $R^{1/t}(t)$ decreases with t. Consequently, $T \in IHRA$ according to Definition 3.32.

(ii) How much would the reliability improve for a period of 2×10^3 days if the statistician adopts 2 OptoEMU sensors with i.i.d. operating times and set them in parallel instead of just a single OptoEMU sensor? (**2.0**)

- **Operation times and common c.d.f.**
  $T_i =$ operation time of the OptoEMU sensor i, $i = 1,\ldots,n$
  $T_i \overset{i.i.d.}{\sim} T$, $i = 1,\ldots,n$
  $F(t) = (1-e^{-t}) \times (1-e^{-2t}) \times I_{(0,+\infty)}(t)$, $i = 1,\ldots,n$

- **New r.v. and reliability function**
  $T_{(n)} = \max_{i=1,\ldots,n} T_i =$ time to failure of the parallel system with n OptoEMU sensors
  $R_{T_{(n)}}(t) \overset{Ex. 2.6}{=} 1 - [1-R(t)]^n$, $t \geq 0$

- **Reliability — using a single sensor**

$$R(2) = (1-e^{-2}) \times (1-e^{-2\times 2}) \simeq 0.151172$$

- **Reliability — using $n = 2$ sensors set in parallel**

$$R_{T_{(2)}}(2) = 1 - [1-R(2)]^2 \simeq 0.279491$$

- **Requested relative improvement**
  When a single sensor is replaced with 3 sensors set in parallel the reliability improves

$$\frac{R_{T_{(2)}}(2) - R(2)}{R(2)} \times 100\% \simeq \frac{0.279491 - 0.151172}{0.151172} \times 100\% \simeq 84.883\%.$$

(iii) After having computed the average operating time of an OptoEMU sensor, find upper bounds for the mean and standard deviation of the operating time of the parallel system in (b)(ii). (**2.5**)

- **Average operating time of an OptoEMU sensor**

$$\begin{aligned}
E(T) &\overset{T\geq 0,\,(2.10)}{=} \int_0^{+\infty} R(t)\,dt \\
&= \int_0^{+\infty} \left[1 - (1-e^{-t}) \times (1-e^{-2t})\right] dt \\
&= \int_0^{+\infty} \left(e^{-t} + e^{-2t} - e^{-3t}\right) dt \\
&= \left(-e^{-t} - \frac{e^{-2t}}{2} + \frac{e^{-3t}}{3}\right)\Bigg|_0^{+\infty} \\
&= 1 + \frac{1}{2} - \frac{1}{3} \\
&= \frac{7}{6} \\
&= \mu^*
\end{aligned}$$

- **Devising the stochastic ageing character of $T_{(n)}$**
  According to Table 3.2 the lifetime of a coherent system — with components with IHRA operation times — is also an IHRA r.v. Moreover, those operation times are independent r.v, thus, positively associated.
  Under these circumstances we can apply Theorem 3.64 to provide an upper bound for the average value of the duration of the parallel system, $\mu = E[T_{(n)}]$.

- **Upper bound for $\mu = E[T_{(n)}]$**

$$\begin{aligned}
\mu &\overset{Th. 3.64}{\leq} \int_0^{+\infty} \left[1 - \prod_{i=1}^n \left(1 - e^{-t/\mu_i}\right)\right] dt \\
&\overset{n=2,\, \mu_i = E(T_i) = \mu^* = 7/6}{=} \int_0^{+\infty} \left[1 - \left(1 - e^{-7t/6}\right)^2\right] dt \\
&= \int_0^{+\infty} \left(2e^{-7t/6} - e^{-7t/3}\right) dt \\
&= \left(-\frac{12e^{-7t/6}}{7} + \frac{3e^{-7t/3}}{7}\right)\Bigg|_0^{+\infty} \\
&= \frac{9}{7}
\end{aligned}$$

- **Upper bound for $SD[T_{(n)}]$**
  Since $T_{(n)} \in IHRA$, we can apply Corollary 3.54 and add that

$$\begin{aligned}
\frac{SD[T_{(n)}]}{E[T_{(n)}]} &\leq 1 \\
SD[T_{(n)}] &\leq E[T_{(n)}] \\
&\leq \frac{9}{7}.
\end{aligned}$$

3. Vacuum tubes[2] produced at a certain plant are assumed to have an underlying exponential life distribution having an unknown average value λ^{-1}.

[2] Although vacuum tubes have been largely replaced by solid-state devices in most amplifying, switching, and rectifying

(a) To estimate λ, 50 vacuum tubes were simultaneously put to test for one year. If failed vacuum tubes are replaced during the test and at the end of the test a total of 5 vacuum tubes have failed, determine the UMVU estimate of the probability that the life of a randomly chosen vacuum tube exceeds 2 years. (**1.5**)

- **Distribution assumption**
  $T_i =$ time to failure (in years) of the i^{th} vacuum tube
  $T_i \stackrel{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$
- **Life test**
  Since
  - vacuum tubes were placed to test for a fixed time, $t_0 = 1$ year
  - the failed vacuum tube were replaced during the test,

  we are dealing with a
  - Type I/item censored testing with replacement.
- **Censored data**
  $n = 50$ (vacuum tube initially put to test)
  $r = 5$ (failures during the test)
  $t_0 = 1$ year (fixed duration of the test)
- **Total time in test**
  Definition 5.17 leads to
  $$\tilde{t} = n\,t_0 = 50 \times 1 = 50.$$
- **Unknown parameter**
  $R_T(2) = e^{-2\lambda}$
- **UMVUE of $R_T(2)$**
  According to Table 5.14 the UMVU estimate of this parameter is
  $$\tilde{R}_T(t) = (1 - \tilde{t}^{-1} \times t)^r \stackrel{t=2}{=} (1 - 50^{-1} \times 2)^5 \simeq 0.815373.$$

(b) Obtain a 95% confidence interval for the reliability for a period of 2 years. (**2.0**)

- **Confidence interval for λ**
  Since we are dealing with a Type I/item censored testing with replacement, we get
  $$CI_{(1-\alpha)\times 100\%}(\lambda) \stackrel{Table\ 5.16}{=} \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2 \times \tilde{t}}\right]$$
  $$CI_{95\%}(\lambda) = [\lambda_L; \lambda_U] \stackrel{r=5,\tilde{t}=50}{=} \left[\frac{F^{-1}_{\chi^2_{(10)}}(0.025)}{2 \times 50}; \frac{F^{-1}_{\chi^2_{(12)}}(0.975)}{2 \times 50}\right] = \left[\frac{3.247}{100}; \frac{23.34}{100}\right] \simeq [0.03247; 0.23340].$$
- **Confidence interval for $R_T(2)$**
  Since $R_T(2) = e^{-2t}$ is a decreasing function of $\lambda > 0$, we can conclude that
  $$CI_{95\%}\left(e^{-2t}\right) = \left[e^{-2\times\lambda_U}; e^{-2\times\lambda_L}\right] \simeq \left[e^{-2\times 0.23340}; e^{-2\times 0.03247}\right] \simeq [0.627005; 0.937124].$$

(c) Now, admit that it has been decided to put a certain number n of tubes on test simultaneously, to stop the test at the 2^{th} failure, that failed vacuum tubes were not replaced during the test. How large should n be if the plant statistician wants the mean length of the testing period not to exceed 3 years when the true value of λ is $\lambda = 0.1$? (**1.5**)

- **Distribution assumption** (same as before)
  $T_i =$ time to failure (in years) of the i^{th} vacuum tube
  $T_i \stackrel{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$
- **Life test**
  Since
  - vacuum tubes were placed to test until the r^{th} failure occurs ($r = 2$)
  - a failed vacuum tube is not replaced during the test,

  we are dealing with a Type II/item censored testing without replacement.
- **Length of the testing period**
  $T_{(r:n)} =$ time of the r^{th} failure out of n possible failures
- **Average length of the testing period**
  $E[T_{(r:n)}] \stackrel{Cor.\ 4.13}{=} \sum_{i=1}^{r} \frac{1}{(n-i+1)\,\lambda}$
- **Requested sample size**
  For $\lambda = 0.1$ and $r = 2$,
  $$n \ : \ E[T_{(r:n)}] \leq 3$$
  $$\sum_{i=1}^{r} \frac{1}{(n-i+1)\,\lambda} \leq 3$$
  $$\sum_{i=1}^{2} \frac{1}{n-i+1} \leq 0.3$$
  $$\frac{1}{n} + \frac{1}{n-1} \leq 0.3$$
  $$n - 1 + n \leq 0.3n^2 - 0.3n$$
  $$0.3n^2 - 2.3 + 1 \geq 0.$$
  Now, note that: the largest solution of $0.3n^2 - 2.3n + 1 = 0$ is
  $$\frac{2.3 + \sqrt{2.3^2 - 4 \times 0.3 \times 1}}{2 \times 0.3} \simeq 7.2040;$$
  its ceiling is 8 and since $0.3 \times 8^2 - 2.3 + 1 = 1.8 \geq 0$ it corresponds to the requested sample size.

applications, there are certain exceptions because they are much less susceptible to transient overvoltages or geomagnetic storms produced by giant solar flares (http://en.wikipedia.org/wiki/Vacuum_tube#Vacuum_tubes_in_the_21st_century).