Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

1st. Test 1st. Semester — **2014/15**

Duration: **1h30m** **2014/11/08** — **8AM**, Room **P8**

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. A safety device relies on n sensors that operate independently with reliabilities $p_1 \leq p_2 \leq \ldots \leq p_n$ and reliability importances $I_r(i)$, $i = 1, 2, \ldots, n$.

   (a) Show that if the sensors are set in series (resp. parallel) then $I_r(1) \geq I_r(2) \geq \ldots \geq I_r(n)$ (resp. $I_r(1) \leq I_r(2) \leq \ldots \leq I_r(n)$). Comment on these results. **(2.5)**

- **Reliabilities of the sensors**
  $p_1 \leq p_2 \leq \ldots \leq p_n$
  Sensors operating in an independent fashion.

- **Structure function and reliability — series system**

$$\begin{aligned}
\phi_{series}(\underline{X}) &\overset{Ex. 1.8}{=} \prod_{j=1}^{n} X_i \\
r_{series}(\underline{p}) &= E\left[\phi_{series}(\underline{X})\right] \\
&\overset{Ex. 1.41}{=} \prod_{i=1}^{n} p_i.
\end{aligned}$$

- **Importance of the reliability of sensor j — series system**

$$\begin{aligned}
I_{r-series}(j) &\overset{(1.29)}{=} \frac{\partial r_{series}(\underline{p})}{\partial p_j} \\
&= \frac{\partial \left(\prod_{i=1}^{n} p_i\right)}{\partial p_j} \\
&= \prod_{i \neq j} p_i.
\end{aligned}$$

- **Comparing the reliability importances — series system**
  Since $p_1 \leq p_2 \leq \ldots \leq p_n$ we get, for $j = 1, 2, \ldots, n-1$,

$$\begin{aligned}
\frac{I_{r-series}(j)}{I_{r-series}(j+1)} &= \frac{\prod_{i \neq j} p_i}{\prod_{i \neq j+1} p_i} \\
&= \frac{\frac{1}{p_j} \prod_{i=1}^{n} p_i}{\frac{1}{p_{j+1}} \prod_{i=1}^{n} p_i} \\
&= \frac{p_{j+1}}{p_j} \\
&\geq 1,
\end{aligned}$$

  and conclude that $I_{r-series}(1) \geq I_{r-series}(2) \geq \ldots \geq I_{r-series}(n)$. QED

- **Comment — series system**
  The sensor with lowest reliability is the most important to the series system, thus reflecting the well-known principle that *a chain is as strong as its weakest link.*

- **Structure function and reliability — parallel system**

$$\begin{aligned}
\phi_{parallel}(\underline{X}) &\overset{Ex. 1.9}{=} 1 - \prod_{j=1}^{n} (1 - X_i) \\
r_{parallel}(\underline{p}) &= E\left[\phi_{parallel}(\underline{X})\right] \\
&\overset{Ex. 1.41}{=} 1 - \prod_{i=1}^{n} (1 - p_i).
\end{aligned}$$

- **Importance of the reliability of sensor j — parallel system**

$$\begin{aligned}
I_{r-parallel}(j) &\overset{(1.29)}{=} \frac{\partial r_{parallel}(\underline{p})}{\partial p_j} \\
&= \frac{\partial \left[1 - \prod_{i=1}^{n} (1 - p_i)\right]}{\partial p_j} \\
&= \prod_{i \neq j} (1 - p_i).
\end{aligned}$$

- **Comparing the reliability importances — parallel system**
  Since $p_1 \leq p_2 \leq \ldots \leq p_n$ we get, for $j = 1, 2, \ldots, n-1$,

$$\begin{aligned}
\frac{I_{r-parallel}(j)}{I_{r-parallel}(j+1)} &= \frac{\prod_{i \neq j} (1 - p_i)}{\prod_{i \neq j+1} (1 - p_i)} \\
&= \frac{\frac{1}{1-p_j} \prod_{i=1}^{n} (1 - p_i)}{\frac{1}{1-p_{j+1}} \prod_{i=1}^{n} (1 - p_i)} \\
&= \frac{1 - p_{j+1}}{1 - p_j} \\
&\leq 1,
\end{aligned}$$

  and conclude that $I_{r-parallel}(1) \leq I_{r-parallel}(2) \leq \ldots \leq I_{r-parallel}(n)$. QED

- **Comment — parallel system**
  The sensor with largest reliability is the most important to the parallel system, this, too, is intuitively reasonable, since if just one component functions, the parallel system functions.

   (b) Admit that $p_i = p$, $i = 1, 2, \ldots, n$, and obtain the approximate perturbation in the reliability of the series (resp. parallel) system due to an increase of 10% in the reliability of every sensor. **(2.0)**

- **Original reliabilities of the sensors**
  $p_j = p$, $j = 1, 2, \ldots, n$

- **Reliabilities of the sensors after a 10% increase**
  $p'_j = p' = 1.1\,p$, $j = 1, 2, \ldots, n$
  $\Delta p_j = p'_j - p_j = p' - p = 0.1p$, $j = 1, 2, \ldots, n$

- **Perturbation in the reliability of the system**
  According to Note 1.61, the perturbation in the reliability of the system — due to changes of Δp_j in the reliability of sensors $j = 1, 2, \ldots, n$ — is given by:

$$\Delta r(\underline{p}) \simeq \sum_{j=1}^{n} I_r(p_j)\, \Delta p_j.$$

  Capitalizing on (a) and on the fact that, $p_j = p$ and $\Delta p_j = 0.1p$, for $j = 1, 2, \ldots, n$, we successively obtain:

- **Perturbation in the reliability of the series system**

$$\begin{aligned}\Delta r_{series}(\underline{p}) &\simeq \sum_{j=1}^{n} I_{r-series}(p_j)\,\Delta p_j \\ &= \sum_{j=1}^{n}\left(\prod_{i\neq j} p_i\right)\,\Delta p_j \\ &= \sum_{j=1}^{n} p^{n-1}\times 0.1p \\ &= 0.1\,n\,p^n;\end{aligned}$$

- **Perturbation in the reliability of the parallel system**

$$\begin{aligned}\Delta r_{parallel}(\underline{p}) &\simeq \sum_{j=1}^{n} I_{r-parallel}(p_j)\,\Delta p_j \\ &= \sum_{j=1}^{n}\left[\prod_{i\neq j}(1-p_i)\right]\,\Delta p_j \\ &= \sum_{j=1}^{n}(1-p)^{n-1}\times 0.1p \\ &= 0.1\,n\,p(1-p)^{n-1}.\end{aligned}$$

(c) Suppose now that the sensors operate with reliabilities p_i, $i = 1, 2, \ldots, n$, and in a positively associated manner. How would you compare the reliabilities of the series and parallel systems? Provide bounds for those reliabilities. **(1.5)**

- **Reliabilities of the sensors**
  p_i, $i = 1, 2, \ldots, n$
  Now, operating in a positively associated manner.

- **Comparing the reliabilities of the series and parallel systems**
  Theorem 1.20 allows us to add that
  $$\phi_{series}(\underline{x}) \leq \phi_{parallel}(\underline{x}),$$
  for all $\underline{x} \in \{0,1\}^n$. Using the monotonicity of the expected value, we can add that
  $$r_{series}(\underline{p}) = E\left[\phi_{series}(\underline{X})\right] \leq E\left[\phi_{parallel}(\underline{X})\right] = r_{parallel}(\underline{p}).$$

- **Bounds for the reliabilities of the series and parallel systems**
  Applying Theorem 1.65[1] and capitalizing on the previous inequality leads to
  $$\prod_{i=1}^{n} p_i \leq r_{series}(\underline{p}) \leq r_{parallel}(\underline{p}) \leq 1 - \prod_{i=1}^{n}(1-p_i).$$

2. A part of a laser scanner depends on $6-\text{out}-\text{of}-10$ system, whose components have independent operation times (in 10^3 days) with common failure function $\lambda(t) = 1 + 2t$, for $t \geq 0$.

(a) Obtain the common median of the operation time of each component, me. **(1.5)**

[1] Bounds for the reliability of a system with positively associated components.

- **Individual operation times and common hazard function**
  T_i = operation time of component i, $i = 1, \ldots, 10$
  T_i are i.i.d. r.v. with common hazard function $\lambda(t) = 1 + 2t$, $t \geq 0$

- **Common reliability function and median**
  Using Prop. 3.3, we get
  $$\begin{aligned}R(t) &= \exp\left[-\int_0^t \lambda(u)\,du\right] \\ &= \exp\left[-\int_0^t (1+2u)\,du\right] \\ &= e^{-(t+t^2)},\ t \geq 0.\end{aligned}$$
  Consequently, we can determine $me = F^{-1}(0.5) = R^{-1}(1-0.5)$. And since $me \in \mathbb{R}_0^+$:
  $$\begin{aligned}R(me) &= 0.5 \\ e^{-(me+me^2)} &= 0.5 \\ me^2 + me + \ln(0.5) &= 0 \\ me &= \frac{-1+\sqrt{1-4\ln(0.5)}}{2} \\ me &\simeq 0.471158.\end{aligned}$$

(b) Derive the reliability function of the operation time of this system and calculate its value for a period of me time units. **(2.0)**

- **Individual operation times and common reliability function**
  T_i are i.i.d. r.v. with common reliability function $R(t) = e^{-(t+t^2)}$, $t \geq 0$

- **Operation time of the system**
  T = duration of the $6-\text{out}-\text{of}-10$ system
  $T \overset{Ex.\,2.7}{=} T_{(n-k+1)} \overset{n=10,\,k=6}{=} T_{(5)}$

- **Reliability function of T**
  $$\begin{aligned}R_T(t) &\overset{(2.8)}{=} F_{Binomial(n,F(t))}(n-k) \\ &= F_{Binomial(10,1-e^{-(t+t^2)})}(10-6) \\ &= \sum_{m=0}^{4}\binom{10}{m}\left[1-e^{-(t+t^2)}\right]^m\left[e^{-(t+t^2)}\right]^{10-m}\end{aligned}$$

- **Requested value**
  $$\begin{aligned}R_T(me) &= F_{Binomial(n,F(me))}(n-k) \\ &= F_{Binomial(10,0.5)}(10-6) \\ &\overset{Tables}{=} 0.3770\end{aligned}$$
  (or 0.376953, according to *Mathematica*).

(c) What can be said about the stochastic ageing character of the operation time of this system? **(1.0)**

- **Devising the stochastic ageing character of T**
  Since $\lambda(t) = 1 + 2t$ is an increasing function in t, we can conclude that
  $$T_i \overset{i.i.d.}{\sim} IHR,\ i = 1, \ldots, n.$$

By applying Prop. 3.25, we can add that any order statistic is also IHR, namely

$T = T_{(5)} \in IHR$.

(d) Calculate the following lower bounds for the reliability function of the operation time of this system for a period of me time units:

(i) ignoring the system topology; (**0.5**)

- **Lower bound for $R_T(me)$ ignoring the system topology**

  Recall that T_i are i.i.d. r.v. with common reliability function $R(t) = e^{-(t+t^2)}$, $t \geq 0$, and that $R(me) = 0.5$.

  Since independent components are also positively associated, we can apply Theorem 2.19 and state that

$$\begin{aligned} R_T(me) &\geq \prod_{i=1}^{n} R_i(me) \\ &= \prod_{i=1}^{n} R(me) \\ &= 0.5^{10} \\ &= 0.000977; \end{aligned}$$

(ii) considering the topology of the system; (**1.0**)

- **Lower bound for $R_T(me)$ considering the system topology**

  Let us remind the reader that a $6-\text{out}-\text{of}-10$ system functions if at least components operate, thus any minimal path has size $\#\mathcal{P}_j = 6$ and they are in total $p = \binom{10}{6}$ minimal paths. Moreover, we can apply Theorem 2.22 and provide the following lower bound:

$$\begin{aligned} R_T(me) &\geq \max_{j=1,\ldots,p} \left\{ \prod_{i \in \mathcal{P}_j} R_i(me) \right\} \\ &= \max_{j=1,\ldots,p} [R(me)]^{\#\mathcal{P}_j} \\ &= 0.5^6 \\ &= 0.015625; \end{aligned}$$

(iii) taking also into account the ageing character of the components and the fact that their common expected time to failure is $\mu^* \simeq 0.545641$. Comment on this and the previous lower bounds. (**2.0**)

- **Lower bound for $R_T(me)$ taking also into account the ageing character of the components, etc.**

  Since

  - $me \simeq 0.471158 < \mu^* \simeq 0.545641$
  - $T_i \overset{i.i.d.}{\sim} IHR$,

  we can apply Theorem 3.58 (with $\mu_i = \mu^*$, $i = 1, \ldots, n$) and provide a lower bound for $R_T(me)$:

$$\begin{aligned} R_T(me) &\geq r_{6-out-of-10}\left(e^{-me/\mu^*}, \ldots, e^{-me/\mu^*}\right) \\ &\overset{Ex. 1.41}{=} 1 - F_{Binomial(10, e^{-me/\mu^*})}(6-1) \\ &\simeq 1 - F_{Binomial(10, 0.421686)}(5) \\ &\overset{Mathematica}{\simeq} 0.204765 \\ &\overset{Tables}{\geq} 1 - F_{Binomial(10, 0.40)}(5) \\ &= 1 - 0.8338 \\ &= 0.1662. \end{aligned}$$

- **Comment**

  The lower bounds for $R_T(me) \overset{(b)}{=} 0.3770$ get less *loose*, as we increase the amount of information on the system we are dealing with:

  i. 0.000977 (ignoring the system topology)
  ii. 0.015625 (considering the system topology)
  iii. 0.1662 (taking also into account the ageing character of the components and the fact that their common expected time to failure is $\mu^* \simeq 0.545641$)

3. A telecommunications network expert collected round-trip delay (RTD) times (in ms).[2]

(a) After having been given the sample, (15, 59, 69, 97, 139, 190, 210, 317, 394, 469), she performed a Kolmogorov-Smirnov goodness-of-fit test using *Mathematica* and got a $p-$value of 0.939523 when an exponential distribution with parameter 0.005 was conjectured. Are the goodness-of-fit test and the conjectured distribution reasonable in light of the data set? (**1.0**)

- **Goodness-of-fit test**

  The Kolmogorov-Smirnov goodness-of-fit test should be performed when we are dealing with:

  - complete data referring to a continuous r.v.;
  - a null hypothesis H_0 involving a continuous distribution (with all known parameters!).

  This seems to be the case!

- **Result of the goodness-of-fit test**

  Recall that the $p-$ value is the largest significance level leading to the non rejection of the null hypothesis. Thus, for these particular data set and null hypothesis:

  - we should not reject $H_0 : T \sim \text{Exponential}(\lambda = 0.005)$ for any significance levels $\alpha_0 \leq p - \text{value} = 0.939523$, namely the usual significance levels $(1\%, 5\%, 10\%)$;
  - we should reject $H_0 : T \sim \text{Exponential}(\lambda = 0.005)$ for any significance levels $\alpha_0 > p - \text{value} = 0.939523$.

  The Exponential$(\lambda = 0.005)$ distribution seems to be very reasonable in light of the data set.

(b) Determine the UMVU estimate of the average value of RTD and the one of the probability that an RTD exceeds 100ms. (**1.0**)

- **UMVU estimates of $E(\mathbf{RTD}) = \lambda^{-1}$ and $P(\mathbf{RTD} > 100) = e^{-100\,\lambda}$**

  Since it is fairly reasonable to admit that the RTD is an exponentially distributed r.v. and $100 < n\bar{t} = 15 + \ldots + 469 = 1959$, we capitalize on the comment after Table 5.10 and on (5.21)

[2] *In telecommunications, the round-trip delay time is the length of time it takes for a signal to be sent plus the length of time it takes for an acknowledgment of that signal to be received* (http://en.wikipedia.org/wiki/Round-trip_delay_time).

to get the requested UMVU estimates:

$$\begin{aligned} UMVUE[E(\text{RTD})] &= \hat{\lambda}^{-1} \\ &= \bar{t} \ \text{(sample mean)} \\ &= 195.9 \\ UMVUE[P(\text{RTD} > t)] &= \left(1 - \frac{t}{n\bar{t}}\right)^{n-1} \\ &\overset{t=100}{=} \left(1 - \frac{100}{10 \times 195.9}\right)^{10-1} \\ &\simeq 0.624029. \end{aligned}$$

(c) After a few inquiries, the telecommunications network expert realized that the previous sample referred to a Type II/item–censored testing without replacement involving (not 10 but) 15 transmission devices. Obtain a point estimate and a 95% confidence interval for the average value of RTD. Do mention any distribution assumptions. (**2.5**)

- **Distribution assumption**
  $T_i \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \ldots, n$.
- **Unknown parameter**
  λ
- **Life test**
  It is mentioned that the data refers to a Type II/item censored testing without replacement.
- **Censored data**
  $(t_{(1)}, \ldots, t_{(r)}) = (15, 59, 69, 97, 139, 190, 210, 317, 394, 469)$
  $n = 15$
  $r = 10$
  $\sum_{i=1}^{r} t_{(i)} = 1959$
- **Cumulative total time in test**

$$\begin{aligned} \tilde{t} &\overset{Def. 5.17}{=} \sum_{i=1}^{r} t_{(i)} + (n - r) \times t_{(r)} \\ &= 1959 + (15 - 10) \times 469 \\ &= 4304 \end{aligned}$$

- **UMVU estimate of $E(\mathbf{RTD}) = \lambda^{-1}$**

$$\begin{aligned} \hat{\lambda} &\overset{Table\ 5.14}{=} \frac{\tilde{t}}{r} \\ &= \frac{4304}{10} \\ &= 430.4 \end{aligned}$$

- **Confidence interval for λ**
  According to Table 5.16,

$$\begin{aligned} CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L; \lambda_U] \\ &= \left[\frac{F_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F_{\chi^2_{(2r)}}(1-\alpha/2)}{2 \times \tilde{t}}\right] \end{aligned}$$

$$\begin{aligned} CI_{95\%}(\lambda) &= \left[\frac{F_{\chi^2_{(20)}}(0.025)}{2 \times 430.4}; \frac{F_{\chi^2_{(20)}}(0.975)}{2 \times 430.4}\right] \\ &\overset{Tables}{=} \left[\frac{9.591}{8608}; \frac{34.17}{8608}\right] \\ &\simeq [0.001114; 0.003970] \end{aligned}$$

- **Another unknown parameter**
  $E(\text{RTD}) = \lambda^{-1}$, which is a decreasing function of λ.
- **Confidence interval for $E(\mathbf{RTD}) = \lambda^{-1}$**

$$\begin{aligned} CI_{95\%}\left(\lambda^{-1}\right) &= \left[\frac{1}{\lambda_U}; \frac{1}{\lambda_L}\right] \\ &\simeq [251.917; 897.508]. \end{aligned}$$

(d) Determine the percentage reduction in the expected duration of this test, having as a reference the expected duration of a test involving just 10 transmission devices. Comment. (**1.5**)

- **Relative reduction of the test duration**
  Considering $T_i \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \ldots, n$ and using result (5.26) for , the relative reduction of the test duration is

$$\begin{aligned} \left[1 - \frac{E(T_{r:n})}{E(T_{r:r})}\right] \times 100\% &= \left[1 - \frac{\sum_{i=1}^{r} \frac{1}{n-i+1}}{\sum_{i=1}^{r} \frac{1}{r-i+1}}\right] \times 100\% \\ &\overset{r=10, n=15}{\simeq} 64.7\%, \end{aligned}$$

  when Type II/item censored testing without replacement has been adopted.
- **Comment**
  This is a very substantial reduction in the test duration, a somewhat expected result because the end of the Type II/item censored testing (without replacement) is determined by approximately 2/3 of the possible n failures.