Department of Mathematics, IST — Probability and Statistics Unit

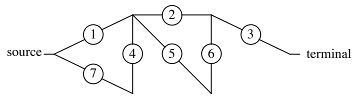
Reliability and Quality Control

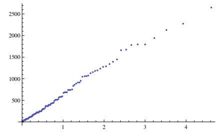
1st. Test ("Recurso") 2nd. Semester — **2012/13**

Duration: **1h30m** **2013/06/28** — **6:30PM**, Room P8

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. *A communication network is set as pictured below on the left and only operates if the source is connected to the terminal.*





Communication network (left, question 1); exponential probability plot (right, question 3).

(a) *Find the minimal path sets and minimal cut sets of this system, and provide an expression (do not simplify it!) for its structure function.* **(2.0)**

- **Minimal path sets**

$$\begin{aligned}\mathcal{P}_1 &= \{1,2,3\}\\ \mathcal{P}_2 &= \{1,5,6,3\} = \{1,3,5,6\}\\ \mathcal{P}_3 &= \{7,4,2,3\} = \{2,3,4,7\}\\ \mathcal{P}_4 &= \{7,4,5,6,3\} = \{3,4,5,6,7\}\\ p^* &= 4 \text{ minimal path sets}\end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned}\mathcal{K}_1 &= \{3\}\\ \mathcal{K}_2 &= \{1,7\}\\ \mathcal{K}_3 &= \{1,4\}\\ \mathcal{K}_4 &= \{2,5\}\\ \mathcal{K}_5 &= \{2,6\}\\ q &= 5 \text{ minimal cut sets}\end{aligned}$$

- **Structure function**

$$\begin{aligned}\phi(\underline{X}) &\stackrel{Th. 1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathcal{P}_j} X_i\right)\\ &= 1 - (1 - X_1X_2X_3) \times (1 - X_1X_3X_5X_6) \times (1 - X_2X_3X_4X_7)\\ &\quad \times (1 - X_3X_4X_5X_6X_7)\end{aligned}$$

Obs. — Equivalently,

$$\begin{aligned}\phi(\underline{X}) &\stackrel{Th. 1.30}{=} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathcal{K}_j}(1 - X_i)\right]\\ &= [1-(1-X_3)] \times [1-(1-X_1)(1-X_7)] \times [1-(1-X_1)(1-X_4)\\ &\quad \times [1-(1-X_2)(1-X_5)] \times [1-(1-X_2)(1-X_6)].\end{aligned}$$

(b) *Admit that each of those 7 components are operational with probabilities $p_i = 0.9$, $i = 1,\ldots,7$, and function independently. Obtain a lower and an upper bound (as strict as reasonably possible!) for the reliability of this system.* **(2.0)**

- **Reliability of the components**
  $p_i = p = 0.9$, $i = 1,\ldots,7$
- **Important**
  Since the 7 components form a coherent system and operate independently, we can apply Theorem 1.68.
- **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned}r(\underline{p}) &\stackrel{Th. 1.68}{\geq} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathcal{K}_j}(1 - p_i)\right]\\ &\stackrel{p_i=p}{=} \prod_{j=1}^{q}\left[1 - (1-p)^{\#\mathcal{K}_j}\right]\\ &= [1-(1-p)] \times [1-(1-p)^2]^4\\ &\stackrel{p=0.9}{=} 0.864536\end{aligned}$$

- **Upper bound for the reliability** $r(\underline{p})$

$$\begin{aligned}r(\underline{p}) &\stackrel{Th., 1.68}{\leq} 1 - \prod_{j=1,\ldots,p^*}\left(1 - \prod_{i\in\mathcal{P}_j} p_i\right)\\ &\stackrel{p_i=p}{=} 1 - \prod_{j=1,\ldots,p^*}\left(1 - p^{\#\mathcal{P}_j}\right)\\ &= 1 - (1-p^3) \times (1-p^4)^2 \times (1-p^5)\\ &\stackrel{p=0.9}{=} 0.986875.\end{aligned}$$

(c) *Assume that the times to failure (in 10^2 days) of those 7 components are i.i.d. random variables with a common IHRA distribution whose expected value is equal to $\mu^* = 1$.*

(i) *Determine a lower and an upper bound (as strict as reasonably possible!) for the expected time to failure of the communication system.* **(3.0)**

- **Individual durations, common stochastic ageing behavior and duration of the system**
  $T_i \stackrel{i.i.d.}{\sim} IHRA$, $i = 1,\ldots,7$
  $E(T_i) = \mu^* = 1$, $i = 1,\ldots,7$
  $T =$ duration of the communication system

- **Important**

  Under these circumstances we can apply Theorem 3.69 to provide a lower and an upper bound for $\mu = E(T)$.

- **Lower bound for $\mu = E(T)$**

$$\begin{aligned}
\mu &\overset{Th. 3.69}{\geq} \max_{j=1,\ldots,p^*} \left[\left(\sum_{i \in \mathcal{P}_j} \mu_i^{-1} \right)^{-1} \right] \\
&\overset{\mu_i=\mu^*}{=} \max_{j=1,\ldots,p^*} \left[\left(\frac{\#\mathcal{P}_j}{\mu^*} \right)^{-1} \right] \\
&= \frac{\mu^*}{\min_{j=1,\ldots,p^*} \#\mathcal{P}_j} \\
&\overset{\mu^*=1}{=} \frac{1}{3}.
\end{aligned}$$

- **Upper bound for $\mu = E(T)$**

$$\begin{aligned}
\mu &\overset{Th. 3.69}{\leq} \min_{j=1,\ldots,q} \int_0^{+\infty} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - e^{-t/\mu_i}) \right] dt \\
&\overset{\mu_i=\mu^*}{=} \min_{j=1,\ldots,q} \int_0^{+\infty} \left[1 - (1 - e^{-t/\mu^*})^{\#\mathcal{K}_j} \right] dt \\
&\overset{\mu^*=1}{=} \int_0^{+\infty} \left[1 - (1 - e^{-t})^{\min_{j=1,\ldots,q} \#\mathcal{K}_j} \right] dt \\
&= \int_0^{+\infty} \left[1 - (1 - e^{-t})^1 \right] dt \\
&= \int_0^{+\infty} e^{-t}\, dt \\
&= 1.
\end{aligned}$$

(ii) *Find an upper bound for the standard deviation of this time to failure.* **(1.5)**

- **Upper bound for $SD(T)$**

  According to Table 3.2 the lifetime of a coherent system — with components with IHRA lifetimes — is also an IHRA r.v.

  Consequently, we can apply Corollary 3.54 and add that

$$\begin{aligned}
\frac{SD(T)}{E(T)} &\leq 1 \\
SD(T) &\leq E(T) = \mu \\
SD(T) &\overset{(i)}{\leq} 1
\end{aligned}$$

2. The logistic distribution has been used to model the time to failure T (in 10^3 km) of shock absorbers.[1] The corresponding c.d.f. is $F(t) = \frac{1}{1+e^{-\frac{t-\mu}{\sigma}}}$, for $-\infty < t < +\infty$ ($-\infty < \mu < +\infty$ and $\sigma > 0$).

   (a) Derive the density, the reliability and the hazard rate functions of T. **(2.0)**

[1]A shock absorber is a mechanical device designed to smooth out or damp shock impulse, and dissipate kinetic energy.

- **Time to failure and its c.d.f.**

  T = time to failure of (in $10^3 km$ of a shock absorber

  $T \sim \text{Logistic}(\mu, \sigma)$

  $F(t) = \frac{1}{1+e^{-\frac{t-\mu}{\sigma}}}$, $-\infty < t < +\infty$ $(-\infty < \mu < +\infty,\ \sigma > 0)$

- **P.d.f. of T**

$$\begin{aligned}
f(t) &= \frac{dF(t)}{dt} \\
&= -\frac{\frac{d e^{-\frac{t-\mu}{\sigma}}}{dt}}{\left(1 + e^{-\frac{t-\mu}{\sigma}}\right)^2} \\
&= \frac{1}{\sigma} \times \frac{e^{-\frac{t-\mu}{\sigma}}}{\left(1 + e^{-\frac{t-\mu}{\sigma}}\right)^2}
\end{aligned}$$

- **Reliability function of T**

$$\begin{aligned}
R(t) &= 1 - F(t) \\
&= 1 - \frac{1}{1 + e^{-\frac{t-\mu}{\sigma}}} \\
&= \frac{e^{-\frac{t-\mu}{\sigma}}}{1 + e^{-\frac{t-\mu}{\sigma}}}
\end{aligned}$$

- **Hazard rate function of T**

$$\begin{aligned}
\lambda(t) &= \frac{f(t)}{R(t)} \\
&= \frac{\frac{1}{\sigma} \times \frac{e^{-\frac{t-\mu}{\sigma}}}{\left(1+e^{-\frac{t-\mu}{\sigma}}\right)^2}}{\frac{e^{-\frac{t-\mu}{\sigma}}}{1+e^{-\frac{t-\mu}{\sigma}}}} \\
&= \frac{1}{\sigma} \times \frac{1}{1 + e^{-\frac{t-\mu}{\sigma}}}.
\end{aligned}$$

(b) What can be said about the stochastic ageing character of T? Can you predict the stochastic ageing character of the failure time of a parallel system with n components with failure times $T_i \overset{i.i.d.}{\sim} T$, $i = 1, \ldots, n$? **(2.5)**

- **Devising the stochastic ageing character of T**

  Since

$$\begin{aligned}
\lambda(t) &= \frac{1}{\sigma} \times \frac{1}{1 + e^{-\frac{t-\mu}{\sigma}}} \\
&= \frac{1}{\sigma} \times F(t)
\end{aligned}$$

  is obviously an increasing function of t (because $F(t)$ is...),

  $T \in IHR$.

  Although Def. 3.14, the notion of IHR distribution is valid for a r.v. taking values in $\mathbb{R}$, according to Barlow *et al.* (1963, p. 377).[2]

[2]Barlow, R.E., Marshall, A.W. and Proschan, F. (1963). Properties of Probability Distributions with Monotone Hazard Rate. *The Annals of Mathematical Statistics* **34**, 375–389.

- **Devising the stochastic ageing character of $T_{(n)}$**

  T_i = time to failure of the i^{th} component, $i = 1, \ldots, n$

  $T_{(n)}$ = time to failure of the parallel system

  According to Prop. 3.23, equation (3.14),

  $$T_i \overset{i.i.d.}{\sim} IHR,\ i = 1, \ldots, n \quad \Rightarrow \quad T_{(n)} \in IHR.$$

- **Obs.**

  Even though Prop. 3.23 is stated for nonnegative r.v., result (3.14) is valid in the present case. In fact,

$$\begin{aligned}
F_{T_{(n)}}(t) &= [F(t)]^n \\
&= \left(1 + e^{-\frac{t-\mu}{\sigma}}\right)^{-n} \\
f_{T_{(n)}}(t) &= \frac{d\,F_{T_{(n)}}(t)}{dt} \\
&= n f(t)[F(t)]^{n-1} \\
&= \frac{n}{\sigma}\left(1 + e^{-\frac{t-\mu}{\sigma}}\right)^{-n-1} \\
\ln[f_{T_{(n)}}(t)] &= \ln(n) - \ln(\sigma) - (n+1)\ln\left(1 + e^{-\frac{t-\mu}{\sigma}}\right) \\
\frac{d^2 \log[f_{T_{(n)}}(t)]}{dt^2} &= \frac{d}{dt}\left[\frac{1}{\sigma}\,\frac{(n+1)e^{-\frac{t-\mu}{\sigma}}}{1 + e^{-\frac{t-\mu}{\sigma}}}\right] \\
&= \frac{n+1}{\sigma}\,\frac{d}{dt}\left(\frac{1}{1 + e^{\frac{t-\mu}{\sigma}}}\right) \\
&= -\frac{n+1}{\sigma^2}\,\frac{e^{\frac{t-\mu}{\sigma}}}{\left(1 + e^{\frac{t-\mu}{\sigma}}\right)^2} \\
&< 0,
\end{aligned}$$

  that is, $f_{T_{(n)}}(t)$ is log-concave, thus, $T_{(n)} \in ILR$, according to Def. 3.32.[3] Consequently, $T_{(n)} \in ILR \overset{Prop. 3.36}{\Rightarrow} T_{(n)} \in IHR.$

(c) Find a lower bound for the reliability for 5000km of the parallel system described in (b), when $\mu_i = \mu^* = 10$, $\sigma = 1$ and $n = 4$. **Note**: Make use of Corollary 3.47 to derive a result similar to the one stated in Theorem 3.56. (**2.0**)

- **Lower bound for $R_{T_{(n)}}(t)$**

  Since $T_i \in IHR$, we can apply Corollary 3.47, derive a result similar to the one stated in Theorem 3.56 and get:

$$\begin{aligned}
R_{T_i}(t) &\geq e^{-t/\mu_i},\ t < \mu_i \\
1 - \prod_{i=1}^{n}[1 - R_{T_i}(t)] &\geq 1 - \prod_{i=1}^{n}\left(1 - e^{-t/\mu_i}\right),\ t < \min_{i=1,\ldots,n} \mu_i \\
R_{T_{(n)}}(t) &\overset{\mu_i=\mu^*}{\geq} 1 - \left(1 - e^{-t/\mu^*}\right)^n,\ t < \mu^* \\
R_{T_{(4)}}(5) &\overset{\mu^*=10,\,t=5,\,n=4}{\geq} 1 - \left(1 - e^{-5/10}\right)^4,\ t < 10 \\
&\geq 0.976031.
\end{aligned}$$

[3] The notion of ILR distribution is also valid for r.v. taking values in $\mathbb{R}$

3. 10 220 observations of the time T (in 1/5000 seconds) between $\alpha-$particle emissions of Americium-241 have been binned into intervals $[t_j, t_{j+1})$, as described in the following frequency table:

Interval	$[0, 100)$	$[100, 300)$	$[300, 500)$	$[500, 700)$	$[700, 1000)$	$[1000, 2000)$	$[2000, 4000)$	$[4000, +\infty)$
Frequency	1609	2424	1770	1306	1213	1528	354	16

(a) Compute estimates of the density, the reliability and the hazard rate functions at $t = 600$. (**1.5**)

- **Non parametric estimates of $f(t)$, $R(t)$ and $\lambda(t)$**

  Let:

  - n = sample size;
  - $N(t)$ = number of survivors at time t;
  - $[t_j, t_{j+1}) = j^{th}$ bin;
  - $\Delta_j = t_{j+1} - t_j$ = range of the j^{th}bin.

  According to Table 5.3, these estimates are, for $t = 600 \in [t_4, t_5) = [500, 700)$, given by:

$$\begin{aligned}
\hat{f}(t) &= \frac{N(t_j) - N(t_{j+1})}{n \times \Delta t_j} \\
&= \frac{\text{number of failure in the } j^{th}\text{bin}}{n \times \Delta t_j} \\
&\overset{j=4}{=} \frac{1306}{10220 \times (700 - 500)} \\
&\simeq 0.000639;
\end{aligned}$$

$$\begin{aligned}
\hat{\lambda}(t) &= \frac{N(t_j) - N(t_{j+1})}{N(t_j) \times \Delta t_j} \\
&\overset{j=4}{=} \frac{1306}{[10220 - (1609 + 2424 + 1770)] \times 200} \\
&= \frac{1306}{(10220 - 5803) \times 200} \\
&= \frac{1306}{4417 \times 200} \\
&\simeq 0.001478.
\end{aligned}$$

  Since $\hat{\lambda}(t) = \frac{\hat{f}(t)}{\hat{R}(t)}$ then

$$\begin{aligned}
\hat{R}(t) &= \frac{\hat{f}(t)}{\hat{\lambda}(t)} \\
&= \frac{N(t_j)}{n} \\
&\overset{j=4}{=} \frac{4417}{10220} \\
&\simeq 0.432192.
\end{aligned}$$

(b) A smaller sample, with 100 non censored observations, was analyzed in more detail and led to the exponential probability plot pictured above on the right. (**1.5**)

After having identified the abscissae and ordinates of this plot, comment on the goodness of fit of the proposed model and calculate a rough estimate of the expected value of T.

- **R.v.**
  T = time (in 1/5000 seconds) between α − particle emissions of Americium-241

- **Conjectured model**
  $\{\text{Exponential}(\lambda) : \lambda > 0\}$

- **Probability plot**
  For any absolutely continuous model, we have
  $$F_T(T_{(i)}) \sim \text{Beta}(i, n-i+1).$$
  Thus, by considering as an estimate of
  $$p_i = F_T(t_{(i)}) = 1 - e^{-\lambda t_{(i)}}$$
  the following expected value
  $$\hat{p}_i = E[F_T(T_{(i)})] = E[\text{Beta}(i, n-i+1)] = \frac{i}{n+1},$$
  we are bound to confront (indirectly and) graphically $\hat{p}_i$ and $F_T(t_{(i)})$, that is,
  $$\begin{aligned} \frac{i}{n+1} &\rightarrow 1 - e^{-\lambda t_{(i)}} \\ 1 - \frac{i}{n+1} &\rightarrow e^{-\lambda t_{(i)}} \\ -\frac{1}{\lambda}\ln\left(1 - \frac{i}{n+1}\right) &\rightarrow t_{(i)}. \end{aligned}$$
  Consequently, the points in the exponential probability plot have abscissae and ordinates:
  $$\left(-\ln\left(1 - \frac{i}{n+1}\right), t_{(i)}\right), \ i = 1, \ldots, n.$$

- **Comment**
  The points in the plot are set in an approximately linear fashion, hence the conjectured model fits very well to the data set.

- **Rough estimate of** $E(T) = \lambda^{-1}$
  It is obtained by determining a rough estimate of the slope (λ^{-1}) of a line fitting the points of the probability plot, namely the ones closer to the origin:
  $$\begin{aligned} \widetilde{\lambda^{-1}} &= \frac{500 - 0}{1 - 0} \\ &= 500. \end{aligned}$$

(c) Obtain a 95% confidence interval (CI) for the median of T, considering that $\sum_{i=1}^{100} t_i = 61266.6$. **(2.0)**
What were the underlying assumptions that allowed you to specify this quantile and obtain the CI?

- **Distribution assumption**
  T_i = time (in 1/5000 seconds) between the i^{th} and the $(i+1)^{th}$ α − particle emission of Americium-241
  $T_i \overset{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$

- **Life test**
  Complete data is available!
  $\sum_{i=1}^{100} t_i = 61266.6$

- **Another unknown parameter**
  $$\begin{aligned} me &= \text{median of } T \\ &= F_T^{-1}(0.5) \\ &= -\frac{1}{\lambda}\ln(1 - 0.5) \end{aligned}$$

- $(1-\alpha) \times 100\%$ **confidence interval (CI) for** me
  This CI can be obtained as follows:
  - **Pivotal quantity**
    $Z = 2\lambda \sum_{i=1}^n T_i \sim \xi^2_{(2n)}$ (check lecture note just before Exercise 5.26)
  - **Percentage points**
    They are represented by a and b, balanced and such that $P(a \leq Z \leq b) = 1 - \alpha$, where $\alpha = 0.05$:
    $$\begin{aligned} a &= F^{-1}_{\chi^2_{(2n)}}(\alpha/2) \\ &= F^{-1}_{\chi^2_{(200)}}(0.025) \\ &\overset{table}{=} 162.7 \\ b &= F^{-1}_{\chi^2_{(2n)}}(1-\alpha/2) \\ &= F^{-1}_{\chi^2_{(200)}}(0.975) \\ &\overset{table}{=} 241.1 \end{aligned}$$
  - **Inverting the inequality** $a \leq Z \leq b$
    Since me is a decreasing function of $\lambda > 0$, we can add that
    $$P(a \leq Z \leq b) = 1 - \alpha$$
    $$P\left(a \leq 2\lambda \sum_{i=1}^n T_i \leq b\right) = 1 - \alpha$$
    $$\vdots$$
    $$P\left(-\tfrac{2\ln(1-0.5)\sum_{i=1}^n T_i}{b} \leq -\tfrac{1}{\lambda}\ln(1-0.5) \leq -\tfrac{2\ln(1-0.5)\sum_{i=1}^n T_i}{a}\right) = 1 - \alpha.$$
  - **CI**
    $$\begin{aligned} CI_{95\%}(me) &= \left[-\frac{2\ln(1-0.5)\sum_{i=1}^n t_i}{b}; -\frac{2\ln(1-0.5)\sum_{i=1}^n t_i}{a}\right] \\ &= \left[-\frac{2\ln(1-0.5) \times 61266.6}{241.1}; -\frac{2\ln(1-0.5) \times 61266.6}{162.7}\right] \\ &\simeq [352.275164; 522.025456]. \end{aligned}$$