Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

2nd. Test ("RECURSO") — 2nd. Semester — **2012/13**
Duration: **1h30m** — **2013/06/28** — **8:15PM**, Room P8

- Please justify your answers.
- This test has **one page** and **four questions**. The total of points is **20.0**.

1. *Elaborate on the adoption and the impact of quality control methods in the Japanese Industry in the 2nd. half of the past century.* (1.0)

- **Adoption and impact of quality control methods in the Japanese Industry in the 2nd. half of the XX century**
  At first, Japan had a widely held reputation for shoddy exports, and their goods were shunned by international markets. This led Japanese organizations to explore new ways of thinking about quality.
  The Japanese welcomed input from foreign companies and lecturers, including two American quality experts:
  - W. Edwards Deming, who had become frustrated with American managers when most programs for statistical quality control were terminated once the war and government contracts came to and end;
  - Joseph M. Juran, who predicted the quality of Japanese goods would overtake the quality of goods produced in the United States by the mid-1970s because of Japans revolutionary rate of quality improvement.

  Rather than relying purely on product inspection, Japanese manufacturers focused on improving all organizational processes through the people who used them. As a result, Japan was able to produce higher-quality exports at lower prices, benefiting consumers throughout the world.

  (http://asq.org/learn-about-quality/history-of-quality/overview/total-quality.html)

2. *The expected number of errors (slips, lapses, small mistakes, mistakes), detected in a monthly sample of $n = 100$ documents from an accounting firm, is monitored by means of an upper one-sided $c-$chart with target value and upper control limit equal to $\lambda_0 = 10$ and $UCL = \lambda_0 + 3\sqrt{\lambda_0}$, respectively.*

(a) *A monthly sample of 100 documents is observed with a total of 15 errors.* (1.0)
Does this sample suggest that the process is out-of-control?

- **Control statistic of the $c-$chart**
  $T_N =$ number of errors detected in a monthly sample of $n = 100$ documents, $N \in \mathbb{N}$
- **Control limits of the upper one-sided $p-$chart**
$$\begin{aligned} LCL &= 0 \\ UCL &= \lambda_0 + 3 \times \sqrt{\lambda_0} \\ &= 10 + 3 \times \sqrt{10} \\ &\simeq 19.4868 \end{aligned}$$
- **Comment**
  Since $t = 15 > UCL = 19.4868$, the observation suggests that the process is in-control.

(b) *Obtain and comment the value of the in-control ARL of this chart.* (3.0)
Calculate the probability that the detection of a shift from λ_0 to $\lambda = 15$ occurs within the first 3 months.

- **Control statistic of the $c-$chart**
  $T_N =$ number of errors detected in a monthly sample of $n = 100$ documents, $N \in \mathbb{N}$
- **Relevant distributions**
  IN-CONTROL: $T_N \sim \text{Poisson}(\lambda_0)$, where $\lambda_0 = 10$
  OUT-OF-CONTROL: $T_N \sim \text{Poisson}(\lambda_0 + \theta)$, where θ ($\theta > 0$) represents the magnitude of an upward shift in the parameter
- **Run length**
  We are dealing with a Shewhart chart therefore the number of samples collected until the chart triggers a signal given θ is represented by $RL(\theta)$ and
$$RL(\theta) \sim \text{Geometric}\left(\xi(\theta)\right).$$
- **Probability of a signal**
$$\begin{aligned} \xi(\theta) &= P(T_N \notin [LCL, UCL] \mid \theta) \\ &\overset{T_N \geq 0,\, LCL=0}{=} P\left(T_N > UCL \mid \theta\right) \\ &= 1 - F_{Poisson(\lambda_0+\theta)}(\lfloor UCL \rfloor) \end{aligned}$$
- **In-control ARL**
$$\begin{aligned} ARL(\theta = 0) &= \frac{1}{\xi(\theta = 0)} \\ &= \frac{1}{1 - F_{Poisson(10+0)}(19)} \\ &= \frac{1}{0.0035} \\ &\simeq 285.714. \end{aligned}$$
- **Comment**
  The average number of samples collected until the chart triggers a false alarm is supposed to be large. 285.714286 samples is an example of a large in-control ARL.
- **Shift**
  $\lambda = 15 = \lambda_0 + \theta$, where $\lambda_0 = 10$ and $\theta = 5$.
- **Requested probability**
$$\begin{aligned} P[RL(\theta) \leq m] &= 1 - \overline{F}_{RL(\theta)}(m) \\ &= 1 - [1 - \xi(\theta)]^m \\ &\overset{m=3, \theta=5, etc.}{=} 1 - \{1 - [1 - F_{Poisson(15)}(19)]\}^3 \\ &= 1 - 0.8752^3 \\ &\simeq 0.329619. \end{aligned}$$

(c) *An upper one-sided CUSUM chart has been recently adopted to detect increases in λ. This chart for Poisson data has been set with no head start, $UCL_C = x = 8$, $\lambda_0 = 10$ and reference value $k = 13$.* (2.0)
Describe how could you obtain the in-control ARL of this CUSUM chart.

- **Upper one-sided CUSUM chart for Poisson data**
  $LCL = 0$
  $UCL_C = x = 8$
  $u = 0$ (no head-start)
  $k = 13$
- **Control statistic**

$$Z_N = \begin{cases} 0, \ N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, \ N \in \mathbb{N} \end{cases}$$

- **Run length**
  It is represented by $RL^u(\theta)$ and has a phase-type distribution!
  Recall that, according to Example/Exercise 10.9 and result (10.8) of the lecture notes, the control statistic is associated to an absorbing Markov chain governed by a transition probability matrix $\mathbf{P}(\theta)$, whose block $\mathbf{Q}(\theta)$ governs the transient states and is obtained by eliminating the last row and the last column of $\mathbf{P}(\theta)$.
- **In-control ARL**
  According to Table 10.3,

$$ARL^0(\theta) = \underline{e}_0^\top \times [\mathbf{I} - \mathbf{Q}(\theta)]^{-1} \times \underline{1},$$

where:

$$\begin{aligned} \underline{e}_0 &= \text{1st. vector of the orthonormal basis of } \mathbb{R}^{x+1}; \\ \mathbf{I} &= \text{identity matrix with rank } (x+1); \\ \underline{1} &= \text{vector of } (x+1) \text{ ones;} \\ \mathbf{Q}(\theta) &= \begin{bmatrix} F_\theta(k) & P_\theta(k+1) & \cdots & P_\theta(k+UCL_C) \\ F_\theta(k-1) & P_\theta(k) & \cdots & P_\theta(k+UCL_C-1) \\ F_\theta(k-2) & P_\theta(k-1) & \cdots & P_\theta(k+UCL_C-2) \\ \vdots & \vdots & \vdots & \vdots \\ F_\theta(k-UCL_C) & P_\theta(k-UCL_C+1) & \cdots & P_\theta(k) \end{bmatrix}, \end{aligned}$$

where F_θ and P_θ represent the c.d.f. and the p.f. of a r.v. with a Poisson($\lambda_0+\theta$) distribution, with $\theta = 0$ because we want to obtain the in-control ARL.

3. *A major cosmetic manufacturer has to monitor the fill weight for bottles of nail polish. The targets of the expected value and variance of this weight are $\mu_0 = 6$ and $\sigma_0^2 = 0.01$, respectively. Assume the readings refer to samples of size n and are taken from the manufacturing process every hour. Furthermore, the shifts in the process mean and standard deviation are represented by $\delta = \sqrt{n}(\mu - \mu_0)/\sigma_0$ ($\delta > 0$) and $\theta = \sigma/\sigma_0$ ($\theta > 1$), respectively.*

   (a) *Assume that an upper one-sided $\bar{X}-$chart for μ with in-control ARL equal to $ARL_\mu(0,1) = 1/\beta = 1000$ samples is used.* (3.0)

   Find the minimum sample size n^* so that a sample triggers a signal with probability of at least 30%, when the process mean shifts from $\mu_0 = 6$ to $\mu = 6.1$.

- **Quality characteristic**
  X = fill weight for bottles of nail polish
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistic**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n

- **Relevant distributions**
  IN-CONTROL: $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, where $\mu_0 = 6$, $\sigma_0^2 = 0.01$ and n to be determined.
  OUT-OF-CONTROL: $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, where $\delta > 0$ (resp. $\theta > 1$) represents the magnitude of a shift in μ (resp. σ).
- **Control limits of the upper one-sided $\bar{X}-$ chart**
  $LCL = 0$
  $UCL = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
- **Probability of triggering a signal**
  Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to:

$$\begin{aligned} \xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL, UCL] \mid \delta, \theta\right) \\ &\overset{LCL=-\infty}{=} 1 - \Phi\left(\frac{UCL - \mu}{\frac{\sigma}{\sqrt{n}}}\right) \\ &= \ldots \\ &= 1 - \Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right), \ \delta > 0, \ \theta > 1. \end{aligned}$$

- **Run length**
  We are dealing with a Shewhart chart, therefore the number of samples collected until the chart triggers a signal given δ and θ, $RL_\mu(\delta,\theta)$, is such that:
  $RL_\mu(\delta,\theta) \sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right).$
- **Obtaining γ_μ**
  The constant γ_μ is such that $ARL_\mu(0,1) = 1/\beta = 1000$, that is,

$$\begin{aligned} \gamma_\mu \ &: \ \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1) \\ & 1 - \Phi(\gamma_\mu) = \frac{1}{ARL_\mu(0,1)} \\ & \gamma_\mu = \Phi^{-1}\left[1 - \frac{1}{ARL_\mu(0,1)}\right] \\ & \gamma_\mu = \Phi^{-1}(0.999) \\ & \gamma_\mu \overset{table}{=} 3.0902. \end{aligned}$$

- **Obtaining $n*$**
  n^* is the smallest positive integer n such that

$$\begin{aligned} P\left(\bar{X}_N > UCL \mid \delta = \frac{6.1-6}{\sqrt{\frac{0.01}{n}}}, \theta = 1\right) &\geq 0.30 \\ 1 - \Phi\left(\frac{\gamma_\mu - \sqrt{n}}{1}\right) &\geq 0.30 \\ \gamma_\mu - \sqrt{n} &\geq \Phi^{-1}(1-0.30) \\ n &\overset{table}{\geq} (3.0902 - 0.5244)^2 \\ n &\overset{table}{\geq} 6.583329, \end{aligned}$$

i.e., $n^* = 7$.

(b) *A statistical consultant recommended an upper one-sided S^2-chart to detect increases in the process* (2.5) *variance. Its upper control limit is equal to $UCL = \sigma_0^2 \times 22.46/(n-1)$, where $n = 7$.*

Obtain and interpret the median of the in-control RL of this chart. **Note**: $F_{\chi^2_{(4)}}(15.3701) = 0.996008$.

- **Control statistic**
  S_N^2 = variance of the N^{th} random sample of size n, $N \in I\!N$
- **Relevant distributions**
  IN-CONTROL: $\frac{(n-1)S_N^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$, where $\sigma_0^2 = 0.01$ and $n = 7$.
  OUT-OF-CONTROL: $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, where θ ($\theta > 1$) represents a shift (a decrease or an increase!) in the standard deviation σ.
- **Control limits of the upper one-sided S^2- chart**
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2 \times 22.46}{n-1}$
- **Probability of triggering a signal**

$$\begin{aligned}\xi_\sigma(\theta) &= P\left(S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right)\\ &= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right]\\ &= \dots\\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{22.46}{\theta^2}\right), \theta > 1\end{aligned}$$

- **Run length**
  $RL_\sigma(\theta) \sim \text{Geometric}(\xi_\sigma(\theta))$
- **Median of the in-control run length**
  It is the smallest positive integer me such that:

$$\begin{aligned}P[RL_\sigma(\theta=1) \le m \mid \theta = 1] &\ge 0.5\\ 1 - [1 - \xi_\sigma(\theta=1)]^m &\ge 0.5\\ m &\ge \frac{\ln(1-0.5)}{\ln[1-\xi_\sigma(\theta=1)]}\\ m &\ge 692.801,\end{aligned}$$

  that is, $me = 693$.
- **Comment**
  In the absence of assignable causes the probability of triggering a false alarm within the first 693 samples is of at least 50%.

(c) *Consider now a general simultaneous scheme for μ and σ^2 composed of a $\bar{X}-$ and S^2-chart, such that $ARL_{\mu,\sigma}(0,1) = 1/\alpha$, where α is a known fixed constant, and $ARL_\mu(0,1) = ARL_\sigma(1) = 1/\eta$. Determine η in terms of α.* (2.0)

- **Deriving η in terms of α**
  Since the ARL of a Shewhart chart is the reciprocal of the probability of triggering a signal, we successively get:

$$\eta \in (0,1) \;:\; \begin{cases} ARL_\mu(0,1) = ARL_\sigma(1) = \frac{1}{\eta}\\ ARL_{\mu,\sigma}(0,1) = \frac{1}{\alpha}\end{cases}$$

$$\begin{cases}\xi_\mu(0,1) = \xi_\sigma(1) = \eta\\ \xi_{\mu,\sigma}(0,1) = \alpha\end{cases}$$

$$\begin{cases}\xi_\mu(0,1) = \xi_\sigma(1) = \eta\\ \xi_\mu(0,1) + \xi_\sigma(1) - \xi_\mu(0,1) \times \xi_\sigma(1) = \alpha\end{cases}$$

$$\begin{cases} -\\ \eta + \eta - \eta \times \eta = \alpha\end{cases}$$

$$\begin{cases} -\\ \eta^2 - 2\eta + \alpha = 0\end{cases}$$

$$\begin{cases} -\\ \eta = \frac{2 \pm \sqrt{4 - 4\alpha}}{2}\end{cases}$$

$$\begin{cases} -\\ \eta = 1 \pm \sqrt{1-\alpha},\end{cases}$$

  I.e., the only root in (0,1) is $\eta^* = 1 - \sqrt{1-\alpha}$.

4. *The cosmetic manufacturer has established an upper specification on the contents of the bottles of nail polish at 6.1 and sends lots of 5000 bottles after having screened them using a (Dodge-Romig) double sampling plan with $n_1 = 34$, $c_1 = 0$; $n_2 = 41$, $c_2 = 2$.*

   (a) *What is the average sample number (ASN) when the lots contain 2% of nonconforming bottles?* (1.5)

   - **Double sampling plan for attributes without rectifying inspection**
     $N = 5000$
     $n_1 = 34$ and $n_2 = 41$ (sample sizes)
     $c_1 = 0$, $c_2 = 2$ (acceptance numbers)
   - **Auxiliary r.v. and their approximate distributions**
     D_i = number of defective units in the i^{th} sample $\overset{a}{\sim}$ Binomial(n_i, p), $i = 1, 2$
   - **Average sample number (ASN)**

$$\begin{aligned}ASN(p) &\overset{(13.21)}{=} n_1 + n_2 \times P(c_1 < D_1 \le c_2)\\ &\simeq n_1 + n_2 \times \sum_{k=c_1+1}^{c_2} P_{Binomial(n_1,p)}(k)\\ &= n_1 + n_2 \times \sum_{k=c_1+1}^{c_2} \binom{n_1}{k} p^k (1-p)^{n_1-k}\\ &\overset{p=0.02,\,etc.}{=} 34 + 41 \times \left[\binom{34}{1} 0.02^1 (1-0.02)^{34-1} + \binom{34}{2} 0.02^2 (1-0.02)^{34-2}\right]\\ &\simeq 34 + 41 \times (0.349116 + 0.117559)\\ &= 34 + 41 \times 0.466675\\ &= 53.133675.\end{aligned}$$

   (b) *Now, admit the rejected lots are screened and all nonconforming bottles of nail polish reworked and returned to the lot.* (2.0)
   Calculate the average total inspection (ATI) when $p = 0.02$.

   - **Important**
     Now, the double sampling plan for attributes has *rectifying inspection*.
   - **Probability of accepting the lot in the first stage of the sampling plan**

$$P_a^I(p) \overset{(13.16)}{=} P(D_1 \le c_1)$$

$$\simeq \quad F_{Binomial(n_1,p)}(c_1)$$
$$\overset{p=0.02,\,etc.}{=} \quad F_{Binomial(34,0.02)}(0)$$
$$= \quad \binom{34}{0} 0.02^0 (1-0.02)^{34-0}$$
$$\simeq \quad 0.503137$$

- **Probability of accepting the lot in the second stage of the sampling plan**

$$P_a^{II}(p) \overset{(13.17)}{=} P(c_1 < D_1 \leq c_2,\ D_1 + D_2 \leq c_2)$$
$$= \sum_{k=c_1+1}^{c_2} P(D_1 = k) \times P(D_2 \leq c_2 - k)$$
$$\simeq \sum_{k=c_1+1}^{c_2} P_{Binomial(n_1,p)}(k) \times F_{Binomial(n_2,p)}(c_2 - k)$$
$$\overset{p=0.02,\,etc.}{=} \sum_{k=1}^{2} \left\{ \binom{34}{k} 0.02^1 (1-0.02)^{34-1} \times \left[\sum_{j=0}^{2-k} \binom{41}{j} 0.02^j (1-0.02)^{41-j} \right] \right\}$$
$$\ldots$$
$$\simeq 0.331430$$

- **Probability of accepting the lot in the double sampling plan**

$$P_a(p) \overset{(13.18)}{=} P_a^I(p) + P_a^{II}(p)$$
$$\simeq 0.503137 + 0.331430$$
$$= 0.834567$$

- **Average total inspection (ATI)**

$$ATI(p) \overset{(13.27)}{=} n_1 \times P_a^I(p) + (n_1 + n_2) \times P_a^{II}(p) + N \times [1 - P_a(p)]$$
$$\simeq 34 \times 0.503137 + (34 + 41) \times 0.331430 + 5000 \times (1 - 0.834567)$$
$$\simeq 869.129.$$

(c) *The statistical consultant suggested the adoption a single sampling plan, with risk points $(p_1, 1-\alpha) = (1\%, 0.98)$ and $(p_2, \beta) = (10\%, 0.1)$ and without rectifying inspection.* (2.0)
Define an appropriate sampling plan and compare it with the one in (a), in terms of the acceptance of incoming lots containing 2% of nonconforming items.

- **Producer's and consumer's risk points**
$(p_1, 1-\alpha) = (1\%, 0.98)$
$(p_2, \beta) = (10\%, 0.1)$

- **Obtaining the acceptance number and sample size**
According to Wetherill and Brown (1991), the acceptance number c and sample size n of a sampling plan for attributes, associated to risk points $(p_1, 1-\alpha)$ and (p_2, β), can be approximately obtained:[1]
  - c should be taken as the smallest integer satisfying
  $$r(c) \leq \frac{p_2}{p_1},$$
  where $r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$;

[1] See page 129 of the lecture notes, in particular, formulae (13.11), (13.10) and (13.12).

  - n should be taken as the smallest integer satisfying
  $$\frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{2p_2} \leq n \leq \frac{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}{2p_1},$$
  namely the ceiling of the lower bound above,
  $$\left\lceil \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{2p_2} \right\rceil.$$

Using the tables to determine $F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta = 0.90)$ and Mathematica to obtain $F^{-1}_{\chi^2_{(2(c+1))}}(\alpha = 0.02)$, we get

c	$r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$	Is $r(c) \leq \frac{p_2}{p_1} = 10$?
0	$\frac{F^{-1}_{\chi^2_{(2(0+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(0+1))}}(0.02)} = \frac{4.605}{0.0404054} \simeq 113.974$	NO!
1	$\frac{F^{-1}_{\chi^2_{(2(1+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(1+1))}}(0.02)} = \frac{7.779}{0.429398} \simeq 18.117$	NO!
2	$\frac{F^{-1}_{\chi^2_{(2(1+1))}}(1-0.1)}{F^{-1}_{\chi^2_{(2(1+1))}}(0.02)} = \frac{10.64}{1.13442} \simeq 9.383$	**YES!**

Consequently, $c = 2$. Moreover,

$$n = \left\lceil \frac{F^{-1}_{\chi^2_{(2(2+1))}}(1-0.1)}{2 \times 0.1} \right\rceil$$
$$\overset{table}{=} \left\lceil \frac{10.64}{2 \times 0.1} \right\rceil$$
$$= \lceil 53.2 \rceil$$
$$= 54.$$

- **Obs.**
The lot should be accepted iff the number of defective bottles of nail polish, in a sample of $n = 54$ bottles, does not exceed $c = 2$.

- **Comparing the two sampling plans**
Since

$$P_a^{simple}(0.02) = F_{Binomial(54,0.02)}(2)$$
$$= 0.98^{54} + 5 \times 0.02 \times 0.98^{53} + \frac{54 times 53}{2} \times 0.02^2 \times 0.92^52$$
$$\simeq 0.906268$$
$$> P_a^{double}(0.02)$$
$$\simeq 0.834587$$

and $AQL = p_1 = 0.01 < 0.02 < 0.10 = p_2 = LTPD$, the single sampling plan seem to be more reasonable in terms of the acceptance of incoming lots containing 2% of nonconforming items.