Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

2nd. Test 2nd. Semester — **2012/13**

Duration: **1h30m** **2013/06/07** — **8AM**, Room P12

- Please justify your answers.
- This test has **one page** and **four questions**. The total of points is **20.0**.

1. *Elaborate on two major contributions of Walter A. Shewhart in the area of Statistical Process Control.* (1.0)

 - **Two of Shewhart's major contributions in SPC**
   - Shewhart recognized that industrial processes yield data and determined this data could be analyzed using statistical techniques to see whether a process is stable and in control, or if it is being affected by special causes that should be fixed.
   - He is known as the father of modern quality control essentially because he invented the control chart, a simple but highly effective statistical tool used to devise if the process is stable or if it is being affected by special causes responsible for changes in the process parameters, such as the process mean or variance.

2. *A busy traffic intersection is monitored using an upper one-sided $p-$chart to detect increases in the percentage of cars illegally passing through a red light in samples of n cars. Consider that the target value and upper control limit of this chart are equal to $p_0 = 0.07$ and $UCL = p_0 + 3 \times \sqrt{p_0(1-p_0)/n}$.*

 (a) *A sample of 88 cars is observed with 15 cars illegally passing through the intersection.* (1.0)
 Does this sample suggest that the process is out-of-control?

 - **Control statistic of the $p-$chart**
 $T_N =$ fraction of defective items in the N^{th} sample of size n, $N \in \mathbb{N}$
 - **Control limits of the upper one-sided $p-$chart**
$$\begin{aligned} LCL &= 0 \\ UCL &= p_0 + 3 \times \sqrt{\frac{p_0(1-p_0)}{n}} \\ &= 0.07 + 3 \times \sqrt{\frac{0.07(1-0.07)}{88}} \\ &= 0.151596 \end{aligned}$$
 - **Comment**
 Since $t = \frac{15}{88} \simeq 0.170455 > UCL = 0.151596$, the observation suggests that the process is out-of-control.

 (b) *Admit that: the probability that a car illegally passes through the intersection has shifted from $p_0 = 0.07$ to $p = 0.15$; now, all samples have the same size $n = 10$.* (2.0)
 Calculate the probability that the detection of this shift requires the collection of at least 3 samples of size $n = 10$.

 - **Control statistic of the $p-$chart**
 $T_N =$ fraction of defective items in the N^{th} sample of size n, $N \in \mathbb{N}$
 - **Relevant distributions**
 In-control: $Y_N = n \times T_N \sim \text{Binomial}(n, p_0)$, where $p_0 = 0.07$
 Out-of-control: $Y_N \sim \text{Binomial}(n, p = p_0 + \theta)$, where θ ($\theta > 0$) represents the magnitude of an upward shift in the parameter
 - **Control limits of the upper one-sided $p-$chart**
$$\begin{aligned} LCL &= 0 \\ UCL &= p_0 + 3 \times \sqrt{\frac{p_0(1-p_0)}{n}} \\ &\stackrel{n=10}{=} 0.07 + 3 \times \sqrt{\frac{0.07(1-0.07)}{10}} \\ &= 0.312054 \end{aligned}$$
 - **Run length**
 We are dealing with a Shewhart chart therefore the number of samples collected until the chart triggers a signal given θ, $RL(\theta)$, is such that:
$$\begin{aligned} RL(\theta) &\sim \text{Geometric}\left(\xi(\theta)\right); \\ \overline{F}_{RL(\theta)}(m) &= P[RL(\theta) > m] \\ &\stackrel{Table\ 9.2}{=} [1-\xi(\theta)]^m, \ m \in \mathbb{N}. \end{aligned}$$
 - **Probability of triggering a signal**
$$\begin{aligned} \xi(\theta) &= P(T_N \notin [LCL, UCL] \mid \theta) \\ &\stackrel{T_N \geq 0,\, LCL=0}{=} P\left(T_N > UCL \mid \theta\right) \\ &= P\left(Y_N = n \times T_N > n \times UCL \mid \theta\right) \\ &= 1 - F_{Binomal(n, p=p_0+\theta)}(n \times UCL) \\ &= 1 - F_{Binomial(10, 0.15)}(3.12054) \\ &= 1 - F_{Binomial(10, 0.15)}(3) \\ &\stackrel{table}{=} 1 - 0.9500 \\ &= 0.0500. \end{aligned}$$
 - **Requested probability**
$$\begin{aligned} P[RL(\theta) \geq 3] &= \overline{F}_{RL(\theta)}(2) \\ &= [1-\xi(\theta)]^2 \\ &= 0.9500^2 \\ &= 0.9025 \end{aligned}$$

 (c) *An upper one-sided CUSUM chart has been recently adopted to detect increases in the expected of cars illegally passing through that intersection in samples of size $n = 10$. This chart for binomial data has been set with no head start, $UCL_C = x = 1$, $np_0 = 0.7$ and reference value $k = 1$.* (3.0)
 Recompute the probability in (b) for this new chart. Comment the result you just obtained in light with the one from (b).

 - **Upper one-sided CUSUM chart for binomial data**
   - Control limits
   $LCL = 0$
   $UCL = x = 1$
   - Reference value
   $k = 1$

– Initial value of the control statistic
$u = 0$ (no head-start)

– Control statistic

$$Z_N = \begin{cases} 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in \mathbb{N} \end{cases}$$

- **Run length**
  It is represented by $RL^u(\theta)$ and has a phase-type distribution!
- **Requested probability**
  According to Table 10.3,

$$\begin{aligned} P\left[RL^0(\theta) \geq 3\right] &= P\left[RL^0(\theta) > 2\right] \\ &= \underline{e}_0^\top \times [\mathbf{Q}(\theta)]^2 \times \underline{1}, \end{aligned}$$

where:

$$\begin{aligned} \underline{e}_0 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \\ \underline{1} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Moreover, since $n = 10$, $p = p_0 + \theta = 0.15$ and $x = k = 1$, we get from formulas (10.8) and (10.10):

$$\begin{aligned} \mathbf{Q}(\theta) &= \begin{bmatrix} F_{Binomial(10,0.15)}(1) & P_{Binomial(10,0.15)}(1+1) \\ F_{Binomial(10,0.15)}(1-1) & P_{Binomial(10,0.15)}(1) \end{bmatrix} \\ &\overset{table}{=} \begin{bmatrix} 0.5443 & 0.8202 - 0.5443 \\ 0.1969 & 0.5443 - 0.1969 \end{bmatrix} \\ &= \begin{bmatrix} 0.5443 & 0.2759 \\ 0.1969 & 0.3474 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} P\left[RL^0(\theta) \geq 3\right] &= [1 \quad 0] \times \begin{bmatrix} 0.5443 & 0.2759 \\ 0.1969 & 0.3474 \end{bmatrix}^2 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \text{sum of the entries of the 1st. line of } [\mathbf{Q}(\theta)]^2 \\ &\simeq 0.350587 + 0.246020 \\ &= 0.596607. \end{aligned}$$

- **Comment**
  The fact that $P\left[RL^0(\theta) \geq 3\right] = 0.596607 < P[RL(\theta) \geq 3] = 0.9025$ suggests that the upper one-sided CUSUM chart for binomial data is quicker that the upper one-sided $p-$chart in the detection of the shift from 0.07 to 0.15.
- **[Obs.**
  Additional calculations allows us to add that $ARL^0(0) = 20.6619 << ARL(0) = 279.631$, thus the upper one-sided CUSUM chart for binomial data triggers much more false alarms than the upper one-sided $p-$chart and should not be used...]

3. *An automatic screw machine turns out round-head bolts with a normally distributed diameter with target mean and variance $\mu_0 = 9mm$ and $\sigma_0^2 = 4 \times 10^{-4}mm$, respectively. Assume the readings refer to samples of size $n = 4$ and are taken from the manufacturing process every hour. Furthermore, the shifts in process mean and standard deviation are represented by $\delta = \sqrt{n}(\mu - \mu_0)/\sigma_0$ $(\delta \neq 0)$ and $\theta = \sigma/\sigma_0$ $(\theta > 0)$, respectively.*

(a) *Assume that a standard $\bar{X}-$chart for μ with in-control ARL equal to $ARL_\mu(0,1) = 1/\alpha = 500$ samples is used and the process mean shifts from $\mu_0 = 9mm$ to $\mu = 9.02mm$.* (3.0)
Compute the probability that this shift will not be detect by the first sample mean after the occurrence of this change in the process mean.

- **Quality characteristic**
  X = diameter of round-head bolts
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistic**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
- **Relevant distributions**
  IN-CONTROL: $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, where $\mu_0 = 9$, $\sigma_0 = 2 \times 10^{-4}$ and $n = 4$
  OUT-OF-CONTROL: $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, where $\delta \neq 0$ (resp. $\theta > 0$) represents the magnitude of a shift in μ (resp. σ)
- **Control limits of the standard $\bar{X}-$ chart**
  $LCL = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
- **Probability of triggering a signal**
  Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to:

$$\begin{aligned} \xi_\mu(\delta, \theta) &= P\left(\bar{X}_N \notin [LCL, UCL] \mid \delta, \theta\right) \\ &= 1 - \left[\Phi\left(\frac{UCL - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{UCL - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\ &= \ldots \\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \ \delta \neq 0, \ \theta > 0. \end{aligned}$$

- **Run length**
  We are dealing with a Shewhart chart, therefore the number of samples collected until the chart triggers a signal given δ and θ, $RL_\mu(\delta, \theta)$, is such that:

$$\begin{aligned} RL_\mu(\delta, \theta) &\sim \text{Geometric}\left(\xi_\mu(\delta, \theta)\right); \\ P[RL_\mu(\delta, \theta) = m] &= [1 - \xi_\mu(\delta, \theta)]^{m-1} \xi_\mu(\delta, \theta), \ m \in \mathbb{N}. \end{aligned}$$

- **Obtaining γ_μ**
  The constant γ_μ is such that $ARL_\mu(0,1) = 500$, that is,

$$\begin{aligned} \gamma_\mu \ : \ & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1) \\ & 1 - [\Phi(\gamma_\mu) - \Phi(-\gamma_\mu)] = \frac{1}{ARL_\mu(0,1)} \\ & \gamma_\mu = \Phi^{-1}\left[1 - \frac{1}{2 \times ARL_\mu(0,1)}\right] \\ & \gamma_\mu = \Phi^{-1}(0.999) \\ & \gamma_\mu \overset{table}{=} 3.0902 \end{aligned}$$

- **Requested probability**
Since $\delta = \frac{\mu-\mu_0}{\frac{\sigma_0}{\sqrt{n}}} = \frac{9.02-9}{\frac{0.02}{\sqrt{4}}} = 2$ and σ is in control (i.e., $\theta = 1$), we obtain

$$\begin{aligned} P[RL_\mu(2,1) > 1] &= 1 - P[RL_\mu(2,1) = 1] \\ &= 1 - \xi_\mu(2,1) \\ &= \left[\Phi\left(\frac{3.0902-2}{1}\right) - \Phi\left(\frac{-3.0902-2}{1}\right)\right] \\ &\simeq \Phi(1.09) - \Phi(-5.09) \\ &\simeq \Phi(1.09) \\ &\overset{table}{=} 0.8621. \end{aligned}$$

(b) *An engineer believes that both decreases and increases in the process variance should be monitored and decides to use a S^2-chart with control limits: $LCL^*_\sigma = \sigma_0^2 \times 0.0349/(n-1)$ and $UCL^*_\sigma = \sigma_0^2 \times 18.9214/(n-1)$. The out-of-control ARL of this chart when $\theta = 0.9$ equals $ARL^*_\sigma(0.9) = 419.796$. What would be the value of this out-of-control ARL if you the engineer had used a S^2- chart with control limits $LCL_\sigma = \sigma_0^2 \times F^{-1}_{\chi^2_{(n-1)}}(\alpha/2)/(n-1)$ and $UCL_\sigma = \sigma_0^2 \times F^{-1}_{\chi^2_{(n-1)}}(1-\alpha/2)/(n-1)$? Comment.* **Note**: *$F_{\chi^2_{(3)}}(0.03) \simeq 0.001369$ and $F_{\chi^2_{(3)}}(20.08) \simeq 0.999837$.* (2.5)

- **Control statistic**
$S_N^2 =$ variance of the N^{th} random sample of size n, $N \in I\!N$

- **Relevant distributions**
IN-CONTROL: $\frac{(n-1)S_N^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$, where $\sigma_0 = 2 \times 10^{-4}$ and $n = 4$
OUT-OF-CONTROL: $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, where θ ($\theta > 0$) represents a shift (a decrease or an increase!) in the standard deviation σ

- **Control limits of the standard S^2- chart**
$LCL_\sigma = \frac{\sigma_0^2}{n-1} \times F^{-1}_{\chi^2_{(n-1)}}\left(\frac{\alpha}{2}\right)$
$UCL_\sigma = \frac{\sigma_0^2}{n-1} \times F^{-1}_{\chi^2_{(n-1)}}\left(1-\frac{\alpha}{2}\right)$

- **Probability of triggering a signal**

$$\begin{aligned} \xi_\sigma(\theta) &= P\left(S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\ &= 1 - \left\{F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,LCL_\sigma}{\sigma^2}\right]\right\} \\ &= \ldots \\ &= 1 - \left\{F_{\chi^2_{(n-1)}}\left[\frac{F^{-1}_{\chi^2_{(n-1)}}\left(1-\frac{\alpha}{2}\right)}{\theta^2}\right] - F_{\chi^2_{(n-1)}}\left[\frac{F^{-1}_{\chi^2_{(n-1)}}\left(\frac{\alpha}{2}\right)}{\theta^2}\right]\right\}, \ \theta > 0 \end{aligned}$$

- **Run length**
$RL_\sigma(\theta) \sim \text{Geometric}\left(\xi_\sigma(\theta)\right)$

- **Requested ARL**

$$\begin{aligned} ARL_\sigma(\theta) &= \frac{1}{\xi_\sigma(\theta)} \\ &\overset{n=4,\alpha=500^{-1}}{=} \frac{1}{1 - \left\{F_{\chi^2_{(3)}}\left[\frac{F^{-1}_{\chi^2_{(3)}}(0.999)}{\theta^2}\right] - F_{\chi^2_{(3)}}\left[\frac{F^{-1}_{\chi^2_{(3)}}(0.001)}{\theta^2}\right]\right\}} \\ &\overset{table,\theta=0.9}{=} \frac{1}{1 - \left\{F_{\chi^2_{(3)}}\left(\frac{16.2662}{0.9^2}\right) - F_{\chi^2_{(3)}}\left(\frac{0.0243}{0.9^2}\right)\right\}} \\ &\simeq \frac{1}{1 - \left[F_{\chi^2_{(3)}}(20.08) - F_{\chi^2_{(3)}}(0.03)\right]} \\ &\simeq \frac{1}{1-(0.999837-0.001369)} \\ &\simeq 652.742 \end{aligned}$$

- **Comment**
$ARL_\sigma(0.9) = 652.742 > ARL^*_\sigma(0.9) = 419.796$, thus the first S^2-chart gives a better protection against a decrease of 10% in the process standard deviation. [Since $ARL^*_\sigma(0.9) = 419.796 < ARL^*_\sigma(1) = 500 = ARL_\sigma(1) < ARL_\sigma(0.9) = 652.742$, we can add that the first chart takes, in average, less time to signal a 10% reduction in σ than to trigger a false alarm, unlike the second chart.]

(c) *Consider now a general simultaneous scheme for μ and σ^2 composed of a $\bar{X}-$ and S^2-chart, with ARL functions $ARL_\mu(\delta,\theta)$ and $ARL_\sigma(\theta)$, respectively.* (2.0)
Prove that the ARL of the simultaneous scheme can be written as follows: $ARL_{\mu,\sigma}(\delta,\theta) = \frac{ARL_\mu(\delta,\theta)\times ARL_\sigma(\theta)}{ARL_\mu(\delta,\theta)+ARL_\sigma(\theta)-1}$.

- **Run length of the simultaneous scheme for μ and σ^2**

$$\begin{aligned} RL_{\mu,\sigma}(\delta,\theta) &= \min\{RL_\mu(\delta,\theta), RL_\sigma(\theta)\} \\ &\sim \text{Geometric}(\xi_{\mu,\sigma}(\delta,\theta)) \end{aligned}$$

- **Probability of triggering a signal**
$\xi_{\mu,\sigma}(\delta,\theta) = \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)$

- **Proof**
The ARL of the simultaneous scheme for μ and σ^2 can be written in terms of the ARL of the individual charts $ARL_\mu(\delta,\theta) = \frac{1}{\xi_\mu(\delta,\theta)}$ and $ARL_\sigma(\theta) = \frac{1}{\xi_\sigma(\theta)}$:

$$\begin{aligned} ARL_{\mu,\sigma}(\delta,\theta) &= \frac{1}{\xi_{\mu,\sigma}(\delta,\theta)} \\ &= \frac{1}{\xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta)\times\xi_\sigma(\theta)} \\ \frac{1}{ARL_{\mu,\sigma}(\delta,\theta)} &= \frac{1}{ARL_\mu(\delta,\theta)} + \frac{1}{ARL_\sigma(\theta)} - \frac{1}{ARL_\mu(\delta,\theta)} \times \frac{1}{ARL_\sigma(\theta)} \\ \frac{1}{ARL_{\mu,\sigma}(\delta,\theta)} &= \frac{ARL_\sigma(\theta) + ARL_\mu(\delta,\theta) - 1}{ARL_\mu(\delta,\theta)\times ARL_\sigma(\theta)} \\ ARL_{\mu,\sigma}(\delta,\theta) &= \frac{ARL_\mu(\delta,\theta)\times ARL_\sigma(\theta)}{ARL_\mu(\delta,\theta) + ARL_\sigma(\theta) - 1}. \end{aligned}$$

4. A company receives lots of 500 items from a certain manufacturer and uses the following double sampling plan for receiving inspection: $n_1 = 20$, $c_1 = 0$; $n_2 = 20$, $c_2 = 2$. Furthermore, rejected lots are screened and all defective items reworked and returned to the lot.
Admit incoming lots contain 3% nonconforming items and compute the approximate probabilities necessary to answer the following questions:

(a) What is the total probability of acceptance of a lot? (2.0)

- **Double sampling plan for attributes with rectifying inspection**
$n_1 = n_2 = 20$ (sample sizes)
$c_1 = 0$, $c_2 = 2$ (acceptance numbers)

- **Auxiliary r.v. and their approximate distributions**
  D_i = number of defective units in the i^{th} sample $\stackrel{a}{\sim}$ Binomial(n_i, p), $i = 1, 2$
- **Probability of accepting the lot in the first stage of the sampling plan**

$$\begin{aligned} P_a^I(p) &\stackrel{(13.16)}{=} P(D_1 \leq c_1) \\ &\simeq F_{Binomial(n_1,p)}(c_1) \\ &\stackrel{p=0.03}{=} F_{Binomial(20,0.03)}(0) \\ &\stackrel{table}{=} 0.5438 \end{aligned}$$

- **Probability of accepting the lot in the second stage of the sampling plan**

$$\begin{aligned} P_a^{II}(p) &\stackrel{(13.17)}{=} P(c_1 < D_1 \leq c_2,\ D_1 + D_2 \leq c_2) \\ &= \sum_{k=c_1+1}^{c_2} P(D_1 = k) \times P(D_2 \leq c_2 - k) \\ &\simeq \sum_{k=c_1+1}^{c_2} P_{Binomial(n_1,p)}(k) \times F_{Binomial(n_2,p)}(c_2 - k) \\ &\stackrel{p=0.03}{=} \sum_{k=1}^{2} [F_{Binomial(20,0.03)}(k) - F_{Binomial(20,0.03)}(k-1)] \times F_{Binomial(20,0.03)}(2-k) \\ &= (0.8802 - 0.5438) \times 0.8802 + (0.9790 - 0.8802) \times 0.5438 \\ &= 0.296100 + 0.053727 \\ &\simeq 0.349827 \end{aligned}$$

- **Probability of accepting the lot in the double sampling plan**

$$\begin{aligned} P_a(p) &\stackrel{(13.18)}{=} P_a^I(p) + P_a^{II}(p) \\ &\simeq 0.5438 + 0.349827 \\ &= 0.893627. \end{aligned}$$

(b) *Calculate the average outgoing quality (AOQ). Comment.* (1.0)

- **Average outgoing quality of a double sampling plan with rectifying inspection**

$$\begin{aligned} AOQ(p) &\stackrel{(13.26)}{=} \frac{p[(N-n_1)\,P_a^I(p) + (N-n_1-n_2)\,P_a^{II}(p)]}{N} \\ &\stackrel{(a)}{=} \frac{0.03 \times [(500-20) \times 0.5438 + (500-20-20) \times 0.349827]}{500} \\ &\simeq 0.025317. \end{aligned}$$

- **Relative reduction in the percentage defective due to the rectifying inspection**

$$\begin{aligned} \left[1 - \frac{AOQ(p)}{p}\right] \times 100\% &= \left(1 - \frac{0.025317}{0.03}\right) \times 100\% \\ &\simeq 15.61\% \end{aligned}$$

- **Comment**
  The relative reduction in the percentage defective is moderate, thus, rectifying inspection is worth doing when $p = 0.03$.

(c) *Now, admit that the company decided to adopt instead a sampling plan by* VARIABLES *with an upper specification limit (U) and known standard deviation σ.* (2.5)

Set such a plan with risk points $(p_1, 1-\alpha) = (1\%, 0.99)$ and $(p_2, \beta) = (15\%, 0.075)$. Compare this sampling plan with the one in (a), in terms of the acceptance of incoming lots contain 3% nonconforming items.

- **Single sampling plan by** VARIABLES **with known variance**
  n_σ (sample size)
  k_σ (acceptance constant)
- **Producer's and consumer's risk points**
  $(p_1, 1-\alpha) = (1\%, 0.99)$
  $(p_2, \beta) = (15\%, 0.075)$
- **Obtaining n_σ and k_σ**
  According to (13.32),

$$(n_\sigma, k_\sigma) \ : \ \begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(1-\alpha) - \Phi^{-1}(\beta)}{\Phi^{-1}(p_2) - \Phi^{-1}(p_1)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(p_2)\Phi^{-1}(1-\alpha) - \Phi^{-1}(p_1)\Phi^{-1}(\beta)}{\Phi^{-1}(\beta) - \Phi^{-1}(1-\alpha)}. \end{cases}$$

$$\begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(0.99) - \Phi^{-1}(0.075)}{\Phi^{-1}(0.15) - \Phi^{-1}(0.01)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(0.15)\Phi^{-1}(0.99) - \Phi^{-1}(0.01)\Phi^{-1}(0.075)}{\Phi^{-1}(0.075) - \Phi^{-1}(0.99)}. \end{cases}$$

$$\begin{cases} n_\sigma \stackrel{table}{=} \left[\frac{2.3263 - (-1.4395)}{(-1.0364) - (-2.3263)}\right]^2 = 8.52319 \\ k_\sigma \stackrel{table}{=} \frac{(-1.0364) \times 2.3263 - (-2.3263) \times (-1.4395)}{(-1.4395) - 2.3263} = 1.52947. \end{cases}$$

  We shall take $n_\sigma = \lceil 8.52319 \rceil = 9$ and $k_\sigma = 1.52947$. [In this case we have

$$\begin{aligned} P_a(p_1) &= \Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_1)\right]\right\} \\ &= \Phi\left\{\sqrt{9}\left[-1.52947 - (-2.3263)\right]\right\} \\ &\simeq \Phi(2.39) \\ &\stackrel{table}{=} 0.9916 \geq 1 - \alpha = 0.99 \\ P_a(p_2) &= \Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_2)\right]\right\} \\ &= \Phi\left\{\sqrt{9}\left[-1.52947 - (-1.0364)\right]\right\} \\ &\simeq \Phi(-1.48) \\ &\stackrel{table}{=} 1 - 0.9306 \\ &= 0.0694 \leq \beta = 0.075.] \end{aligned}$$

- **Probability of accepting the lot in the sampling plan by** VARIABLES **with known variance**

$$\begin{aligned} P_a(p) &= \Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p)\right]\right\} \\ &\stackrel{p=0.03}{=} \Phi\left\{\sqrt{9}\left[-1.52947 - (-1.8808)\right]\right\} \\ &\simeq \Phi(1.05) \\ &\stackrel{table}{=} 0.8531 \end{aligned}$$

- **Comment**
  Since the probability of acceptance of incoming lots contain 3% nonconforming items is larger for the double sampling plan for attributes — and this sampling plan is more sophisticated and requires more observations than the (single!) sampling plan by VARIABLES — we should favor this last sampling plan.