Department of Mathematics, IST — Probability and Statistics Unit

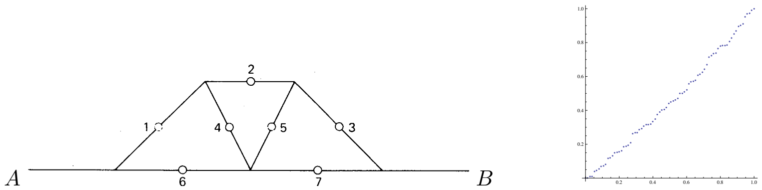
Reliability and Quality Control

t. Test 2nd. Semester — **2012/13**

ıration: **1h30m** **2013/05/04** — **8AM**, Room P12

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. *Consider 7 pipes of a domestic central heating system numbered and set as pictured on the left. Moreover, admit pipes are subjected to obstruction and the system only operates if water flows from A to B.*



Pipes of a domestic central heating system (left, question 1); TTT plot (right, question 3).

(a) *Find the minimal path sets and minimal cut sets of this system, and provide an expression (do not simplify it!) for the structure function of the domestic central heating system.* (2

- **Minimal path sets**

$$\begin{aligned}
\mathcal{P}_1 &= \{6,7\} \\
\mathcal{P}_2 &= \{1,2,3\} \\
\mathcal{P}_3 &= \{1,4,7\} \\
\mathcal{P}_4 &= \{6,5,3\} = \{3,5,6\} \\
\mathcal{P}_5 &= \{1,2,5,7\} \\
\mathcal{P}_6 &= \{1,4,5,3\} = \{1,3,4,5\} \\
\mathcal{P}_7 &= \{6,4,2,3\} = \{2,3,4,6\} \\
p^* &= 7 \text{ minimal path sets}
\end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned}
\mathcal{K}_1 &= \{1,6\} \\
\mathcal{K}_2 &= \{3,7\} \\
\mathcal{K}_3 &= \{2,4,6\} \\
\mathcal{K}_4 &= \{2,5,7\} \\
\mathcal{K}_5 &= \{1,4,5,7\} \\
\mathcal{K}_6 &= \{6,4,5,3\} = \{3,4,5,6\} \\
q &= 6 \text{ minimal cut sets}
\end{aligned}$$

- **Structure function**

$$\begin{aligned}
\phi(\underline{X}) &\stackrel{Th.1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathcal{P}_j} X_i\right) \\
&= 1 - (1-X_6X_7)\times(1-X_1X_2X_3)\times(1-X_1X_4X_7)\times(1-X_3X_5X_6) \\
&\quad \times(1-X_1X_2X_5X_7)\times(1-X_1X_3X_4X_5)\times(1-X_2X_3X_4X_6).
\end{aligned}$$

Obs. — Equivalently,

$$\begin{aligned}
\phi(\underline{X}) &\stackrel{Th.1.30}{=} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathcal{K}_j}(1-X_i)\right] \\
&= [1-(1-X_1)(1-X_6)]\times[1-(1-X_3)(1-X_7)] \\
&\quad \times[1-(1-X_2)(1-X_4)(1-X_6)]\times[1-(1-X_2)(1-X_5)(1-X_7)] \\
&\quad \times[1-(1-X_1)(1-X_4)(1-X_5)(1-X_7)] \\
&\quad \times[1-(1-X_3)(1-X_4)(1-X_5)(1-X_6)].
\end{aligned}$$

(b) *Admit that each of those 7 pipes are unobstructed with probabilities $p_i = 0.95$, $i = 1,\ldots,7$, and operate in a positively associated fashion. Obtain a lower and an upper bound (as strict as reasonably possible!) for the reliability of this system.* (2

- **Reliability of the components**

  $p_i = p = 0.95$, $i = 1,\ldots,7$

- **Important**

  Since the 7 components form a coherent system and operate in a positively associated fashion, we can apply Theorem 1.70, in particular result (1.40) (min-max bounds!).

- **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned}
r(\underline{p}) &\stackrel{Th.1.70}{\geq} \max_{j=1,\ldots,p^*}\left\{\prod_{i\in\mathcal{P}_j} p_i\right\} \\
&\stackrel{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\
&\stackrel{p\in(0,1)}{=} p^{\min_{j=1,\ldots,p^*}\#\mathcal{P}_j} \\
&= p^2 \\
&\stackrel{p=0.95}{=} 0.9025.
\end{aligned}$$

- **Upper bound for the reliability** $r(\underline{p})$

$$\begin{aligned}
r(\underline{p}) &\stackrel{Th.,1.70}{\leq} \min_{j=1,\ldots,q}\left[1 - \prod_{i\in\mathcal{K}_j}(1-p_i)\right] \\
&\stackrel{p_i=p}{=} \min_{j=1,\ldots,q}\left[1-(1-p)^{\#\mathcal{K}_j}\right] \\
&\stackrel{p\in(0,1)}{=} p^{\min_{j=1,\ldots,q}\#\mathcal{K}_j} \\
&= 1-(1-p)^2 \\
&\stackrel{p=0.95}{=} 0.9975.
\end{aligned}$$

(c) *Assume now that the times to obstruction (in 10^2 days) of each of those 7 components are positively associated random variables with common Gamma distribution with shape parameter $\alpha = 3$ and unitary scale parameter. Determine a lower and an upper bound (as strict as reasonably possible!) for the reliability function of the domestic central heating system for a period of 532 days.* (2

Note: $F_{Gamma(\alpha,\lambda)}(x) = F_{\chi^2_{(2\alpha)}}(2\lambda x)$.

- **Individual durations, common distribution and duration of the system**

  T_i = time to obstruction of pipe i $i = 1, \ldots, 7$

  T_i, $i = 1, \ldots, 7$, positively associated r.v.

  $T_i \sim \text{Gamma}(\alpha = 3, \lambda = 1)$

  $R_i(t) = R(t) = 1 - F_{\chi^2_{(2\alpha)}}(2\lambda t) = 1 - F_{\chi^2_{(6)}}(2t)$, $t \geq 0$

  T = duration of the system

- **Important**

  Under these circumstances we can apply Theorem 2.22 to provide a lower and an upper bound for the reliability function $R_T(t)$.

- **Lower bound for the reliability function** $R_T(t)$

$$
\begin{aligned}
R_T(t) &\overset{T2.22}{\geq} \max_{j=1,\ldots,p^*}\left[\prod_{i\in\mathcal{P}_j} R_i(t)\right] \\
&\overset{R_i(t)=R(t)}{=} \max_{j=1,\ldots,p^*}[R(t)]^{\#\mathcal{P}_j} \\
&= [R(t)]^{\min_{j=1,\ldots,p^*}\#\mathcal{P}_j} \\
&= [R(t)]^2 \\
&= \left[1 - F_{\chi^2_{(6)}}(2t)\right]^2 \\
&\overset{t=5.32}{=} \left[1 - F_{\chi^2_{(6)}}(10.64)\right]^2 \\
&\overset{table}{=} (1-0.90)^2 \\
&\simeq 0.01.
\end{aligned}
$$

- **Upper bound for the reliability function** $R_T(t)$

$$
\begin{aligned}
R_T(t) &\overset{T2.22}{\leq} \min_{j=1,\ldots,q}\left\{1 - \prod_{i\in\mathcal{K}_j}[1 - R_i(t)]\right\} \\
&\overset{R_i(t)=R(t)}{=} \min_{j=1,\ldots,q}\left\{1 - [1 - R(t)]^{\#\mathcal{K}_j}\right\} \\
&= 1 - [1 - R(t)]^{\min_{j=1,\ldots,q}\#\mathcal{K}_j} \\
&= 1 - [1 - R(t)]^2 \\
&= 1 - \left[F_{\chi^2_{(6)}}(2t)\right]^2 \\
&\overset{t=5.32}{=} 1 - \left[F_{\chi^2_{(6)}}(10.64)\right]^2 \\
&\overset{table}{=} 1 - 0.90^2 \\
&\simeq 0.19.
\end{aligned}
$$

2. *Admit that the times to failure (in 10^4 days) of n in-line repeater amplifiers[1] are i.i.d. random variables with common c.d.f. $F(t) = 1 - (t+1)^{-\theta}$, for $t \geq 0$ ($\theta > 1$).*

(a) *Admit that those n in-line repeater amplifiers were put to use at the same time. Derive the reliability and hazard rate functions of the time until the first failure.* (2

- **Individual times to failure and common c.d.f.**

  T_i = time to failure of the i^{th} in-line repeater amplifier

  $T_i \overset{i.i.d.}{\sim} T$, $i = 1, \ldots, n$, where T has c.d.f. given by

$$
F(t) = \begin{cases} 0, & t < 0 \\ 1 - (t+1)^{-\theta}, & t \geq 0 \quad (\theta > 1) \end{cases}
$$

- **New r.v.**

  $T_{(1)} = \min_{i=1,\ldots,n} T_i$ = time to the 1st. failure (of those n in-line repeater amplifiers)

- **Reliability function of** $T_{(1)}$

$$
\begin{aligned}
R_{T_{(1)}}(t) &\overset{(2.3)}{=} [R(t)]^n \\
&= [1 - F(t)]^n \\
&= (t+1)^{-n\theta}, \; t \geq 0
\end{aligned}
$$

- **P.d.f. of** $T_{(1)}$

$$
\begin{aligned}
f_{T_{(1)}}(t) &= -\frac{dR_{T_{(1)}}(t)}{dt} \\
&= -\frac{d(t+1)^{-n\theta}}{dt} \\
&= n\theta(t+1)^{-n\theta-1}, \; t \geq 0
\end{aligned}
$$

- **Hazard rate function of** $T_{(1)}$

$$
\begin{aligned}
\lambda_{T_{(1)}}(t) &= \frac{f_{T_{(1)}}(t)}{R_{T_{(1)}}(t)} \\
&= \frac{n\theta(t+1)^{-n\theta-1}}{(t+1)^{-n\theta}} \\
&= \frac{n\theta}{t+1}, \; t \geq 0.
\end{aligned}
$$

- **Obs.** — $T_{(1)}$ belongs to the same family of distributions as the T_i.

(b) *What can be said about the stochastic ageing character of the times to failure T_i and of the time between consecutive failures $T_{(i)} - T_{(i-1)}$? Is the p.d.f. of $T_{(i)} - T_{(i-1)}$ a monotone function? (Do not derive the p.d.f. of $T_{(i)} - T_{(i-1)}$ in any case!)* (3

- **Devising the stochastic ageing character of** T_i

  Since

$$
\lambda_{T_{(1)}}(t) \overset{(3.8)}{=} n\,\lambda_T(t)
$$

  we get

$$
\lambda_T(t) = \frac{\theta}{t+1}, \; t \geq 0,
$$

[1] *In-line repeater amplifiers were used in submarine communications cables.*

which is a decreasing function of t.[2] Consequently,

$$T_i \in DHR,\ i = 1, \ldots, n.$$

- **Devising the stochastic ageing character of $T_{(i)} - T_{(i-1)}$**

  $T_{(i)} - T_{(i-1)} =$ time between consecutive failures

  According to Prop. 3.26 (eq. (3.17)),

  $$T_i \in DHR,\ i = 1, \ldots, n \quad \Rightarrow \quad T_{(i)} - T_{(i-1)} \in DHR,\ i = 1, \ldots, n.$$

- **Devising the monotone behavior of the p.d.f. of $T_{(i)} - T_{(i-1)}$**

  According to Prop. 3.19,

  $$T_{(i)} - T_{(i-1)} \in DHR,\ i = 1, \ldots, n \quad \Rightarrow \quad f_{T_{(i)}-T_{(i-1)}}(t) \downarrow_t .$$

(c) *Calculate $E(T_i)$ and $E[T_{(1)}]$. Compare $E[T_{(1)}]$ to an upper bound obtained taking advantage of the stochastic ageing character of the times to failure T_i.* (3

- **Expected value of T_i**

$$\begin{aligned}
E(T_i) &= \mu_i \\
&\overset{T_i \geq 0}{=} \int_0^{+\infty} R(t)\, dt \\
&= \int_0^{+\infty} (t+1)^{-\theta}\, dt \\
&= \left. \frac{(t+1)^{-\theta+1}}{-\theta+1} \right|_0^{+\infty} \\
&\overset{\theta \geq 1}{=} \frac{1}{\theta - 1} \\
&= \mu
\end{aligned}$$

- **Expected value of $T_{(1)}$**

$$\begin{aligned}
E[T_{(1)}] &\overset{T_i \geq 0}{=} \int_0^{+\infty} R_{T_{(1)}}(t)\, dt \\
&= \int_0^{+\infty} (t+1)^{-n\theta}\, dt \\
&= \left. \frac{(t+1)^{-n\theta+1}}{-n\theta+1} \right|_0^{+\infty} \\
&\overset{n\theta \geq 1}{=} \frac{1}{n\theta - 1}
\end{aligned}$$

- **Important**

  Since the times to failure T_i independent DHR r.v. they are also

  - positively associated (see comment after Def. 1.62)
  - NWUE (by Prop. 3.36).

  Moreover, they are identically distributed.

  Thus, we can apply Theorem 3.62 and provide an upper bound for $E[T_{(1)}]$ by reverting the inequality in (3.55).

[2] Similarly, $\lambda_T(t) = \frac{f_T(t)}{R_T(t)} = \frac{-\frac{dR_{T_{(1)}}(t)}{dt}}{R_T(t)} = \frac{\theta}{t+1}$, $t \geq 0$, a decreasing function of t.

- **Upper bound for $E[T_{(1)}]$**

$$\begin{aligned}
E[T_{(1)}] &= \mu_S \\
&\leq \left(\sum_{i=1}^{n} \mu_i^{-1} \right)^{-1} \\
&\overset{\mu_i = \mu}{=} \left(\frac{n}{\mu} \right)^{-1} \\
&= \frac{\mu}{n} \\
&= \frac{1}{n(\theta - 1)} \\
&= \frac{1}{n\theta - n}.
\end{aligned}$$

- **Comment**

  $E[T_{(1)}] = \frac{1}{n\theta - 1}$ is indeed smaller than or equal to $\frac{1}{n\theta - n}$, for $n \in \mathbb{N}$.

3. (a) *After having put 75 avionic systems to test, a statistician from the reliability department of a light aircraft manufacturer obtained the TTT plot pictured above on the right. What sort of stochasting ageing does this TTT plot suggest for the time to failure of the avionic system?* (0

- **Failure times**

  $T_i =$ time to failure of the i^{th} avionic system

- **Suggested stochastic ageing**

  The points of the TTT plot suggest a 45^o line.

  According to Note 5.5, this suggests in turn that the data should be modeled by an

  - Exponential distribution,

  that is, a memoryless distribution thus, with

  - constant hazard rate (CHR).

(b) *Suppose that: 100 bulbs are placed to test for one year; failed bulbs are replaced during the test; and at the end of the test a total of 10 bulbs have failed.*

Answer the following questions after having specified some convenient distribution assumption:

(i) *Estimate the probability that (a randomly chosen) bulb will fail after 6 months of operation?* (2

- **Distribution assumption**

  $T_i =$ time to failure (in years) of the i^{th} bulb

  $T_i \overset{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda),\ i = 1, \ldots, n$

- **Life test**

  Since

  - bulbs were placed to test for a fixed time, $t_0 = 1$ year
  - the failed bulbs were replaced during the test,

  we are dealing with a

  ∘ Type I/item censored testing with replacement.

- **Censored data**
  $n = 100$ (bulbs initially put to test)
  $r = 10$ (failures during the test)
  $t_0 = 1$ year (fixed duration of the test)

- **Total time in test**

$$\tilde{t} \overset{D5.17}{=} nt_0 = 100 \times 1 = 100$$

- **Unknown parameter**
  λ

- **Maximum likelihood estimate (MLE) of λ**
  According to Table 5.13, the ML estimator of λ is given by $\frac{1}{T}$, if $R = 0$, and $\frac{R}{T}$, if $R > 0$. Since $r = 10 > 0$, we get

$$\hat{\lambda} = \frac{r}{\tilde{t}} = \frac{10}{100} = 0.1$$

- **Another unknown parameter**

$$p = P(T > 6 \text{ months}) = R_T(0.5) = R_{Exp(\lambda)}(0.5) = e^{-0.5\lambda}$$

- **MLE of p**
  Invoking the invariance of the ML estimators, we get

$$\hat{p} = e^{-0.5\hat{\lambda}} = e^{-0.5\times 0.1} \simeq 0.951329.$$

  The UMVUE of p in Table 5.14 could be used instead and the result is quite similar: $(1 - \tilde{t}^{-1} \times t)^r = 0.951110$.

(ii) *Obtain a 90% confidence interval for the warranty period that should be advertised if one wishes that 95% of the bulbs will survive the warranty period? Comment the result of (b)(i) in light of the interval estimate you just obtained.* (**2**

- **Another unknown parameter**

$$\xi = \text{desired warranty period}$$

$$\xi : P(T > \xi) = R_T(\xi) = 0.95$$

$$\xi = F_T^{-1}(1 - 0.95) = R_T^{-1}(0.95)$$

$$\xi = -\frac{1}{\lambda}\ln(0.95)$$

- **Confidence interval for λ**
  According to Table 5.16,

$$CI_{(1-\alpha)\times 100\%}(\lambda) = [\lambda_L; \lambda_U] = \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2 \times \tilde{t}}\right]$$

$$CI_{90\%}(\lambda) = \left[\frac{F^{-1}_{\chi^2_{(20)}}(0.05)}{2 \times 100}; \frac{F^{-1}_{\chi^2_{(22)}}(0.95)}{2 \times 100}\right] = \left[\frac{10.85}{200}; \frac{33.92}{200}\right] \simeq [0.05425; 0.1696]$$

- **Confidence interval for ξ**
  Since ξ is a decreasing function of $\lambda > 0$, we can conclude that

$$CI_{90\%}(\xi) = \left[-\frac{1}{\lambda_U}\ln(0.95); -\frac{1}{\lambda_L}\ln(0.95)\right] = \left[-\frac{1}{0.1696}\ln(0.95); -\frac{1}{0.05425}\ln(0.95)\right] \simeq [0.302437; 0.945499] \text{ (in years)} \simeq [3.629243; 11.345988] \text{ (in months).}$$

- **Comment**
  6 months is obvious a warranty period in the previous interval (and therefore a reasonable value) because the estimated reliability for a 6 month period exceeds 95%.