Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

2nd. Test 1st. Semester — **2011/12**
Duration: **1h30m** **2012/01/20** — **8AM**, Room P1

- Please justify your answers.
- This test has **one page** and **four questions**. The total of points is **20.0**.

1. *Elaborate on the importance of craftsmen unions/guilds in Medieval Europe to ensure product and service quality.* (1.0)

   - **Importance of craftsmen guilds in Medieval Europe**
     From the end of the 13th century to the early 19th century, craftsmen across medieval Europe were organized into unions called guilds. These guilds were responsible for developing strict rules for product and service quality. Inspection committees enforced the rules by marking flawless goods with a special mark or symbol.
     Craftsmen themselves often placed a second mark on the goods they produced. At first this mark was used to track the origin of faulty items. But over time the mark came to represent a craftsman's good reputation. For example, stonemasons marks symbolized each guild members obligation to satisfy his customers and enhance the trades reputation.
     Inspection marks and master-craftsmen marks served as proof of quality for customers throughout medieval Europe. This approach to manufacturing quality was dominant until the Industrial Revolution in the early 19th century.[1]

2. *A p chart is going to be set to control the fraction of nonconforming switches in samples of size n.*

   (a) *How large should n be so that the p chart will have a positive lower control limit $LCL = p_0 - \gamma\sqrt{p_0(1-p_0)/n}$? What is the importance of a positive lower control limit in such a context?* (1.5)

   - **Obtaining a positive lower control limit**

$$
\begin{aligned}
n \ : \ & LCL > 0 \\
& p_0 - \gamma\sqrt{\frac{p_0(1-p_0)}{n}} > 0 \\
& \sqrt{\frac{p_0(1-p_0)}{n}} < \frac{p_0}{\gamma} \\
& n > p_0(1-p_0) \times \frac{\gamma^2}{p_0^2} \\
& n > \frac{(1-p_0)\times\gamma^2}{p_0}
\end{aligned}
$$

   - **Importance of a a positive lower control limit**
     When dealing with a p chart it is essential to have a positive lower control limit in order to detect in a fairly quick fashion a decrease in the fraction of nonconforming items (i.e., quality improvement), namely with $ARL(\delta < 0) < ARL(0)$
     In case the lower control limit is not positive, we deal with an upper one-sided p chart, whose ARL in the presence of a decrease is surely and unreasonably larger than the in-control ARL (i.e., $ARL(\delta < 0) < ARL(0)$).

[1] Taken from http://www.asq.org/learn-about-quality/history-of-quality/overview/guilds.html (now a non operational link).

   (b) *If 20 switches are inspected each day, the target fraction nonconforming is $p_0 = 0.02$ and $\gamma = 3$, what is the median of the time this chart needs to detect a shift in p to 0.04? Interpret this result.* (2.0)

   - **Control statistic**
     Y_N = fraction of nonconforming switches in the N^{th}sample of 20 switches, $N \in I\!N$
   - **Distributions**
     $n \times Y_N \sim \text{Binomial}(n, p = p_0)$, IN CONTROL, where $n = 20$, $p_0 = 0.02$
     $n \times Y_N \sim \text{Binomial}(n, p = p_0 + \delta)$, OUT OF CONTROL, where δ $(0 < \delta < 1 - p_0)$ represents the magnitude of the shift in p
   - **Control limits of the p chart**

$$
\begin{aligned}
LCL &= \max\left\{0, p_0 - \gamma\sqrt{\frac{p_0(1-p_0)}{n}}\right\} \\
&= \max\left\{0, 0.02 - 3 \times \sqrt{\frac{0.02\times(1-0.02)}{20}}\right\} \\
&= \max\{0, -0.0739149\} \\
&= 0 \\
UCL &= p_0 + \gamma\sqrt{\frac{p_0(1-p_0)}{n}} \\
&= 0.02 + 3 \times \sqrt{\frac{0.02\times(1-0.02)}{20}} \\
&= 0.113915
\end{aligned}
$$

   - **Probability of triggering a signal**

$$
\begin{aligned}
\xi(\delta) &= P(Y_N \notin [LCL, UCL] \mid \delta) \\
&\overset{Y_N\geq 0, LCL=0}{=} P(n \times Y_N > n \times UCL \mid \delta) \\
&= 1 - F_{Binomial(n,p=p_0+\delta)}(n \times UCL) \\
&= 1 - F_{Binomial(20,0.04)}(20 \times 0.113915) \\
&= 1 - F_{Binomial(20,0.04)}(2.2783) \\
&= 1 - F_{Binomial(20,0.04)}(2) \\
&\overset{table}{=} 1 - 0.9561 \\
&= 0.0439
\end{aligned}
$$

   - **Run length**
     We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:
     $RL(\delta) \sim \text{Geometric}\,(\xi(\delta))\,.$
   - **Median of $RL(\delta)$**

$$
\begin{aligned}
F^{-1}_{RL(\delta)}(\alpha) &\overset{Table\ 9.2}{=} \inf\left\{m \in I\!N : F_{RL(\delta)}(m) \geq \alpha\right\} \\
&= \quad 1 - [1-\xi(\delta)]^m \geq \alpha \\
&= \quad [1-\xi(\delta)]^m \leq 1-\alpha \\
&= \quad m \times \ln[1-\xi(\delta)] \leq \ln(1-\alpha) \\
&\overset{\ln[1-\xi(\delta)]<0}{=} \quad m \geq \frac{\ln(1-\alpha)}{\ln[1-\xi(\delta)]}
\end{aligned}
$$

$$\begin{aligned} &\overset{\alpha=0.5}{=} && m \geq \frac{\ln(1-0.5)}{\ln(1-0.0439)} \\ &= && m \geq 15.453459 \\ &= \quad 16 \end{aligned}$$

- **Comment**
  When the expected value of the fraction of nonconforming switches duplicates, in samples of $n = 20$ switches, the probability of triggering a valid signal within the first 16 samples is at least 50%, i.e., most of time.

(c) *A statistician has just proposed an upper one-sided CUSUM to detect increases in the expected number of nonconforming switches in samples of size $n = 20$.*
How should she set this new chart and how could she obtain the median of the time this chart needs to detect that shift from $p_0 = 0.02$ to 0.04? (3.5)

- **Reference value, k, of the upper one-sided CUSUM chart for binomial data**
  It should be the closest integer to

$$\begin{aligned} n \times \frac{\ln\left(\frac{1-p_0}{1-p_1}\right)}{\ln\left[\frac{(1-p_0)\times p_1}{(1-p_1)\times p_0}\right]} &= 20 \times \frac{\ln\left[\frac{1-0.02}{1-0.04}\right]}{\ln\left[\frac{(1-0.02)\times 0.04}{(1-0.04)\times 0.02}\right]} \\ &= 0.577760 \end{aligned}$$

  (see result (10.14)), that is, $k = 1$.

- **Control statistic** (no head-start)

$$Z_N = \begin{cases} u = 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y'_N - k)\}, & N \in I\!N, \end{cases}$$

  (see Example 10.9 and (10.4)), where $Y'_N = n \times Y_N$ represents the number of nonconforming switches in the N^{th}sample of 20 switches, $N \in I\!N$

- **Control limits**
  $LCL = 0$ because we are dealing with an upper one-sided CUSUM chart for binomial data.
  The upper control limit should be an integer, $UCL = x$, such that the in-control ARL is fairly large, e.g. 200 samples. Obtaining x requires some numerical work...

- **Out-of-control transition probability matrix**
  Let

$$\begin{aligned} \mathbf{Q}(\delta) &= [p_{ij}(\theta)]_{i,j=0}^{x} \\ &= \begin{bmatrix} F_\delta(k) & P_\delta(k+1) & P_\delta(k+2) & \cdots & P_\delta(k+x) \\ F_\delta(k-1) & P_\delta(k) & P_\delta(k+1) & \cdots & P_\delta(k+x-1) \\ F_\delta(k-2) & P_\delta(k-1) & P_\delta(k) & \cdots & P_\delta(k+x-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_\delta(k-x) & P_\delta(k-x+1) & P_\delta(k-x+2) & \cdots & P_\delta(k) \end{bmatrix}, \end{aligned}$$

  where $\delta = p_1 - p_0 = 0.02$ and F_δ and P_δ represent the c.d.f. and the p.f. of a r.v. with a Binomial$(n = 20, p_0 + \delta = p_1)$ distribution.
  Then $RL^0(\delta)$, the RL of this upper one-sided CUSUM chart, has a phase-type distribution with parameters $(\underline{e}_0, \mathbf{Q}(\delta))$, where $\underline{e}_0 = (1, 0, \ldots, 0)$ is the first vector of the orthonormal basis of $I\!R^{x+1}$.

- **Obtaining the median of the out-of-control RL**
  This RL-related quantity is equal to

$$F^{-1}_{RL(\delta)}(\alpha) \overset{Table\ 9.2}{=} \inf\left\{m \in I\!N : F_{RL^0(\delta)}(m) \geq \alpha\right\},$$

  where $\alpha = 0.5$ and $F_{RL^0(\delta)}(m) = 1 - \underline{e}_0^\top \times [\mathbf{Q}(\delta)]^m \times \underline{1}$, $m \in I\!N$, with $\underline{1} = (1, 1, \ldots, 1)$, a column vector of $x + 1$ ones.

  In the CUSUM case, there is no explicit solution for $F_{RL^0(\delta)}(m) \geq \alpha$, thus, the median of RL can be only obtained by extensive search, i.e., testing the values of m one by one until the condition $F_{RL^0(\delta)}(m) \geq \alpha$ is fulfilled.

3. *The fill volume of a soft drink beverage bottles is an important quality characteristic. The volume can be approximately measured by placing a gauge over the crown and comparing the height of the liquid in the neck of the bottle — a reading of zero corresponds to the correct fill height.*

   Assume that the reading has a normal distribution with parameters with target values $\mu_0 = 0$ and $\sigma_0 = 0.1$, and that samples of size $n = 9$ are taken from the manufacturing process every hour.

   (a) *After having set up an upper one-sided S^2 chart that has an in-control ARL equal to 1000, obtain the probability that a shift from $\sigma_0 = 0.1$ to $\sigma = \sqrt{26.12/11.03} \times \sigma_0$ is detected by this chart within 3 hours.* (2.5)

- **Quality characteristic**
  X = reading (difference between the height of the liquid and the gauge)
  $X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistic**
  S^2_N = variance of the N^{th} random sample of size n, $N \in I\!N$

- **Distribution**
  $\frac{(n-1)S^2_N}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
  $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL, where $\sigma_0 = 0.1$, $n = 9$ θ, and $(\theta > 1)$ represents a shift (an increase!) in the standard deviation σ.

- **Control limits of the upper one-sided S^2- chart**
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probability of triggering a signal**
  Taking into account the distribution of the control statistic, the individual chart for σ triggers a signal with probability equal to

$$\begin{aligned} \xi_\sigma(\theta) &= P\left(S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\ &= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] \\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right), \ \theta \geq 1. \end{aligned}$$

- **Run length of the chart**
  We are dealing with a Shewhart chart, therefore the number of samples collected until the chart triggers a signal given $\theta = \frac{\sigma}{\sigma_0}$, $RL_\sigma(\theta)$, satisfies

$$\begin{aligned} RL_\sigma(\theta) &\sim \text{Geometric}\left(\xi_\sigma(\theta)\right) \\ P\left[RL_\sigma(\theta) \leq m\right] &= 1 - \left[1 - \xi_\sigma(\theta)\right]^m, \ m \in I\!N. \end{aligned}$$

- **Obtaining γ_σ and the upper control limit**

$$\begin{aligned} \gamma_\sigma \quad : \quad & ARL_\sigma(1) = 1000 \\ & \frac{1}{\xi_\sigma(1)} = 1000 \\ & 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = 0.001 \\ & \gamma_\sigma = F_{\chi^2_{(9-1)}}(1 - 0.001) \end{aligned}$$

$$\gamma_\sigma \overset{table}{=} 26.12$$

$$\begin{aligned} UCL_\sigma &= \frac{\sigma_0^2}{n-1} \times \gamma_\sigma \\ &= \frac{0.1^2}{9-1} \times 26.12 \\ &= 0.032650. \end{aligned}$$

- **Requested probability**

$$\begin{aligned} P\left[RL_\sigma(\theta) \le m\right] &= 1 - [1-\xi_\sigma(\theta)]^m \\ &= 1 - \left\{1 - \left[1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right)\right]\right\}^m \\ &= 1 - \left[F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right)\right]^m \\ &= 1 - \left[F_{\chi^2_{(9-1)}}\left(\frac{26.12}{\sqrt{26.12/11.03}^2}\right)\right]^3 \\ &= 1 - \left[F_{\chi^2_{(8)}}(11.03)\right]^3 \\ &\overset{table}{=} 1 - 0.8^3 \\ &= 0.488. \end{aligned}$$

(b) *A standard $\bar{X}$ chart with* 3.0902 *sigma limits has been also adopted and is run along with the chart upper one-sided S^2 chart.*

Calculate the probability that this joint scheme for μ and σ detects the shift described in line (a), assuming that μ remains in-control.

What is the probability of a misleading signal of Type III in such a case? (3.0)

- **An additional control statistic**
  $\bar{X}_N =$ mean of the N^{th} random sample of size n

- **Distribution**
  $\bar{X}_N \sim \text{Normal}\left(\mu=\mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 0$, $\sigma_0 = 0.1$ and $n = 9$
  $\bar{X}_N \sim \text{Normal}\left(\mu=\mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \ne 0$) represents the magnitude of the shift (a decrease or an increase!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ.

- **Control limits of the standard $\bar{X}$ chart**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}} = -0.103007$
  $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}} = 0.103007$, because $\mu_0 = 0$, $\sigma_0 = 0.1$, $n = 9$ and $\gamma_\mu = 3.0902$.

- **Probability of triggering a signal**
  Taking into account the distribution of this control statistic, the individual chart for μ triggers a signal with probability equal to

$$\begin{aligned} \xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta,\theta\right) \\ &= 1 - \left[\Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \ \delta \in I\!R, \ \theta \ge 1. \end{aligned}$$

- **Probability of a signal by the joint scheme for μ and σ**
  The joint scheme triggers a signal if either of the individual charts triggers an alarm. Moreover, the control statistics of the individual charts are independent given (δ,θ). As a consequence, the joint scheme for μ and σ triggers a signal with probability equal to:

$$\begin{aligned} \xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \delta,\theta\right) \\ &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta). \end{aligned}$$

When μ remains in control and a shift from σ_0 to $\sigma = \theta\sigma_0$ has occurred, we get:

$$\begin{aligned} \xi_\mu(\delta,\theta) &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right] \\ &\overset{(\delta,\theta)=(0,\sqrt{26.12/11.03})}{=} 1 - \left[\Phi\left(\frac{3.0902 - 0}{\sqrt{26.12/11.03}}\right) - \Phi\left(\frac{-3.0902 - 0}{\sqrt{26.12/11.03}}\right)\right] \\ &\simeq 1 - [\Phi(2.01) - \Phi(-2.01)] \\ &= 2 \times [1 - \Phi(2.01)] \\ &\overset{table}{=} 2 \times (1 - 0.9778) \\ &= 0.0444 \end{aligned}$$

$$\begin{aligned} \xi_\sigma(\theta) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right) \\ &\overset{\theta=\sqrt{26.12/11.03}}{=} 1 - F_{\chi^2_{(9-1)}}\left(\frac{26.12}{\sqrt{26.12/11.03}^2}\right) \\ &\overset{(a)}{=} 1 - 0.8 \\ &= 0.2. \end{aligned}$$

Then a signal is triggered by the joint scheme, when $(\delta,\theta) = \left(0, \sqrt{26.12/11.03}\right)$, with probability:

$$\begin{aligned} \xi_{\mu,\sigma}\left(0, \sqrt{26.12/11.03}\right) &= \xi_\mu\left(0, \sqrt{26.12/11.03}\right) + \xi_\sigma\left(\sqrt{26.12/11.03}\right) \\ &\quad -\xi_\mu\left(0, \sqrt{26.12/11.03}\right) \times \xi_\sigma\left(\sqrt{26.12/11.03}\right) \\ &\simeq 0.0444 + 0.2 - 0.0444 \times 0.2 \\ &= 0.23552. \end{aligned}$$

- **Probability of a misleading signal of type III**

$$\begin{aligned} PMS_{III}(\theta) &\overset{Table\ 10.12}{=} \frac{1 - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}{\left[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)\right]^{-1} - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]} \\ &= \frac{1 - \left[\Phi\left(\frac{3.0902}{\sqrt{\frac{26.12}{11.03}}}\right) - \Phi\left(-\frac{3.0902}{\sqrt{\frac{26.12}{11.03}}}\right)\right]}{\left[F_{\chi^2_{(8)}}\left(\frac{26.12}{\sqrt{\frac{26.12}{11.03}}^2}\right)\right]^{-1} - \left[\Phi\left(\frac{3.0902}{\sqrt{\frac{26.12}{11.03}}}\right) - \Phi\left(-\frac{3.0902}{\sqrt{\frac{26.12}{11.03}}}\right)\right]} \\ &\simeq \frac{1 - [\Phi(2.01) - \Phi(-2.01)]}{\left[F_{\chi^2_{(8)}}(11.03)\right]^{-1} - [\Phi(2.01) - \Phi(-2.01)]} \\ &\overset{(a)}{=} \frac{0.0444}{0.8^{-1} - (1 - 0.0444)} \\ &\simeq 0.151020. \end{aligned}$$

(c) *Derive and interpret the following result for the Shewhart case:* $PMS_{III}(\theta) = \frac{\xi_\mu(0,\theta) \times [1-\xi_\sigma(\theta)]}{\xi_{\mu,\sigma}(0,\theta)}$. (1.5)

- **Proof**
  Let us remind the reader that, for the Shewhart case, we have

$$
\begin{aligned}
\xi_\mu(0,\theta) &= 1-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)] \\
\xi_\sigma(\theta) &= 1-F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2) \\
\xi_{\mu,\sigma}(0,\theta) &= \xi_\mu(0,\theta)+\xi_\sigma(\theta)-\xi_\mu(0,\theta)\times\xi_\sigma(\theta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
PMS_{III}(\theta) &= \frac{1-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]}{\left[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)\right]^{-1}-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]} \\
&= \frac{\{1-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]\}\times\left\{1-\left[1-F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)\right]\right\}}{1-(1-\{1-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]\})\times\left\{1-\left[1-F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)\right]\right\}} \\
&= \frac{\xi_\mu(0,\theta)\times[1-\xi_\sigma(\theta)]}{1-[1-\xi_\mu(0,\theta)]\times[1-\xi_\sigma(\theta)]} \\
&= \frac{\xi_\mu(0,\theta)\times[1-\xi_\sigma(\theta)]}{\xi_\mu(0,\theta)+\xi_\sigma(\theta)-\xi_\mu(0,\theta)\times\xi_\sigma(\theta)} \\
&= \frac{\xi_\mu(0,\theta)\times[1-\xi_\sigma(\theta)]}{\xi_{\mu,\sigma}(0,\theta)}.
\end{aligned}
$$

- **Interpretation**
  In case we are dealing with a joint Shewhart scheme for μ and σ, the probability of a misleading signal of Type III coincides with the probability of a signal being triggered by the individual chart for μ but not by the individual chart for σ, conditional to a signal triggered by the joint scheme.

4. *A soft drink producer has established an upper specification on the contents of the 33cl bottles at 37cl. If 1% (or less) of the contents of these bottles are below this limit the producer wishes to accept and dispatch the lot with probability (of at least) 0.95 (i.e., $p_1 = 1\%, 1-\alpha = 0.95$), while if 10% (or more) of the contents of the bottles are above this limit the producer would like to accept and dispatch the lot with probability (of at most) 0.15 (i.e., $p_2 = 10\%, \beta = 0.15$).*

   (a) *Find an appropriate sampling plan for* ATTRIBUTES. (3.5)

   - **Producer's and consumer's risk points**
     $(p_1, 1-\alpha) = (1\%, 0.95)$
     $(p_2, \beta) = (10\%, 0.15)$
   - **Obtaining the acceptance number and sample size**
     According to Wetherill and Brown (1991), the acceptance number c and sample size n of a sampling plan for attributes, associated to risk points $(p_1, 1-\alpha)$ and (p_2, β), can be approximately obtained:[2]
     - c should be taken as the smallest integer satisfying
       $$r(c) \leq \frac{p_2}{p_1},$$
       where $r(c) = \frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{F^{-1}_{\chi^2_{2(c+1)}}(\alpha)}$;
     - n should be taken as the smallest integer satisfying
       $$\frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{2p_2} \leq n \leq \frac{F^{-1}_{\chi^2_{2(c+1)}}(\alpha)}{2p_1},$$
       namely the ceiling of the lower bound above,
       $$\left\lceil \frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{2p_2} \right\rceil.$$

     Since

c	$r(c) = \frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{F^{-1}_{\chi^2_{2(c+1)}}(\alpha)}$	Is $r(c) \leq \frac{p_2}{p_1} = 10$?
0	$\frac{F^{-1}_{\chi^2_{2(0+1)}}(1-0.15)}{F^{-1}_{\chi^2_{2(0+1)}}(0.05)} \stackrel{table}{=} \frac{3.794}{0.103} = 36.835$	NO!
1	$\frac{F^{-1}_{\chi^2_{2(1+1)}}(1-0.15)}{F^{-1}_{\chi^2_{2(1+1)}}(0.05)} \stackrel{table}{=} \frac{6.745}{0.711} = 9.487$	**YES**!

     we get $c = 1$. Moreover,

$$
\begin{aligned}
n &= \left\lceil \frac{F^{-1}_{\chi^2_{2(1+1)}}(1-0.15)}{2\times 0.1} \right\rceil \\
&\stackrel{table}{=} \left\lceil \frac{6.745}{2\times 0.1} \right\rceil \\
&= \lceil 33.725 \rceil \\
&= 34.
\end{aligned}
$$

   - **[Obs.** The lot should be accept iff the number of cans with content larger than 37cl, in a sample of $n = 34$ cans, does not exceed $c = 1$.]

   (b) *Assume that the variance of the bottle contents is known and that the producer adopted instead a sampling plan by* VARIABLES *with $n_\sigma = 7$ and $k_\sigma = 1.685411$.*
   Obtain the lot acceptance probabilities of this sampling plan by variables for $p_1 = 1\%$ and $p_2 = 10\%$.[3] *Which sampling plan would you favour?* (1.5)

   - **Requested lot acceptance probabilities**

p	$P_a(p)\Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma-\Phi^{-1}(p)\right]\right\}$
$p_1 = 0.01$	$\Phi\left\{\sqrt{7}\left[-1.685411-\Phi^{-1}(0.01)\right]\right\} \simeq \Phi(1.70) \stackrel{table}{=} 0.9554 \geq 1-\alpha = 0.95$
$p_2 = 0.10$	$\Phi\left\{\sqrt{7}\left[-1.685411-\Phi^{-1}(0.10)\right]\right\} \simeq \Phi(-1.07) \stackrel{table}{=} 1-0.8577 = 0.1423 \leq \beta = 0.15$

   - **[Obs.:** The lot should be accept iff $Q = \frac{U-\bar{x}}{\sigma} \geq k_\sigma$, where Q is the quality index, $U = 37$ is the upper specification limit, $\bar{x}$ represents the mean of a sample with size $n_\sigma = 7$, and $k_\sigma = 1.685411$.]
   - **Favourite sampling plan**
     The sampling plan by variables (SPV) requires a considerably smaller sample ($n_\sigma = 7$) than the sampling plan for attributes $n = 34$, thus, should the SPV be favoured.

[2]See page 129 of the lecture notes, in particular, formulae (13.11), (13.10) and (13.12).

[3]Result (13.34) should be read as follows: $P_a(p) = \cdots = \Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma-\Phi^{-1}(p)\right]\right\}$.