Department of Mathematics, IST — Probability and Statistics Unit

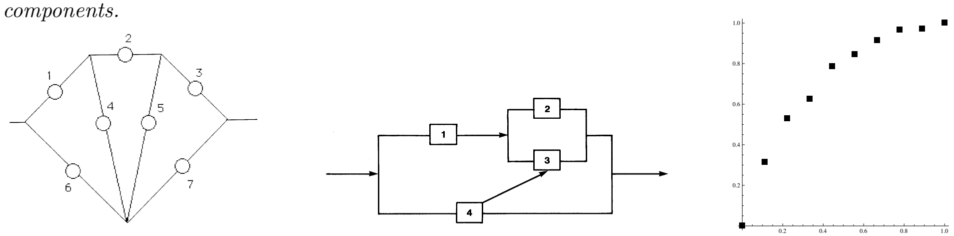
Reliability and Quality Control

t. Test 1st. Semester — **2011/12**

ıration: **1h30m** **2011/11/05** — **8AM**, Room P1

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. *Admit that a transponder*[1] *crucially depends on a diamond structure (pictured on the left) with 7 components.*



Diamond structure (left, question 1); crosslinked system (center, question 2); TTT plot (right, question 3).

(a) *Prove that component 4 is not irrelevant, and provide an expression (do not simplify it!) for the structure function of the diamond structure of the transponder.* (2

- **Relevance of component 4**

Component 4 is not irrelevant because there is at least a state vector $\underline{x}$ such that:

$$\phi(0_4,\underline{x}) \neq \phi(1_4,\underline{x}).$$

In fact, for

$$\underline{x} = (1,0,0,1,0,0,1),$$

we get

$$\begin{aligned}\phi(0_4,\underline{x}) &= \phi(1,0,0,0,0,0,1) = 0\\ &\neq \phi(1_4,\underline{x}) = \phi(1,0,0,1,0,0,1) = 1,\end{aligned}$$

because the system does not function when only components 1 and 7 operate, whereas it operates with components 1, 4 and 7 functioning.

- **Minimal path sets**

$$\begin{aligned}\mathcal{P}_1 &= \{1,2,3\}\\ \mathcal{P}_2 &= \{1,4,7\}\\ \mathcal{P}_3 &= \{1,4,5,3\} = \{1,3,4,5\}\\ \mathcal{P}_4 &= \{1,2,5,7\}\\ \mathcal{P}_5 &= \{6,7\}\\ \mathcal{P}_6 &= \{6,4,2,3\} = \{2,3,4,6\}\\ \mathcal{P}_7 &= \{6,5,3\} = \{3,5,6\}\\ p^* &= 7 \text{ minimal path sets}\end{aligned}$$

[1] *Aircraft have transponders to assist in identifying them on radar and on other aircraft's collision avoidance systems.*

- **Structure function**

$$\begin{aligned}\phi(\underline{X}) &\overset{T1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathcal{P}_j} X_i\right)\\ &= 1-(1-X_1X_2X_3)(1-X_1X_4X_7)(1-X_1X_3X_4X_5)(1-X_1X_2X_5X_7)\\ &\quad \times(1-X_6X_7)(1-X_2X_3X_4X_6)(1-X_3X_5X_6).\end{aligned}$$

(b) *Now, admit that each of those 7 components have reliability $p_i = p = 0.975, i = 1,\ldots,7$, and operate in an independent fashion. Obtain a lower and an upper bound (as strict as possible) for the reliability of the diamond structure.* (2

- **Components**

$p_i = p = 0.975$, $i = 1,\ldots,7$

Since the 7 components form a coherent system and operate independently, we can apply Theorem 1.68.

- **Minimal cut sets**

$$\begin{aligned}\mathcal{K}_1 &= \{1,6\}\\ \mathcal{K}_2 &= \{3,7\}\\ \mathcal{K}_3 &= \{2,4,6\}\\ \mathcal{K}_4 &= \{1,4,5,7\}\\ \mathcal{K}_5 &= \{3,4,5,6\}\\ \mathcal{K}_6 &= \{2,5,7\}\\ q &= 6 \text{ minimal cut sets}\end{aligned}$$

- **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned}r(\underline{p}) &\overset{T1.68}{\geq} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathcal{K}_j}(1-p_i)\right]\\ &\overset{p_i=p}{=} \prod_{j=1}^{q}\left[1-(1-p)^{\#\mathcal{K}_j}\right]\\ &= \left[1-(1-p)^2\right]^2 \times \left[1-(1-p)^3\right]^2 \times \left[1-(1-p)^4\right]^2\\ &\overset{p=0.975}{=} 0.998718.\end{aligned}$$

- **Upper bound for the reliability**

$$\begin{aligned}r(\underline{p}) &\overset{T1.68}{\leq} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathcal{P}_j} p_i\right)\\ &\overset{p_i=p}{=} 1 - \prod_{j=1}^{p^*}\left(1-p^{\#\mathcal{P}_j}\right)\\ &= 1-(1-p^2)\times(1-p^3)^3\times(1-p^4)^3\\ &\overset{p=0.975}{=} 0.999999982.\end{aligned}$$

- **Obs.**

Theorem 1.70 (Min-Max for positively associated) leads to a worse lower bound:

$$r(\underline{p}) \overset{T1.70}{\geq} \max_{j=1,\ldots,p^*} \prod_{i\in\mathcal{P}_j} p_i$$

$$
\begin{aligned}
& \overset{p_i=p}{=} && \max_{j=1,\dots,p^*} p^{\#\mathcal{P}_j} \\
& = && p^{\min_{j=1,\dots,p^*} \#\mathcal{P}_j} \\
& = && p^2 \\
& \overset{p=0.975}{=} && 0.950625.
\end{aligned}
$$

A better upper bound can be obtained by using Theorem 1.70 (Min-Max):

$$
\begin{aligned}
r(\underline{p}) & \overset{T1.70}{\leq} && \min_{j=1,\dots,q} \left[1 - \prod_{i \in \mathcal{K}_j} (1-p_i)\right] \\
& = && \min_{j=1,\dots,q} \left[1 - (1-p)^{\#\mathcal{K}_j}\right] \\
& = && 1 - (1-p)^{\min_{j=1,\dots,q} \#\mathcal{K}_j} \\
& = && 1 - (1-p)^2 \\
& \overset{p=0.975}{=} && 0.999375.
\end{aligned}
$$

(c) *Assume now that the durations (in 10^3 days) of the 7 components are positively associated random variables with common Weibull distribution with scale parameter $\delta = 1$ and shape parameter $\alpha = 2$. Determine a lower and an upper bound (as strict as possible) for the reliability function of the same structure for a period of 2 years.* (2

- **Individual durations, common distribution and duration of the system**

  T_i, $i = 1, \dots, 7$ positively associated r.v.
  $T_i \overset{i.i.d.}{\sim} Weibull(\delta = 1, \alpha = 2)$
  $R_i(t) = R(t) = e^{-t^2}$, $t \geq 0$

  $T =$ duration of the system

  Under these circumstances we can apply Theorem 2.22 to provide a lower and an upper bound for $R_T(t)$.

- **Lower bound for the reliability function** $R_T(t)$

$$
\begin{aligned}
R_T(t) & \overset{T2.22}{\geq} && \max_{j=1,\dots,p^*} \left[\prod_{i \in \mathcal{P}_j} R_i(t)\right] \\
& \overset{R_i(t)=R(t)}{=} && \max_{j=1,\dots,p^*} [R_i(t)]^{\#\mathcal{P}_j} \\
& = && [R(t)]^{\min_{j=1,\dots,p^*} \#\mathcal{P}_j} \\
& = && [R(t)]^2 \\
& = && \left(e^{-t^2}\right)^2 \\
& \overset{t=\frac{2\times 365}{1000}}{=} && e^{-2\times\left(\frac{2\times 365}{1000}\right)^2} \\
& \simeq && 0.344452.
\end{aligned}
$$

- **Upper bound for the reliability function** $R_T(t)$

$$
\begin{aligned}
R_T(t) & \overset{T2.22}{\leq} && \min_{j=1,\dots,q} \left\{1 - \prod_{i \in \mathcal{K}_j} [1 - R_i(t)]\right\} \\
& \overset{R_i(t)=R(t)}{=} && \min_{j=1,\dots,q} \left\{1 - [1-R(t)]^{\#\mathcal{K}_j}\right\} \\
& = && 1 - [1-R(t)]^{\min_{j=1,\dots,q} \#\mathcal{K}_j}
\end{aligned}
$$

$$
\begin{aligned}
& = && 1 - [1-R(t)]^2 \\
& = && 1 - \left(1 - e^{-t^2}\right)^2 \\
& \overset{t=\frac{2\times 365}{1000}}{=} && 1 - \left[1 - e^{-\left(\frac{2\times 365}{1000}\right)^2}\right]^2 \\
& \simeq && 0.829349.
\end{aligned}
$$

2. *A crosslinked system comprises 4 silicon photodiode*[2] *detectors, as pictured above in the center. The durations (in 10^4 hours) of these 4 detectors, T_i ($i = 1, \dots, 4$) are i.i.d. random variables with common reliability function $R(t) = t^{-3}$, for $t \geq 1$, and $R(t) = 1$, for $t < 1$.*

   (a) *Obtain the reliability function of the crosslinked system for a period of 87600 hours.* (3

   - **Individual durations and common reliability function**

     T_i, $i = 1, \dots, 4$, i.i.d. r.v. with reliability function

     $$R_{T_i}(t) = R(t) = \begin{cases} 1, & t < 1 \\ t^{-3}, & t \geq 1 \end{cases}$$

   - **Duration of the system**

     $T = \max\{\min\{T_1, \max\{T_2, T_3\}\}, T_4\}$

   - **Reliability function of** $\max\{T_2, T_3\}$

     According to Example 2.6, it is equal to

     $$R_{\max\{T_2,T_3\}}(t) \overset{R_{T_i}(t)=R(t)}{=} 1 - [1-R(t)]^2$$

   - **Reliability function of** $\min\{T_1, \max\{T_2, T_3\}\}$

     Following Example 2.5, we get

     $$
     \begin{aligned}
     R_{\min\{T_1,\max\{T_2,T_3\}\}}(t) & = && R_{T_1}(t) \times R_{\max\{T_2,T_3\}}(t) \\
     & \overset{R_{T_i}(t)=R(t)}{=} && R(t) \times \left\{1 - [1-R(t)]^2\right\}
     \end{aligned}
     $$

   - **Reliability function of** T

     $$
     \begin{aligned}
     R_T(t) & = && R_{\max\{\min\{T_1,\max\{T_2,T_3\}\},T_4\}}(t) \\
     & = && 1 - \left[1 - R_{\min\{T_1,\max\{T_2,T_3\}\}}(t)\right] \times [1 - R_{T_4}(t)] \\
     & \overset{R_{T_i}(t)=R(t)}{=} && 1 - \left(1 - R(t) \times \left\{1 - [1-R(t)]^2\right\}\right) \times [1 - R(t)]
     \end{aligned}
     $$

   - **Reliability for a period of 87600 hours**

     $$
     \begin{aligned}
     R_T(8.76) & = && 1 - \left(1 - R(8.76) \times \left\{1 - [1 - R(8.76)]^2\right\}\right) \times [1 - R(8.76)] \\
     & = && 1 - \left\{1 - 8.76^{-3} \times \left[1 - \left(1 - 8.76^{-3}\right)^2\right]\right\} \times (1 - 8.76^{-3}) \\
     & \simeq && 0.001492.
     \end{aligned}
     $$

   Alternative method

   - **Individual durations and common reliability function**

     T_i, $i = 1, \dots, 7$, i.i.d. r.v. with reliability function

     $$R_i(t) = R(t) = \begin{cases} 1, & t < 1 \\ t^{-3}, & t \geq 1 \end{cases}$$

[2] *A photodiode exhibits sensitivity to light, for instance by varying its electrical resistance like a photoresistor.*

- **Minimal path sets**

$$\mathcal{P}_1 = \{1,2\}$$
$$\mathcal{P}_2 = \{1,3\}$$
$$\mathcal{P}_3 = \{4\}$$

- **Structure function**

$$\begin{aligned}
\phi(\underline{X}) &\overset{T1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathcal{P}_j} X_i\right) \\
&= 1-(1-X_1X_2)(1-X_1X_3)(1-X_4) \\
&= 1-(1-X_1X_2-X_1X_3+X_1^2X_2X_3)(1-X_4) \\
&\overset{X_1^2=_{st}X_1}{=} 1-(1-X_1X_2-X_1X_3+X_1X_2X_3-X_4+X_1X_2X_4+X_1X_3X_4 \\
&\quad -X_1X_2X_3X_4) \\
&= X_4+X_1X_2+X_1X_3-X_1X_2X_3-X_1X_2X_4-X_1X_3X_4+X_1X_2X_3X_4.
\end{aligned}$$

- **Reliability**

Since $\underline{X} = (X_1,\ldots,X_4)$, where $X_i \overset{indep}{\sim} \text{Bernoulli}(p_i)$, $i=1,2,3,4$, we get:

$$\begin{aligned}
r(\underline{p}) &= r(p_1,\ldots,p_4) \\
&= E[\phi(\underline{X})] \\
&= p_4+p_1p_2+p_1p_3-p_1p_2p_3-p_1p_2p_4-p_1p_3p_4+p_1p_2p_3p_4 \\
&\overset{p_i=p}{=} p+2p^2-3p^3+p^4
\end{aligned}$$

- **Reliability function**

Considering T the duration of the crosslinked system, we have:

$$\begin{aligned}
R_T(t) &= P(T>t) \\
&\overset{N2.8}{=} r(R_1(t),\ldots,R_4(t)) \\
&= r(R(t),\ldots,R(t)) \\
&= R(t)+2[R(t)]^2-3[R(t)]^3+[R(t)]^4 \\
&\overset{R(t)=t^{-3}}{=} t^{-3}+2(t^{-3})^2-3(t^{-3})^3+(t^{-3})^4 \\
&\overset{t=8.76}{=} 0.001492.
\end{aligned}$$

(b) *Are the durations T_i DHRA? What can be said about the stochastic ageing of the duration of the crosslinked system?* (3

- **Individual durations and common reliability function**

T_i, $i=1,\ldots,7$, i.i.d. r.v. with reliability function

$$R_i(t) = R(t) = \begin{cases} 1, & t<1 \\ t^{-3}, & t\geq 1 \end{cases}$$

- **Common p.d.f.**

$$\begin{aligned}
f(t) &= -\frac{dR(t)}{dt} \\
&= \begin{cases} 0, & t<1 \\ 3t^{-4}, & t\geq 1 \end{cases}
\end{aligned}$$

- **Common hazard rate function**

$$\begin{aligned}
\lambda(t) &= \frac{f(t)}{R(t)} \\
&= \begin{cases} 0, & t<1 \\ \frac{3t^{-4}}{t^{-3}} = \frac{3}{t}, & t\geq 1 \end{cases}
\end{aligned}$$

- **Obs.**

$T_i \notin DHR$ because $\lambda(t)$ is not a decreasing function for $t\geq 0$ (even though it is decreasing for $t\geq 1$).

- **Investigating the DHRA character of the T_i's**

$$\begin{aligned}
\frac{1}{t}\Lambda(t) &= \frac{1}{t}\int_0^t \lambda(u)\,du \\
&= \begin{cases} 0, & t<1 \\ \frac{1}{t}\int_1^t \frac{3}{u}\,du = \frac{3\ln(t)}{t}, & t\geq 1 \end{cases} \\
\frac{d\frac{1}{t}\Lambda(t)}{dt} &= \frac{3}{t^2}\left[1-\ln(t)\right],\ t\geq 1 \\
&= \begin{cases} \geq 0, & 1\leq t\leq e \\ \leq 0, & t\geq e \end{cases}
\end{aligned}$$

- **Conclusion**

$\frac{1}{t}\Lambda(t)$ is not a monotonous function for $t\geq 0$, thus $T_i \notin DHRA$.

- **Stochastic ageing of T**

According to Table 3.2, the $DHRA$ property of the components of a system is not necessarily preserved after the formation of a coherent system. Thus, even if the T_i were $DHRA$ we could not add anything about the stochastic ageing of T, the duration of the (coherent) crosslinked system.

(c) *Calculate the common value of $\mu_i = E(T_i) = \mu^*$ and obtain a lower limit for the expected value of the duration of the parallel sub-system with components 2 and 3, falsely assuming that $T_i \in DHRA$.* (2

- **Common expected value**

$$\begin{aligned}
\mu^* &\overset{T_i\geq 0}{=} \int_0^{+\infty} R(t)\,dt \\
&= \int_0^1 1\,dt + \int_1^{+\infty} t^{-3}\,dt \\
&= 1 - \left.\frac{t^{-2}}{2}\right|_1^{+\infty} \\
&= \frac{3}{2}
\end{aligned}$$

- **Lower bound for $\mu_P = E(\max T_2, T_3)$**

Since T_2 and T_3 are independent r.v. (therefore positively associated) and we are falsely assuming that they are both $DHRA$, we can apply Theorem 3.64 (Equation (3.57) with the inequality reversed) and get

$$\begin{aligned}
\mu_P &= E(\max T_2, T_3) \\
&\geq \int_0^{+\infty}\left[1-\prod_{i=1}^n\left(1-e^{-t/\mu_i}\right)\right] dt \\
&\overset{n=2,\,\mu_i=\mu^*=3/2}{=} \int_0^{+\infty}\left[1-\left(1-e^{-2t/3}\right)^2\right] dt \\
&= \int_0^{+\infty}\left(2e^{-2t/3}-e^{-4t/3}\right) dt
\end{aligned}$$

$$= \quad -3e^{-2t/3}\Big|_0^{+\infty} + \frac{3}{4}e^{-4t/3}\Big|_0^{+\infty}$$

$$= \quad 3 - \frac{3}{4}$$

$$= \quad \frac{9}{4}.$$

3. (a) *A sample of 9 specimens of a titanium alloy were subjected to a fatigue test to determine time to crack initiation. The observed times of crack initiation (in units of 10^3 cycles) were 18, 32, 39, 53, 59, 68, 77, 78, 93, and the TTT plot pictured above on the right.*

i) *Exemplify the obtention of the TTT plot, by using just 4 points and the fact that the total time in test is 517 (in units of 10^3 cycles).* (**1**

- **Failure times**
  T_i = time (in units of 10^3)of crack initiation of specimen i, $i = 1, \ldots, 9$
- **Complete data**
  $\underline{t} = (18, 32, 39, 53, 59, 68, 77, 78, 93)$
- **Total time on test up to time $t_{(i)}$**

$$\tau\left(t_{(i)}\right) = \sum_{j=1}^{i}(n-j+1)\left[t_{(j)} - t_{(j-1)}\right]$$

- **Abcissae of the TTT plot**
  $\frac{i}{n}$, $i = 0, 1, \ldots, n$
- **Ordinates of the TTT plot**
  $\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$, $i = 1, \ldots, n$, where $\tau\left(t_{(n)}\right) = 517$; equal to 0, for $i = 0$.
- **Four points of the TTT plot**

i	$\frac{i}{n}$	$\tau\left(t_{(i)}\right) = \sum_{j=1}^{i}(n-j+1)\left[t_{(j)} - t_{(j-1)}\right]$	$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$
0	0	0	0
1	$\frac{1}{9} = 0.(1)$	$(9-1+1) \times 18 = 162$	$\frac{162}{517} \simeq 0.313$
2	$\frac{2}{9} = 0.(2)$	$162 + (9-2+1) \times (32-18) = 274$	$\frac{274}{517} \simeq 0.530$
3	$\frac{3}{9} = 0.(3)$	$274 + (9-3+1) \times (39-32) = 323$	$\frac{323}{517} \simeq 0.625$

ii) *What sort of stochasting ageing does this TTT plot suggest for the time to crack initiation?* (**0**

- **Suggested stochastic ageing**
  The TTT plot is concave. This suggests a IHR behavior of the time to crack initiation.

(b) *A life test for a new insulating material used 25 specimens. The specimens were tested simultaneously at $30kV$, the test was run until 15 of the specimens failed, and the ordered failure times ($t_{(i)}$) recorded as: 1.08, 12.20, 17.80, 19.10, 26.00, 27.90, 28.20, 32.20, 35.90, 43.50, 44.00, 45.20, 45.70, 46.30, 47.80 (with $\sum_{i=1}^{15} t_{(i)} = 472.88$).* (**2**
After having specified some convenient distribution assumption, obtain a 95% confidence interval for the 80% quantile of the failure time (T) distribution, $F_T^{-1}(0.80)$.

- **Distribution assumption**
  $T_i \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \ldots, 25$
- **Life test**
  Since the end of the test was determined by $r = 15^{th}$ failure and nothing in this exercise suggests that the $n = 25$ specimens were replaced during the life test (they "were tested simultaneously"), we are dealing with a
  - Type II/item censored testing without replacement.
- **Unknown parameter**
  λ
- **Censored data**
  $(t_{(1)}, \ldots, t_{(r)}) = (1.08, 12.20, \ldots, 46.30, 47.8015)$
  $n = 25$
  $r = 15$
  $\sum_{i=1}^{15} t_{(i)} = 472.88$
- **Cumulative total time in test**

$$\tilde{t} \overset{D5.17}{=} \sum_{i=1}^{r} t_{(i)} + (n-r) \times t_{(r)} = 472.88 + (25-15) \times 47.80 = 950.88$$

- **Confidence interval for λ**

$$CI_{(1-\alpha)\times 100\%}(\lambda) = [\lambda_l; \lambda_U] = \left[\frac{F_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F_{\chi^2_{(2r)}}(1-\alpha/2)}{2 \times \tilde{t}}\right]$$

$$CI_{95\%}(\lambda) = \left[\frac{F_{\chi^2_{(30)}}(0.025)}{2 \times 950.88}; \frac{F_{\chi^2_{(30)}}(0.975)}{2 \times 950.88}\right] = \left[\frac{16.79}{1901.76}; \frac{46.98}{1901.76}\right] \simeq [0.008829; 0.024703]$$

- **Another unknown parameter**
  $F_T^{-1}(0.80) = -\frac{1}{\lambda}\ln(1-0.80)$, which is a decreasing function of $\lambda > 0$.
- **Confidence interval for $F_T^{-1}(0.80)$**

$$CI_{95\%}\left(F_T^{-1}(0.80)\right) = \left[\frac{1}{\lambda_U}\ln(1-0.80); \frac{1}{\lambda_L}\ln(1-0.80)\right] \simeq [65.150376; 182.296882].$$