Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

1st. TEST (“Época de Recurso”) 1st. Semester — **2011/12**

Duration: **1h30m** **2012/02/04** — **8AM**, Room P1

- Please justify your answers.
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. *Assume that a part of a domestic wastewater treatment station constitutes a system, with 6 components and structure function given by:*

$$\begin{aligned}\phi(\underline{X}) &= 1-(1-X_1X_2X_3X_6)\times(1-X_1X_2X_5X_6)\times(1-X_1X_4X_5X_6)\times(1-X_1X_3X_4X_6)\\ &= [1-(1-X_1)]\times[1-(1-X_2)(1-X_4)]\times[1-(1-X_3)(1-X_5)]\times[1-(1-X_6)].\end{aligned}$$

(a) *Identify the minimal path sets and minimal cut sets, and draw a reliability block diagram as close as possible of the system.* (**2.0**)

- **Structure function**

  By considering $X_i \sim \text{Bernoulli}(p_i)$, $i=1,\dots,6$ and applying results (1.13) and (1.14), we can conclude that the structure function of this system equals

$$\begin{aligned}\phi(\underline{X}) &\overset{(1.13)}{=} 1-\prod_{j=1}^{p^*}\left(1-\prod_{i\in\mathcal{P}_j}X_i\right)\\ &\overset{(1.14)}{=} \prod_{j=1}^{q}\left[1-\prod_{i\in\mathcal{K}_j}(1-X_i)\right],\end{aligned}$$

  where $\mathcal{P}_j$ $(j=1,\dots,p^*)$ and $\mathcal{K}_j$ $(j=1,\dots,q)$ represent the p^* minimal path sets and the q minimal cut sets, respectively.

- **Minimal path sets**

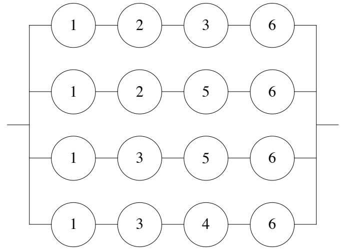
$$\begin{aligned}\mathcal{P}_1 &= \{1,2,3,6\}\\ \mathcal{P}_2 &= \{1,2,5,6\}\\ \mathcal{P}_3 &= \{1,4,5,6\}\\ \mathcal{P}_3 &= \{1,3,4,6\}\\ p^* &= 4 \text{ minimal path sets}\end{aligned}$$

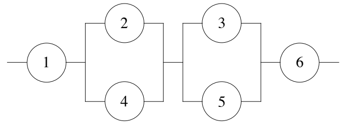
- **Minimal cut sets**

$$\begin{aligned}\mathcal{K}_1 &= \{1\}\\ \mathcal{K}_2 &= \{2,4\}\\ \mathcal{K}_3 &= \{3,5\}\\ \mathcal{K}_4 &= \{6\}\\ q &= 4 \text{ minimal cut sets}\end{aligned}$$

- **Reliability block diagram** (in terms of minimal path/cut sets)

  By capitalizing on Theorem 1.30 and on the minimal path/cut sets, we can provide two representations of the system:



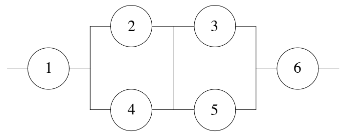


Since

- the first reliability block diagram in terms of minimal path sets has repeated components in the different series sub-systems and
- the reliability block diagram in terms of minimal cut sets has no repeated components in the different parallel sub-systems,

this last representation seems to the closest to the original system.

- **Obs.** — **Reliability block diagram** (the original system!)



(b) *Now, suppose that each of those 6 components are independent and have reliability $p_i = p = 0.95$, $i=1,\dots,6$. Calculate the reliability of the system.* (**2.0**)

- **Reliabilities of the components**

  $p_i = p = 0.95$, $i=1,\dots,6$

  $\underline{p} = (p_1,\dots,p_6)$

- **Reliability of the system**

  Taking into account

  - the reliabilities of the components,
  - the fact that they operate in an independent fashion, and
  - the structure function

$$\begin{aligned}\phi(\underline{X}) &\overset{(a)}{=} [1-(1-X_1)]\times[1-(1-X_2)(1-X_4)]\times[1-(1-X_3)(1-X_5)]\\ &\quad\times[1-(1-X_6)],\end{aligned}$$

  where $X_i \overset{i.i.d.}{\sim} \text{Bernoulli}(p_i = p = 0.95)$, $i=1,\dots,6$, we get the reliability of the system

$$\begin{aligned}r(\underline{p}) &= E[\phi(\underline{X})]\\ &= E\{[1-(1-X_1)]\times[1-(1-X_2)(1-X_4)]\times[1-(1-X_3)(1-X_5)]\end{aligned}$$

$$
\begin{aligned}
& \times[1-(1-X_6)]\} \\
\overset{X_i\, indep}{=} \; & E(X_1) \times [1-E(1-X_2)E(1-X_4)] \times [1-E(1-X_3)E(1-X_5)] \times E(X_6) \\
= \; & p_1 \times [1-(1-p_2)(1-p_4)] \times [1-(1-p_3)(1-p_5)] \times p_6 \\
\overset{p_i=p}{=} \; & p^2 \times [1-(1-p)^2]^2 \\
\overset{p=0.95}{\simeq} \; & 0.897993.
\end{aligned}
$$

(c) *Obtain a lower and an upper bound (as strict as possible) for the reliability of the system, in case the 6 components operate in a positively associated fashion.* **(2.5)**

- **Components**

  $p_i = p = 0.95,\ i = 1,\ldots,6$

  Since the 6 components form a coherent system and operate in a positively associated fashion, we can apply Theorem 1.70, namely result (1.42).

- **Minimal path sets**

$$
\begin{aligned}
\mathcal{P}_1 &= \{1,2,3,6\} \\
\mathcal{P}_2 &= \{1,2,5,6\} \\
\mathcal{P}_3 &= \{1,4,5,6\} \\
\mathcal{P}_3 &= \{1,3,4,6\} \\
p^* &= 4 \text{ minimal path sets}
\end{aligned}
$$

- **Minimal cut sets**

$$
\begin{aligned}
\mathcal{K}_1 &= \{1\} \\
\mathcal{K}_2 &= \{2,4\} \\
\mathcal{K}_3 &= \{3,5\} \\
\mathcal{K}_4 &= \{6\} \\
q &= 4 \text{ minimal cut sets}
\end{aligned}
$$

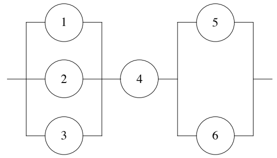
- **Lower bound for the reliability** $r(\underline{p})$

$$
\begin{aligned}
r(\underline{p}) & \overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i \in \mathcal{P}_j} p_i \\
& \overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\
& \overset{\#\mathcal{P}_j=4,\forall j}{=} p^4 \\
& \overset{p=0.95}{=} 0.95^4 \\
& \simeq 0.814506.
\end{aligned}
$$

- **Upper bound for the reliability**

$$
\begin{aligned}
r(\underline{p}) & \overset{(1.42)}{\leq} \min_{j=1,\ldots,q} \left[1 - \prod_{i \in \mathcal{K}_j} (1-p_i)\right] \\
& \overset{p_i=p}{=} \min_{j=1,\ldots,q} \left[1-(1-p)^{\#\mathcal{K}_j}\right] \\
& = 1-(1-p)^{\min_{j=1,\ldots,q} \#\mathcal{K}_j} \\
& = 1-(1-p)^1 \\
& \overset{p=0.95}{=} 0.95.
\end{aligned}
$$

2. *The figure below is a reliability block diagram for a part of a computer system:*



Assume that the durations (in 10^3 hours) of the 6 components, T_i $(i = 1,\ldots,6)$, are independent random variables with common Gamma$(\alpha = 5, \lambda = 1)$ distribution.

(a) *Obtain the reliability function of this part of the computer system for a period of 9155 hours.* **(3.0)**

Note: $F_{Gamma(\alpha,\lambda)}(x) = F_{\chi^2_{(2\alpha)}}(2\lambda x)$.

- **Individual durations (in 10^3 hours) and common reliability function**

  $T_i \overset{i.i.d.}{\sim}$ Gamma$(\alpha = 5, \lambda = 1)\ i = 1,\ldots,6$ with common reliability function

$$
\begin{aligned}
R_{T_i}(t) &= R(t) \\
&= \begin{cases} 1, & t < 0 \\ 1 - F_{Gamma(\alpha,\lambda)}(t) = 1 - F_{\chi^2_{(2\alpha)}}(2\lambda t), & t \geq 0 \end{cases}
\end{aligned}
$$

- **Duration of the system**

  $T = \min\{\max\{T_1,T_2,T_3\}, T_4, \max\{T_5,T_6\}\}$

- **Reliability functions of** $\max\{T_1,T_2,T_3\}$ **and** $\max\{T_5,T_6\}$

  According to Example 2.6, namely result (2.5), the reliability functions of these two independent r.v. are equal to

$$
\begin{aligned}
R_{\max\{T_1,T_2,T_3\}}(t) & \overset{R_{T_i}(t)=R(t)}{=} 1-[1-R(t)]^3 \\
R_{\max\{T_5,T_6\}}(t) & \overset{R_{T_i}(t)=R(t)}{=} 1-[1-R(t)]^2.
\end{aligned}
$$

- **Reliability function of T and requested reliability**

  Inspired by Example 2.5, we can conclude that the reliability function of the minimum of the independent r.v. $\max\{T_1,T_2,T_3\}$, T_4 and $\max\{T_5,T_6\}$ is the product of their reliability functions. If to this we add the fact that, for $\alpha = 5$, $\lambda = 1$ and $t = 9.155$,

$$
\begin{aligned}
R(t) &= 1 - F_{\chi^2_{(2\alpha)}}(2\lambda t) \\
&= 1 - F_{\chi^2_{(10)}}(18.31) \\
& \overset{table}{=} 1 - 0.95 \\
&= 0.05,
\end{aligned}
$$

  we successively get

$$
\begin{aligned}
R_T(t) &= R_{\min\{\max\{T_1,T_2,T_3\},T_4,\max\{T_5,T_6\}\}}(t) \\
&= R_{\max\{T_1,T_2,T_3\}}(t) \times R_{T_4}(t) \times R_{\max\{T_5,T_6\}}(t) \\
&= \left\{1-[1-R(t)]^3\right\} \times R(t) \times \left\{1-[1-R(t)]^2\right\} \\
&= \left[1-(1-0.05)^3\right] \times 0.05 \times \left[1-(1-0.05)^2\right] \\
&= 0.000695.
\end{aligned}
$$

Alternative method

- **Individual durations and common reliability function**
  $T_i \stackrel{i.i.d.}{\sim} \text{Gamma}(\alpha = 5, \lambda = 1)$ $i = 1, \ldots, 6$ with common reliability function

$$\begin{aligned} R_{T_i}(t) &= R(t) \\ &= \begin{cases} 1, & t < 0 \\ 1 - F_{Gamma(\alpha,\lambda)}(t) = 1 - F_{\chi^2_{(2\alpha)}}(2\lambda t), & t \geq 0 \end{cases} \end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned} \mathcal{K}_1 &= \{1, 2, 3\} \\ \mathcal{P}_2 &= \{4\} \\ \mathcal{P}_3 &= \{5, 6\} \\ q &= 3 \text{ minimal cut sets} \end{aligned}$$

- **Structure function**

$$\begin{aligned} \phi(\underline{X}) &\stackrel{(1.14)}{=} \prod_{j=1}^{q} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - X_i)\right], \\ &= [1 - (1 - X_1)(1 - X_2)(1 - X_3)] \times [1 - (1 - X_4)] \times [1 - (1 - X_5)(1 - X_6)]. \end{aligned}$$

- **Reliability**
  Since $\underline{X} = (X_1, \ldots, X_6)$, where $X_i \stackrel{indep}{\sim} \text{Bernoulli}(p_i = p)$, $i = 1, \ldots, 6$, we obtain

$$\begin{aligned} r(\underline{p}) &= r(p_1, \ldots, p_6) \\ &= E[\phi(\underline{X})] \\ &= E\{[1 - (1 - X_1)(1 - X_2)(1 - X_3)] \times [1 - (1 - X_4)] \times [1 - (1 - X_5)(1 - X_6)]\} \\ &= [1 - (1 - p_1)(1 - p_2)(1 - p_3)] \times p_4 \times [1 - (1 - p_5)(1 - p_6)] \\ &= [1 - (1 - p)^3] \times p \times [1 - (1 - p)^2] \end{aligned}$$

- **Reliability function of T and requested reliability**
  Considering T the duration of the system and noting that, for $\alpha = 5$, $\lambda = 1$ and $t = 9.155$,

$$\begin{aligned} R(t) &= 1 - F_{\chi^2_{(2\alpha)}}(2\lambda t) \\ &= 1 - F_{\chi^2_{(10)}}(18.31) \\ &\stackrel{table}{=} 1 - 0.95 \\ &= 0.05, \end{aligned}$$

  we have:

$$\begin{aligned} R_T(t) &= P(T > t) \\ &\stackrel{N2.8}{=} r(R_1(t), \ldots, R_4(t)) \\ &= r(R(t), \ldots, R(t)) \\ &= \{1 - [1 - R(t)]^3\} \times R(t) \times \{1 - [1 - R(t)]^2\} \\ &= \left[1 - (1 - 0.05)^3\right] \times 0.05 \times \left[1 - (1 - 0.05)^2\right] \\ &= 0.000695. \end{aligned}$$

(b) *Are the durations of the components IHR? What can be said about the stochastic ageing of the duration of this part of the computer system?* **(3.0)**

- **Individual durations**
  $T_i \stackrel{i.i.d.}{\sim} \text{Gamma}(\alpha = 5, \lambda = 1)$, $i = 1, \ldots, 6$.

- **Stochastic ageing of T_i**
  First note that $\alpha = 5 > 1$. Therefore, according to the sufficient conditions derived in Exercise 3.18,[1]

$$T_i \stackrel{i.i.d.}{\sim} IHR, \; i = 1, \ldots, 6.$$

  Now, if we apply Proposition 3.23, namely result (3.14), we can also add that

$$\begin{aligned} \max\{T_1, T_2, T_3\} &\in IHR \\ \max\{T_5, T_6\} &\in IHR. \end{aligned}$$

  Moreover, the system can me written is a series system with 3 independent sub-systems whose durations are independent IHR r.v. Thus, by applying now result (3.11) from Proposition 3.23, we can finally state that

$$\begin{aligned} T &= \min\{\max\{T_1, T_2, T_3\}, T_4, \max\{T_5, T_6\}\} \\ &\in IHR. \end{aligned}$$

(c) *Determine a lower bound and an upper bound for the expected value of the duration of this part of the computer system.* **(3.0)**

- **Preliminaries**
  We are dealing with a coherent system characterized as follows:
  - $T_i \stackrel{i.i.d.}{\sim} IHR, \; i = 1, \ldots, 6 \quad \stackrel{Prop. 3.36}{\leftrightarrow} \quad T_i \stackrel{i.i.d.}{\sim} IHRA, \; i = 1, \ldots, 6$;
  - $\mu_i = E(T_i) = \mu^* = E[\text{Gamma}(\alpha = 5, \lambda = 1)] = \frac{\alpha}{\lambda} = 5$;
  - the minimal path sets are

$$\begin{aligned} \mathcal{P}_1 &= \{1, 4, 5\} \\ \mathcal{P}_2 &= \{1, 4, 6\} \\ \mathcal{P}_3 &= \{2, 4, 5\} \\ \mathcal{P}_4 &= \{2, 4, 6\} \\ \mathcal{P}_5 &= \{3, 4, 5\} \\ \mathcal{P}_6 &= \{3, 4, 6\} \\ q &= 6 \text{ minimal path sets;} \end{aligned}$$

  - the minimal cut sets are

$$\begin{aligned} \mathcal{K}_1 &= \{1, 2, 3\} \\ \mathcal{P}_2 &= \{4\} \\ \mathcal{P}_3 &= \{5, 6\} \\ q &= 3 \text{ minimal cut sets.} \end{aligned}$$

  Now, we can apply Theorem 3.69, and conclude obtain the a lower bound and an upper bound for $E(T)$...

[1] Or by proving that $T_i \stackrel{i.i.d.}{\sim} ILR$, $i = 1, \ldots, 6$, i.e., the common p.d.f. is log-concave and then applying Proposition 3.36 to conclude that the r.v. are IHR.

- **Lower bound for $E(T)$**

$$\begin{aligned}\mu &= E(T)\\ &\geq \max_{j=1,\dots,p}\left\{\left(\sum_{i\in\mathcal{P}_j}\mu_i^{-1}\right)^{-1}\right\}\\ &\overset{\mu_i=\mu^*}{=} \max_{j=1,\dots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\}\\ &= \frac{\mu^*}{\min_{j=1,\dots,p}\{\#\mathcal{P}_j\}}\\ &= \frac{\mu^*}{3}\\ &= \frac{5}{3}.\end{aligned}$$

- **Upper bound for $E(T)$**

$$\begin{aligned}\mu &= E(T)\\ &\leq \min_{j=1,\dots,q}\int_0^{+\infty}\left[1-\prod_{i\in\mathcal{K}_j}\left(1-e^{-t/\mu_i}\right)\right]dt\\ &\overset{\mu_i=\mu^*}{=} \min_{j=1,\dots,q}\int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^{\#\mathcal{K}_j}\right]dt\\ &= \int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^{\min_{j=1,\dots,q}\#\mathcal{K}_j}\right]dt\\ &= \int_0^{+\infty}\left[1-\left(1-e^{-t/\mu^*}\right)^1\right]dt\\ &= \int_0^{+\infty} e^{-t/\mu^*}\,dt\\ &= \left(\mu^*\,e^{-t/\mu^*}\right)\Big|_0^{+\infty}\\ &= \mu^*\\ &= 5.\end{aligned}$$

3. *The time in minutes to breakdown (failure) for an insulating fluid is under study. After 100 minutes, there were 7 breakdowns at the following times (in minutes): 7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30.*

   (a) *What do you think about the suggestion of using an exponential distribution to model the data? Obtain the p-value of an appropriated hypotheses test.* (**2.0**)

- **Life test**
  Since the test had a scheduled end after exactly $t_0 = 100$ minutes and the exercise suggests just an insulating fluid repeatedly tested, we are dealing with a
  - Type I/item censored testing with replacement.

- **R.v.**
  $T_{(i)}$ = time of the i^{th} breakdown of the insulating fluid
  $Z_i = T_{(i)} - T_{(i-1)}$ = time between the i^{th} and $(i-1)^{th}$ breakdown of the insulating fluid
  $Z_i \overset{i.i.d.}{\sim} Z,\ i \in \mathbb{N}$

- **Censored data**
  $n = 1$
  $r = 10$ breakdowns during the life test
  $(t_{(1)},\dots,t_{(r)}) = (7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30)$
  $(z_1,\dots,z_r) = (7.74, 17.05-7.74, 20.46-17.05, 21.02-20.46, 22.66-21.02, 43.40-22.66, 47.30-43.40) = (7.74, 9.31, 3.41, 0.56, 1.64, 20.74, 3.9)$

- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= n\times t_0\\ &= 1\times 100\\ &= 100Km\end{aligned}$$

- **Hypotheses**
  $H_0 : Z \sim \text{Exponential}(\lambda)$
  $H_1 : Z \sim \text{Weibull}(\lambda^{-1},\alpha),\ \alpha \neq 1$

- **Significance level**
  α_0

- **Test statistic** (Bartlett's test)

$$\begin{aligned}B_r &\overset{(5.19)}{=} \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r Z_i}{r}\right)-\frac{1}{r}\sum_{i=1}^r\ln(Z_i)\right]\\ &\overset{a}{\sim}_{H_0} \chi^2_{(r-1)}\end{aligned}$$

- **Rejection region of H_0**

$$W = \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right)\cup\left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right)$$

- **Decision** (based on the p-value)
  The observed value of the test statistic is

$$\begin{aligned}b_r &= \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r z_i}{r}\right)-\frac{1}{r}\sum_{i=1}^r\ln(z_i)\right]\\ &\simeq \frac{2\times 7}{1+\frac{7+1}{6\times 7}}\times\left[\ln\left(\frac{47.3}{7}\right)-\frac{1}{7}\times 9.812122\right]\\ &\simeq 5.984294.\end{aligned}$$

  Since the rejection region is two-sided

$$p-value = 2\times\min\{p^-,p^+\}$$

  where

$$\begin{aligned}p^- &= F_{B_r|H_0}(b_r)\\ &\simeq F_{\chi^2_{(r-1)}}(5.984294)\\ &= F_{\chi^2_{(6)}}(5.984294)\\ &\overset{Excel}{=} 0.575048\\ &[\in (0.500, 0.600)]\\ p^+ &= 1-F_{B_r|H_0}(b_r)\\ &= 1-p^-\\ &\simeq 1-0.575048\end{aligned}$$

$$= 0.424952$$
$$[\in (0.400, 0.500)]$$

[because $F^{-1}_{\chi^2_{(6)}}(0.500) = 5.346 < 5.984294 < 6.211 = F^{-1}_{\chi^2_{(6)}}(0.600)$].

Therefore we should:

- not reject H_0 for any significance level $\alpha \leq 2 \times 42.4952\% \simeq 95\%$, namely at all usual significance levels $(1\%, 5\%, 10\%)$;
- reject H_0 for any significance level $\alpha > 95\%$.

[We should: not reject H_0 for any significance level $\alpha \leq 80.0\%$, namely at all usual significance levels $(1\%, 5\%, 10\%)$.]

(b) *After having specified a convenient distribution assumption, obtain a UMVU estimate and a 90% confidence interval for the reliability of the time to breakdown for a period of 50 minutes.* (**2.5**)

- **Distribution assumption**
  $Z_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda),\ i = 1, \ldots, 7.$
  This is fairly reasonable since we did not reject H_0 in (a).

- **Unknown parameter**
  $R_Z(t) = e^{-\lambda t}$, which is a decreasing function of $\lambda > 0$

- **Unbiased estimate of $R_Z(t)$**
  According to Table 5.14, the UMVUE of $R_T(t)$ is, for $t = 50 < \tilde{t} = 100$ and $r > 0$, equal to

$$\begin{aligned}
\tilde{R}_Z(t) &= \left(1 - \tilde{t}^{-1} \times t\right)^r \\
&= \left(1 - \frac{1}{100} \times 50\right)^7 \\
&\simeq 0.007813.
\end{aligned}$$

- **Confidence interval for λ**
  According to Table 5.16 of the lecture notes,

$$\begin{aligned}
CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L; \lambda_U] \\
&= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2 \times \tilde{t}}\right] \\
CI_{90\%}(\lambda) &\stackrel{(a)}{=} \left[\frac{F^{-1}_{\chi^2_{(14)}}(0.05)}{2 \times 1 \times 100}; \frac{F^{-1}_{\chi^2_{(16)}}(0.95)}{2 \times 1 \times 100}\right] \\
&= \left[\frac{6.571}{200}; \frac{26.30}{200}\right] \\
&= [0.032855; 0.1315].
\end{aligned}$$

- **Confidence interval for $R_Z(t)$**

$$\begin{aligned}
CI_{90\%}\left(R_Z(t)\right) &= \left[e^{-\lambda_U \times t}; e^{-\lambda_L \times t}\right] \\
&\stackrel{t=50}{=} \left[e^{-0.1315 \times 50}; e^{-0.032855 \times 50}\right] \\
&\simeq [0.001395; 0.193447].
\end{aligned}$$