Department of Mathematics, IST — Probability and Statistics Unit

Reliability and Quality Control

2nd. TEST ("Época de Recurso") 1st. Semester — **2011/12**

Duration: **1h30m** **2012/02/04** — **10AM**, Room P1

- Please justify your answers.
- This test has **one page** and **four questions**. The total of points is **20.0**.

1. *Elaborate on the role of quality control during World War II in the USA.* (**1.0**)

   - **Role of quality control during World War II in the USA**

     After entering World War II in December 1941, the United States enacted legislation to help gear the civilian economy to military production. At that time, military contracts were typically awarded to the manufacturer that submitted the lowest bid. Products were inspected on delivery to ensure conformance to requirements.

     During this period, quality became an important safety issue. Unsafe military equipment was clearly unacceptable, and the U.S. armed forces inspected virtually every unit produced to ensure that it was safe for operation. This practice required huge inspection forces and caused problems in recruiting and retaining competent inspection personnel.

     To ease the problems without compromising product safety, the armed forces began to use sampling inspection to replace unit-by-unit inspection. With the aid of industry consultants, particularly from Bell Laboratories, they adapted sampling tables and published them in a military standard, known as Mil-Std-105. These tables were incorporated into the military contracts so suppliers clearly understood what they were expected to produce.

     The armed forces also helped suppliers improve quality by sponsoring training courses in Walter Shewharts statistical quality control (SQC) techniques.

     (Source: http://www.asq.org/learn-about-quality/history-of-quality/overview/wwii.html)

2. *An upper one-sided c chart is used to monitor the number of surface imperfections on porcelain enameled water heater cabinets. Suppose the target of the expected value of the number of those imperfections in samples of size n is equal to $\lambda_0 = 2$.*

   (a) *Find the smallest upper control limit that ensures an in-control average run length (ARL) of at least 500 samples.* (**1.5**)

   - **Control statistic**

     Y_N = number of surface imperfections in the N^{th} sample of n porcelain enameled water heater cabinets, $N \in \mathbb{N}$

   - **Distribution**

     $Y_N \sim \text{Poisson}(\lambda_0)$, IN CONTROL, where $\lambda_0 = 2$

     $Y_N \sim \text{Poisson}(\lambda = \lambda_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of the shift in λ

   - **Control limits of the upper one-sided c chart**

$$
\begin{aligned}
LCL &= 0 \text{ (because we are dealing with an upper one-sided chart)}\\
UCL &= \lambda_0 + \gamma\sqrt{\lambda_0}\\
&= 2 + \gamma \times \sqrt{2}
\end{aligned}
$$

   - **Probability of triggering a signal**

$$
\begin{aligned}
\xi(\delta) &= P(Y_N \notin [LCL, UCL] \mid \delta)\\
&\overset{Y_N \geq 0, LCL=0}{=} P(Y_N > UCL \mid \delta)\\
&= 1 - F_{Poisson(\lambda=\lambda_0+\delta)}(UCL)
\end{aligned}
$$

   - **Run length and average run length**

     We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution and average run length:

$$
\begin{aligned}
RL(\delta) &\sim \text{Geometric}(\xi(\delta))\\
ARL(\delta) &\overset{Table\ 9.2}{=} \frac{1}{\xi(\delta)}.
\end{aligned}
$$

   - **Finding the requested upper control limit**

$$
\begin{aligned}
UCL \ :\ & ARL(0) \geq ARL^* = 500\\
& \xi(0) \leq \frac{1}{ARL^*}\\
& 1 - F_{Poisson(\lambda_0)}(UCL) \leq \frac{1}{ARL^*}\\
& UCL \geq F^{-1}_{Poisson(\lambda_0)}\left(1 - \frac{1}{ARL^*}\right)\\
& UCL \geq \inf\left\{m \in \mathbb{N}_0 : F_{Poisson(2)}(m) \geq 0.998\right\}\\
& UCL \geq 7
\end{aligned}
$$

     because

$$
\begin{aligned}
F_{Poisson(2)}(6) &\overset{table}{=} 0.9955 \leq 0.998\\
F_{Poisson(2)}(7) &\overset{table}{=} 0.9989 \geq 0.998.
\end{aligned}
$$

   (b) *If the expected value of the number of imperfections shifts from $\lambda_0 = 2$ to $\lambda_1 = 6$, what is the probability of a valid signal within the first 10 samples? Comment.* (**2.0**)

   - **Shift**

     From $\lambda_0 = 2$ to $\lambda = \lambda_0 + \delta = 6$, i.e., $\delta = 4$.

   - **Probability of triggering a signal**

     If we adopt $UCL = 7$, we get

$$
\begin{aligned}
\xi(4) &\overset{(a)}{=} 1 - F_{Poisson(\lambda=2+4)}(7)\\
&\overset{table}{=} 1 - 0.7440\\
&= 0.2560.
\end{aligned}
$$

   - **Requested probability**

     Since $RL(\delta) \sim \text{Geometric}(\xi(\delta))$, we get

$$
\begin{aligned}
P[RL(\delta) \leq m] &\overset{Table\ 9.2}{=} 1 - [1 - \xi(\delta)]^m, \ m \in \mathbb{N}\\
&= 1 - (1 - 0.2560)^{10}\\
&= 0.948033.
\end{aligned}
$$

   - **Comment**

     The new value of the parameter ($\lambda_1 = 6$) triples the target value ($\lambda_0 = 2$); unsurprisingly, the probability of a signal when we collect a sample ($\xi(4)$) is large and therefore the probability of a signal within the first 10 samples is extremely high.

(c) *A manager has just proposed an upper one-sided CUSUM chart to detect increases from λ_0 to λ_1. How should he/she set this alternative chart with a reference value (an integer reference value close to) $\frac{\lambda_1-\lambda_0}{ln(\lambda_1/\lambda_0)}$ and how could he/she obtain the probability requested in (b)?* (**3.5**)

- **Reference value, k, of the upper one-sided CUSUM chart for Poisson data**
  It should be the closest integer to
  $$\frac{\lambda_1-\lambda_0}{ln(\lambda_1/\lambda_0)} = \frac{6-2}{ln(6/2)} = 3.64096,$$
  that is, $k = 4$.

- **Control statistic** (no head-start)
  $$Z_N = \begin{cases} u = 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in I\!N, \end{cases}$$
  which is similar to the control statistic of an upper one-sided CUSUM chart for binomial data in Example 10.9, equation (10.4).

- **Control limits**
  $LCL = 0$ because we are dealing with an upper one-sided CUSUM chart for Poisson data.
  The upper control limit should be an integer, $UCL_{CUSUM} = x$, such that the in-control ARL is fairly large, e.g. 200 samples. Obtaining x requires some numerical work...

- **RL distribution**
  $RL^0(\delta)$, the RL of this upper one-sided CUSUM chart, has a phase-type distribution with parameters $(\underline{e}_0, \mathbf{Q}(\delta))$, where $\underline{e}_0 = (1, 0, \ldots, 0)$ is the first vector of the orthonormal basis of $I\!R^{x+1}$ and
  $$\mathbf{Q}(\delta) = [p_{ij}(\delta)]_{i,j=0}^{x} = \begin{bmatrix} F_\delta(k) & P_\delta(k+1) & P_\delta(k+2) & \cdots & P_\delta(k+x) \\ F_\delta(k-1) & P_\delta(k) & P_\delta(k+1) & \cdots & P_\delta(k+x-1) \\ F_\delta(k-2) & P_\delta(k-1) & P_\delta(k) & \cdots & P_\delta(k+x-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_\delta(k-x) & P_\delta(k-x+1) & P_\delta(k-x+2) & \cdots & P_\delta(k) \end{bmatrix},$$
  where $\delta = \lambda_1 - \lambda_0 = 4$ and F_δ and P_δ represent the c.d.f. and the p.f. of a r.v. with a Poisson$(\lambda_0 + \delta = \lambda_1)$ distribution.

- **Obtaining the probability requested in (b)**
  This RL-related quantity is equal to
  $$P\left[RL^0(\delta) \leq m\right] \overset{Table\,9.2}{=} 1 - \underline{e}_0^\top \times [\mathbf{Q}(\delta)]^m \times \underline{1},\ m \in I\!N,$$
  where $m = 10$ and $\underline{1} = (1, 1, \ldots, 1)$, a column vector of $x+1$ ones. Obtaining this quantity requires some programming but only depends on a power of a matrix and other trivial matrix operations.

3. *An automatic screw machine turns out round-head bolts with a specified diameter of 9.00 ± 0.04 mm. The process has been operating in control at $\mu_0 = 9$ and $\sigma_0 = 0.02$, and samples of size $n = 4$ are taken from the manufacturing process every hour. A standard $\bar{X}$ chart has been adopted and is run along with an upper one-sided S^2 chart with an in-control ARL, $ARL_\sigma(1)$, equal to 1000.*

   (a) *Determine the limits of the standard $\bar{X}$ chart in such a way that the in-control ARL of the joint scheme for μ and σ, $ARL_{\mu,\sigma}(0,1)$, equals 500 samples.* (**3.0**)

- **Quality characteristic**
  X = diameter of a round-dhead bolt
  $X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistic**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  S_N^2 = variance of the N^{th} random sample of size n, $N \in I\!N$

- **Distribution**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 9$, $\sigma_0 = 0.02$ and $n = 4$
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or an increase!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ.
  $\frac{(n-1)S_N^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
  $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL

- **Control limits of the individual charts**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probabilities of triggering signals**
  Taking into account the distribution of the control statistics, the individual charts for μ and σ trigger signals with probabilities equal to
  $$\begin{aligned} \xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta,\theta\right) \\ &= 1 - \left[\Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right],\ \delta \in I\!R,\ \theta \geq 1, \\ \xi_\sigma(\theta) &= P\left(S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\ &= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] \\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right),\ \theta \geq 1, \end{aligned}$$
  respectively.
  The joint scheme triggers a signal if either of the individual charts triggers an alarm. Moreover, the control statistics of the individual charts are independent given (δ, θ). As a consequence, the joint scheme for μ and σ triggers a signal with probability equal to:
  $$\begin{aligned} \xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \delta,\theta\right) \\ &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta). \end{aligned}$$

- **Run length of the joint scheme**
  We are dealing with a joint Shewhart scheme, therefore the number of samples collected until the scheme triggers a signal given (δ, θ) satisfies
  $$RL_{\mu,\sigma}(\delta,\theta) \sim \text{Geometric}\left(\xi_{\mu,\sigma}(\delta,\theta)\right)$$

$$ARL_{\mu,\sigma}(\delta,\theta) = \frac{1}{\xi_{\mu,\sigma}(\delta,\theta)}.$$

- **Obtaining γ_μ**

  Since $RL_\sigma(\theta) \sim$ Geometric $(\xi_\sigma(\theta))$, and the in-control ARL of the chart for σ is equal to 1000, i.e.,

$$ARL_\sigma(1) = 1000$$

$$\xi_\sigma(1) = \frac{1}{1000},$$

  we have

$$\begin{aligned}
\gamma_\mu \; : \; & ARL_{\mu,\sigma}(0,1) = 500 \\
& \xi_\mu(0,1) + \xi_\sigma(1) - \xi_\mu(0,1) \times \xi_\sigma(1) = \frac{1}{500} \\
& \xi_\mu(0,1) = \frac{\frac{1}{500} - \xi_\sigma(1)}{1-\xi_\sigma(1)} \\
& 1 - [\Phi(\gamma_\mu) - \Phi(-\gamma_\mu)] = \frac{\frac{1}{500} - \xi_\sigma(1)}{1-\xi_\sigma(1)} \\
& \gamma_\mu = \Phi^{-1}\left(1 - \frac{\frac{\frac{1}{500}-\frac{1}{1000}}{1-\frac{1}{1000}}}{2}\right) \\
& \gamma_\mu \simeq \Phi^{-1}(0.999499) \\
& \gamma_\mu \overset{table}{=} 3.29
\end{aligned}$$

- **Control limits of the individual chart for μ**

$$\begin{aligned}
LCL_\mu &= \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\
&= 9 - 3.29 \times \frac{0.02}{\sqrt{4}} \\
&= 8.9671 \\
UCL_\mu &= \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\
&= 9 + 3.29 \times \frac{0.02}{\sqrt{4}} \\
&= 9.0329
\end{aligned}$$

(b) *Obtain the probability that a shift from $\sigma_0 = 0.02$ to $\sigma = \sqrt{16.27/6.251} \times \sigma_0$ is detected by this joint scheme (exactly) at the fifth sample, assuming that μ remains in-control.* **(3.0)**

What is the probability of a misleading signal of Type III in this case?

- **Remark**

  Before we proceed we need to obtain the constant γ_σ.

$$\begin{aligned}
\gamma_\sigma \; : \; & ARL_\sigma(1) = ARL^* = 1000 \\
& \xi_\sigma(1) = \frac{1}{ARL^*} \\
& 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = \frac{1}{ARL^*} \\
& \gamma_\sigma = F_{\chi^2_{(4-1)}}(1-0.001) \\
& \gamma_\sigma \overset{table}{=} 16.27.
\end{aligned}$$

- **Probability of a signal by the joint scheme for μ and σ**

  The joint scheme triggers a signal if either of the individual charts triggers an alarm. Moreover, the control statistics of the individual charts are independent given (δ,θ). As a consequence, the joint scheme for μ and σ triggers a signal with probability equal to:

$$\begin{aligned}
\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \delta,\theta\right) \\
&= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta).
\end{aligned}$$

  When μ remains in control and a shift from σ_0 to $\sigma = \theta\sigma_0$ has occurred, we obtain:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right] \\
&\overset{(\delta,\theta)=(0,\sqrt{16.27/6.251})}{=} 1 - \left[\Phi\left(\frac{3.29-0}{\sqrt{16.27/6.251}}\right) - \Phi\left(\frac{-3.29-0}{\sqrt{16.27/6.251}}\right)\right] \\
&\simeq 1 - [\Phi(2.04) - \Phi(-2.04)] \\
&= 2 \times [1 - \Phi(2.04)] \\
&\overset{table}{=} 2 \times (1 - 0.9793) \\
&= 0.0414 \\
\xi_\sigma(\theta) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right) \\
&\overset{\theta=\sqrt{16.27/6.251}}{=} 1 - F_{\chi^2_{(4-1)}}\left(\frac{16.27}{\sqrt{16.27/6.251}^2}\right) \\
&= 1 - F_{\chi^2_{(3)}}(6.251) \\
&\overset{table}{=} 1 - 0.9 \\
&= 0.1.
\end{aligned}$$

  Then a signal is triggered by the joint scheme, when $(\delta,\theta) = \left(0,\sqrt{16.27/6.251}\right)$, with probability:

$$\begin{aligned}
\xi_{\mu,\sigma}\left(0,\sqrt{16.27/6.251}\right) &= \xi_\mu\left(0,\sqrt{16.27/6.251}\right) + \xi_\sigma\left(\sqrt{16.27/6.251}\right) \\
&\quad -\xi_\mu\left(0,\sqrt{16.27/6.251}\right) \times \xi_\sigma\left(\sqrt{16.27/6.251}\right) \\
&\simeq 0.0414 + 0.1 - 0.0414 \times 0.1 \\
&= 0.13726.
\end{aligned}$$

- **Requested probability**

$$\begin{aligned}
P\left[RL_{\mu,\sigma}\left(0,\sqrt{16.27/6.251}\right) = 5\right] &\overset{Table\ 9.2}{=} \left[1 - \xi_{\mu,\sigma}\left(0,\sqrt{16.27/6.251}\right)\right]^{5-1} \\
&\quad \times \xi_{\mu,\sigma}\left(0,\sqrt{16.27/6.251}\right) \\
&\simeq (1-0.13726)^4 \times 0.13726 \\
&\simeq 0.076044
\end{aligned}$$

- **Probability of a misleading signal of type III**

$$PMS_{III}(\theta) \overset{Table\ 10.12}{=} \frac{1 - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}{\left[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)\right]^{-1} - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}$$

$$
\begin{aligned}
&= \frac{1-\left[\Phi\left(\frac{3.29}{\sqrt{\frac{16.27}{6.251}}}\right)-\Phi\left(-\frac{3.29}{\sqrt{\frac{16.27}{6.251}}}\right)\right]}{\left[F_{\chi^2_{(3)}}\left(\frac{26.12}{\sqrt{\frac{26.12}{6.251}}}\right)\right]^{-1}-\left[\Phi\left(\frac{3.29}{\sqrt{\frac{26.12}{11.03}}}\right)-\Phi\left(-\frac{3.29}{\sqrt{\frac{26.12}{11.03}}}\right)\right]} \\
&\simeq \frac{1-[\Phi(2.04)-\Phi(-2.04)]}{\left[F_{\chi^2_{(8)}}(6.251)\right]^{-1}-[\Phi(2.04)-\Phi(-2.04)]} \\
&\overset{(a)}{=} \frac{0.0414}{0.9^{-1}-(1-0.0414)} \\
&\simeq 0.271456.
\end{aligned}
$$

(c) *Prove that the following result holds for a joint Shewhart scheme for μ and σ: $PMS_{IV}(\delta) = \frac{[1-\xi_\mu(\delta,1)]\times\xi_\sigma(1)}{\xi_{\mu,\sigma}(\delta,1)}$.* **(1.0)**

- **Proof**

Let us remind the reader that, for the Shewhart case, we have

$$
\begin{aligned}
\xi_\mu(\delta,1) &= 1-[\Phi(\gamma_\mu-\delta)-\Phi(-\gamma_\mu-\delta)] \\
\xi_\sigma(1) &= 1-F_{\chi^2_{(n-1)}}(\gamma_\sigma) \\
\xi_{\mu,\sigma}(\delta,1) &= \xi_\mu(\delta,1)+\xi_\sigma(1)-\xi_\mu(\delta,1)\times\xi_\sigma(1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
PMS_{IV}(\delta) &\overset{Table\ 10.12}{=} \frac{1-F_{\chi^2_{(n-1)}}(\gamma_\sigma)}{[\Phi(\gamma_\mu-\delta)-\Phi(-\gamma_\mu-\delta)]^{-1}-F_{\chi^2_{(n-1)}}(\gamma_\sigma)} \\
&= \frac{[\Phi(\gamma_\mu-\delta)-\Phi(-\gamma_\mu-\delta)]\times\left[1-F_{\chi^2_{(n-1)}}(\gamma_\sigma)\right]}{1-[\Phi(\gamma_\mu-\delta)-\Phi(-\gamma_\mu-\delta)]\times F_{\chi^2_{(n-1)}}(\gamma_\sigma)} \\
&= \frac{[1-\xi_\mu(\delta,1)]\times\xi_\sigma(1)}{1-[1-\xi_\mu(\delta,1)]\times[1-\xi_\sigma(1)]} \\
&= \frac{[1-\xi_\mu(\delta,1)]\times\xi_\sigma(1)}{\xi_\mu(\delta,1)+\xi_\sigma(1)-\xi_\mu(\delta,1)\times\xi_\sigma(1)} \\
&= \frac{[1-\xi_\mu(\delta,1)]\times\xi_\sigma(1)}{\xi_{\mu,\sigma}(\delta,1)}.
\end{aligned}
$$

4. *Suppose that a vendor ships components in lots of size $N = 5000$. A single sampling plan with rectifying inspection is being used with $n = 50$ and $c = 2$; rejected lots are screened and all defective items reworked and returned to the lot.*

(a) *Find the level p of lot quality that will be accepted approximately 9.48% of the time.* **(1.5)**

- **Single sampling plan (for attributes)**

$N = 5000$ (lot size)
$n = 50$ (sample size)
$c = 2$ (acceptance number)

- **Auxiliary r.v. and its approximate distributions**

$$
\begin{aligned}
D &= \text{number of nonconforming components in the sample} \\
&\overset{a}{\sim} \text{Binomial}(n,p) \\
&\overset{n>20,p<0.1}{\simeq} \text{Poisson}(np)
\end{aligned}
$$

- **Obtaining the requested level p of lot quality**

$$
\begin{aligned}
p \ &: \ P(D\leq c) \simeq 0.0948 \\
&F_{Poisson(\lambda=np)}(c) \simeq 0.0948
\end{aligned}
$$

Now, consulting the tables of the c.d.f. of the Poisson distribution, namely checking what value of the parameter λ satisfies $F_{Poisson(\lambda=np)}(c) \simeq 0.0948$, we obtain

$$
\begin{aligned}
F_{Poisson(\lambda=np=5.40)}(2) &\simeq 0.0948 \\
50\times p &= 5.40 \\
p &= 0.108.
\end{aligned}
$$

(b) Suppose that incoming lots are $p = 0.005$ nonconforming. Calculate the average outgoing quality (AOQ) at this point. Comment. **(1.0)**

- **Average outgoing quality (AOQ) of a single sampling plan with rectifying inspection**

$$
\begin{aligned}
AOQ(p) &\overset{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\
&= \frac{p\,(N-n)}{N}\times P(D\leq c) \\
&\simeq \frac{p\,(N-n)}{N}\times F_{Binomial(n,p)}(c) \\
&= \frac{0.005\times(5000-50)}{100}\times\sum_{d=0}^{2}\binom{50}{d}\times 0.005^d\times(1-0.005)^{100-x}
\end{aligned}
$$

[Alternatively,

$$
\begin{aligned}
AOQ(p) &\overset{Mathematica}{\simeq} 0.004940 \\
&\overset{n>20,p<0.1}{\simeq} \frac{p\,(N-n)}{N}\times F_{Poisson(np)}(c) \\
&= \frac{0.005\times(5000-50)}{100}\times F_{Poisson(50\times0.005=0.25)}(2) \\
&\overset{table}{=} \frac{0.005\times(5000-50)}{100}\times 0.9978 \\
&= 0.00493911.]
\end{aligned}
$$

- **Comment**

With a $p = 0.005$ nonconforming, rectifying inspection is useless because it is responsible for an insignificant relative decrease in the fraction nonconforming of

$$
\begin{aligned}
\frac{|p-AOQ|}{p}\times100\% &= \frac{|0.005-0.004940|}{0.005}\times100\% \\
&\simeq 1.203499\%.
\end{aligned}
$$

(c) *Now, admit that the vendor decided to adopt instead a sampling plan by* VARIABLES *with an upper specification limit (U) and known standard deviation σ.* **(2.5)**

Set such a plan with risk points $(p_1, 1-\alpha) = (1\%, 0.95)$ and $(p_2,\beta) = (10\%, 0.15)$.

State and interpret the acceptance rule of this plan.

- **Sampling plan by** VARIABLES **with known variance**

n_σ (sample size)
k_σ (acceptance constant)
σ (known standard deviation)

U (upper specification limit)

- **Producer's and consumer's risk points**
  $(p_1, 1-\alpha) = (1\%, 0.95)$
  $(p_2, \beta) = (10\%, 0.15)$

- **Obtaining n_σ and k_σ**
  According to (13.32),

$$(n_\sigma, k_\sigma) \; : \; \begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(1-\alpha)-\Phi^{-1}(\beta)}{\Phi^{-1}(p_2)-\Phi^{-1}(p_1)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(p_2)\Phi^{-1}(1-\alpha)-\Phi^{-1}(p_1)\Phi^{-1}(\beta)}{\Phi^{-1}(\beta)-\Phi^{-1}(1-\alpha)}. \end{cases}$$

$$\begin{cases} n_\sigma = \left[\frac{\Phi^{-1}(0.95)-\Phi^{-1}(0.15)}{\Phi^{-1}(0.1)-\Phi^{-1}(0.01)}\right]^2 \\ k_\sigma = \frac{\Phi^{-1}(0.1)\Phi^{-1}(0.95)-\Phi^{-1}(0.01)\Phi^{-1}(0.15)}{\Phi^{-1}(0.15)-\Phi^{-1}(0.95)}. \end{cases}$$

$$\begin{cases} n_\sigma \overset{table}{=} \left[\frac{1.6449-(-1.0364)}{(-1.2816)-(-2.3263)}\right]^2 = 6.587303 \\ k_\sigma \overset{table}{=} \frac{(-1.2816)\times 1.6449-(-2.3263)\times(-1.0364)}{(-1.0364)-1.6449} = 1.685407. \end{cases}$$

  If we take $n_\sigma = \lceil 6.587303 \rceil = 7$ and $k_\sigma = 1.685407$ then

$$\begin{aligned} P_a(p_1) &= \Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_1)\right]\right\} \\ &= \Phi\left\{\sqrt{7}\left[-1.685407 - \Phi^{-1}(0.01)\right]\right\} \\ &\simeq \Phi(1.69) \\ &\overset{table}{=} 0.9545 \\ &\geq 1-\alpha = 0.95 \\ P_a(p_2) &= \Phi\left\{\sqrt{n_\sigma}\left[-k_\sigma - \Phi^{-1}(p_2)\right]\right\} \\ &= \Phi\left\{\sqrt{7}\left[-1.685407 - \Phi^{-1}(0.10)\right]\right\} \\ &\simeq \Phi(-1.07) \\ &\overset{table}{=} 1 - 0.8577 \\ &= 0.1423 \\ &\leq \beta = 0.15. \end{aligned}$$

- **Acceptance rule**
  For this sampling plan, we have

$$Q = \frac{U - \bar{X}}{\sigma} \geq k_\sigma,$$

  where Q is called the quality index and $\bar{X}$ represents the mean of the random sample of size n_σ.

- **Interpretation of the acceptance rule**
  If $Q = \frac{U-\bar{X}}{\sigma} \geq k_\sigma$, we would accept the lot because the sample data imply that the lot mean is sufficiently far below the upper specification limit (U) to ensure that the lot fraction nonconforming is satisfactory.