Fiabilidade e Controlo de Qualidade — LMAC

Exame de Época Especial — 2o. Semestre – 2006/07

Duração: 3 horas — 8/Setembro/07 (Sábado) – P12, 09h

1. *Seis dispositivos a laser foram sujeitos a testes tendo sido registadas falhas eléctricas graves ao fim de 63, 114, 14820, 16105, 17393 e 18707 horas.*

   (a) *Averigue a adequação do modelo exponencial a este conjunto de dados ao nível de significância de 5%.* (1.5)

   - **R.v.**
     T_i = time (in hours) of the i^{th} serious electrical failure in a laser device
     $T_i \stackrel{i.i.d.}{\sim} T$, $i = 1, \ldots, n$
   - **Life test / data**
     Since the test was NOT scheduled to end after exactly t_0 time units or after r failures, and the exercise does NOT suggest that the $n = 6$ devices were replaced during this life test, we are dealing with COMPLETE DATA: $\underline{t} = (63, 114, 14820, 16105, 17393, 18707)$.
   - **Hypotheses**
     $H_0 : T \sim \text{Exponential}(\lambda)$
     $H_1 : T \sim \text{Weibull}(\lambda^{-1}, \alpha)$, $\alpha \neq 1$
   - **Significance level**
     $\alpha_0 = 5\%$
   - **Test statistic** (Bartlett's test)

     $$\begin{aligned} B_r &\stackrel{(5.17)}{=} \frac{2r}{1 + \frac{r+1}{6r}} \left\{ \ln\left[\frac{\sum_{i=1}^{r} T_{(i)}}{r}\right] - \frac{1}{r}\sum_{i=1}^{r} \ln\left[T_{(i)}\right] \right\} \\ &\stackrel{a}{\sim}_{H_0} \chi^2_{(r-1)} \end{aligned}$$

     where $r = n$ and the $T_{(i)}$ represents the time of the i^{th} failure time.
   - **Rejection region of H_0**

     $$\begin{aligned} W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right) \\ &\stackrel{r=10,\,\alpha_0=0.05}{=} (0, 0.831) \cup (12.83, +\infty) \end{aligned}$$
   - **Decision**
     Since
     - $\sum_{i=1}^{6} t_{(i)} = 67202$
     - $\sum_{i=1}^{6} \ln\left[t_{(i)}\right] = 47.707$

     the observed value of the test statistic is

     $$\begin{aligned} b_r &= \frac{2r}{1 + \frac{r+1}{6r}} \left\{ \ln\left[\frac{\sum_{i=1}^{r} t_{(i)}}{r}\right] - \frac{1}{r}\sum_{i=1}^{r} \ln\left[t_{(i)}\right] \right\} \\ &= \frac{2 \times 6}{1 + \frac{6+1}{6 \times 6}} \left\{ \ln\left[\frac{67202}{6}\right] - \frac{1}{6} \times 47.707 \right\} \\ &= 13.683 \\ &\notin (0, 0.831) \cup (12.83, +\infty). \end{aligned}$$

     Thus, we should reject H_0 for any significance level $\alpha \geq 5\%$.

   (b) *Considerando as hipóteses de trabalho que entender convenientes, obtenha uma estimativa pontual e outra intervalar para a mediana do tempo até registo de falha eléctrica grave.* (1.5)

   - **Obs.**
     The p-value is approximately equal to

     $$\begin{aligned} 2 \times \min\{F_{\chi^2_{(r-1)}}(t), 1 - F_{\chi^2_{(r-1)}}(t)\} &= 2 \times \min\{0.982246, 0.0177535\} \\ &= 0.035507. \end{aligned}$$

     Thus, we should:
     – reject H_0 for any significance level $\alpha > 3.55\%$ (such as 5% and 10%);
     – not reject H_0 for any significance level $\alpha \leq 3.55\%$ (such as 1%).
   - **Distribution assumption**
     $T_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \ldots, 6$.
     This is fairly reasonable since we did not reject H_0 for all the usual significance levels $(1\%, 5\%, 10\%)$.
   - **Unknown parameter**
     λ
   - **ML estimate of λ**

     $$\begin{aligned} \hat{\lambda} &\stackrel{(5.20)}{=} \frac{n}{\sum_{i=1}^{n} t_i} \\ &= \frac{n}{\sum_{i=1}^{n} t_{(i)}} \\ &= \frac{6}{67202} \\ &\simeq 0.000089283 \end{aligned}$$
   - **Confidence interval for λ**
     – **Pivotal quantity for λ**
       According to p. 139 of the lecture notes,

       $$\begin{aligned} Z &= 2\lambda \sum_{i=1}^{n} T_i \\ &\sim \chi^2_{(2n)}. \end{aligned}$$

     – **Percentage points (balanced ones**
       Considering the confidence level $(1-\alpha) \times 100\% = 95\%$ and $n = 6$, we obtain

       $$\begin{aligned} a_\alpha &= F^{-1}_{\chi^2_{(2n)}}(1 - \alpha/2) \\ &= 4.404 \\ b_\alpha &= F^{-1}_{\chi^2_{(2n)}}(1 - \alpha/2) \\ &= 23.34. \end{aligned}$$

     – **Inverting the double inequality** $a_\alpha \leq Z \leq b_\alpha$

       $$\begin{aligned} P(a_\alpha \leq Z \leq b_\alpha) &= 1 - \alpha \\ P\left(a_\alpha \leq 2\sum_{i=1}^{n} T_i \leq b_\alpha\right) &= 1 - \alpha \\ P\left(\frac{a_\alpha}{2\sum_{i=1}^{n} T_i} \leq \lambda \leq \frac{b_\alpha}{2\sum_{i=1}^{n} T_i}\right) &= 1 - \alpha \end{aligned}$$

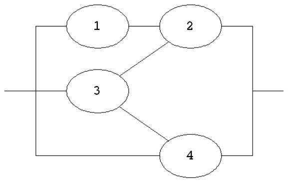
     – **Confidence interval for λ**

       $$CI_{(1-\alpha)\times 100\%}(\lambda) = [\lambda_L, \lambda_U]$$

$$\begin{aligned}
&= \left[\frac{a_\alpha}{2\sum_{i=1}^n t_i}, \frac{b_\alpha}{2\sum_{i=1}^n t_i}\right] \\
CI_{95\%}(\lambda) &= \left[\frac{4.404}{2\times 67202}, \frac{23.34}{2\times 67202}\right] \\
&\simeq [0.0000327669, 0.000173656]
\end{aligned}$$

- **Another unknown parameter**
  $F_T^{-1}(0.5) = -\frac{1}{\lambda}\ln(1-0.5)$, which is a decreasing function of $\lambda > 0$.
- **ML estimate of $F_T^{-1}(0.5)$**
  By invoking the invariance property of the MLE, we get:
  $$\begin{aligned}
  \hat{F}_T^{-1}(0.5) &= -\frac{1}{\hat{\lambda}}\ln(1-0.5) \\
  &= -\frac{1}{0.000089283}\ln(1-0.5) \\
  &\simeq 7763.48
  \end{aligned}$$
- **Confidence interval for $F_T^{-1}(0.5)$**
  $$\begin{aligned}
  CI_{95\%}\left(F_T^{-1}(0.5)\right) &= \left[\frac{1}{\lambda_U}\ln(1-0.5); \frac{1}{\lambda_L}\ln(1-0.5)\right] \\
  &\simeq [3991.51; 21153.9].
  \end{aligned}$$

2. *Considere que o sistema eléctrico no esquema abaixo é constituído por componentes independentes com tempos até falha (em meses) T_i com distribuição normal$(\mu = 10, \sigma^2 = 2^2)$, $i = 1,2,3,4$.*



(a) *Demonstre que os tempos até falha são v.a. ILR.* (1.0)

- **Individual durations and common distribution**
  $T_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu^* = 10, \sigma^2 = 2^2)$, $i = 1,\dots,4$
- **Stochastic ageing of T_i**
  Since
  $$\begin{aligned}
  \ln\left(f_{T_i}(t)\right) &= \ln\left\{\frac{1}{\sqrt{2\pi}\sigma}\exp\left[\frac{1}{2}\left(\frac{t-\mu^*}{\sigma}\right)^2\right]\right\} \\
  &= -\frac{1}{2}\ln(2\pi) - \ln(\sigma) - \frac{1}{2}\left(\frac{t-\mu^*}{\sigma}\right)^2 \\
  \frac{d\,\ln\left(f_{T_i}(t)\right)}{dt} &= -\frac{t-\mu^*}{\sigma} \\
  \frac{d^2\,\ln\left(f_{T_i}(t)\right)}{dt^2} &= -\frac{1}{\sigma} \\
  &< 0
  \end{aligned}$$
  we can conlude that
  $$T_i \stackrel{i.i.d.}{\sim} ILR,\ i = 1,\dots,4,$$
  according to Definition 3.33.

(b) *Obtenha um limite inferior para o valor esperado da duração do sistema eléctrico.*
Determine também um limite inferior o mais estrito possível — que tire partido do carácter de envelhecimento estocástico de $T_i, i = 1,2,3,4$ — para a fiabilidade do sistema eléctrico para um período de 5 meses. (2.5)

- **Duration of the electrical system**
  T
- **Lower bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - minimal path sets
    $$\begin{aligned}
    \mathcal{P}_1 &= \{1,2\} \\
    \mathcal{P}_2 &= \{2,3\} \\
    \mathcal{P}_3 &= \{4\} \\
    p &= 3 \text{ minimal path sets}
    \end{aligned}$$
  - the n components have durations $T_i \stackrel{i.i.d.}{\sim} ILR$, $i = 1,\dots,n$, thus, by Proposition 3.36, $T_i \stackrel{i.i.d.}{\sim} NBUE$, $i = 1,\dots,n$;
  - the expected value of the duration of each of the n components is equal to $\mu_i = \mu^* = 10$.

  Now, we can apply Theorem 3.65, and state:
  $$\begin{aligned}
  \mu &= E(T) \\
  &\geq \max_{j=1,\dots,p}\left\{\left(\sum_{i\in\mathcal{P}_j}\mu_i^{-1}\right)^{-1}\right\} \\
  &\stackrel{\mu_i=\mu^*}{=} \max_{j=1,\dots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\} \\
  &= \max_{j=1,\dots,p}\left\{\frac{\mu^*}{\#\mathcal{P}_j}\right\} \\
  &= \frac{\mu^*}{\min_{j=1,\dots,p}\#\mathcal{P}_j} \\
  &\simeq \frac{10}{1} \\
  &\simeq 10.
  \end{aligned}$$
- **Lower bound for $R_T(t)$**
  This coherent system is also characterized by:
  - components with independent — and, by Proposition 3.36, IHR — durations, with common reliability functions
    $$\begin{aligned}
    R_i(t) &= R(t) \\
    &\stackrel{(4.19)}{=} 1 - \Phi\left(\frac{t-\mu^*}{\sigma}\right);
    \end{aligned}$$
  - structure function given by
    $$\phi(\underline{X}) \stackrel{(1.13)}{=} 1 - \prod_{j=1}^{p}\left(1 - \prod_{i\in\mathcal{P}_j}X_i\right),$$
    where $X_i \stackrel{i.i.d.}{\sim} \text{Bernoulli}(p_i)$, $i = 1,2,3,4$ and $X_i^k \stackrel{d}{=} X_i$, $k \in I\!N$, thus, equal to
    $$\phi(\underline{X}) = 1 - (1 - X_1X_2)(1 - X_2X_3)(1 - X_4)$$

$$
\begin{aligned}
&= 1-(1-X_2X_3-X_1X_2+X_1X_2^2X_3)(1-X_4)\\
&\stackrel{d}{=} 1-(1-X_2X_3-X_1X_2+X_1X_2X_3)(1-X_4);
\end{aligned}
$$

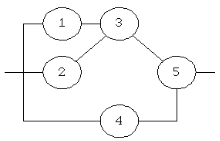
– reliability equal to

$$
\begin{aligned}
r(\underline{p}) &= E\left[\phi(\underline{X})\right]\\
&= E\left[1-(1-X_2X_3-X_1X_2+X_1X_2X_3)(1-X_4)\right]\\
&\stackrel{X_i \stackrel{i.i.d.}{\sim} Bernoulli(p_i)}{=} 1-(1-p_2p_3-p_1p_2+p_1p_2p_3)(1-p_4).
\end{aligned}
$$

As a consequence, we can apply Proposition 3.58, and state that, for $t \leq \min_{i=1,\ldots,n}$,

$$
\begin{aligned}
R_T(t) &\geq r\left(e^{-\frac{t}{\mu_1}},\ldots,e^{-\frac{t}{\mu_n}}\right)\\
&= 1-\left(1-e^{-\frac{t}{\mu_2}}e^{-\frac{t}{\mu_3}}-e^{-\frac{t}{\mu_1}}e^{-\frac{t}{\mu_2}}+e^{-\frac{t}{\mu_1}}e^{-\frac{t}{\mu_2}}e^{-\frac{t}{\mu_3}}\right)\times\left(1-e^{-\frac{t}{\mu_4}}\right)\\
&\stackrel{t=5,\mu_i=10}{\simeq} 0.808234.
\end{aligned}
$$

3. *Um circuito integrado possui cinco componentes dispostas de acordo com a figura abaixo.*



(a) *Obtenha a fiabilidade deste circuito integrado caso a fiabilidade comum das 5 componentes seja igual a 0.95.* (1.5)

- **Reliabilities of the components**
  $p_i = p = 0.95,\ i = 1,\ldots,5$

- **Minimal path sets**

$$
\begin{aligned}
\mathcal{P}_1 &= \{1,3,5\}\\
\mathcal{P}_2 &= \{2,3,5\}\\
\mathcal{P}_3 &= \{4,5\}\\
p^* &= 3 \text{ minimal path sets}
\end{aligned}
$$

- **Structure function**
  Note that, for $X_i \sim \text{Bernoulli}(p_i)$, $X_i^k \stackrel{d}{=} X_i$, $k \in I\!N$. Thus:

$$
\begin{aligned}
\phi(\underline{X}) &\stackrel{(1.13)}{=} 1-\prod_{j=1}^{p^*}\left(1-\prod_{i\in\mathcal{P}_j}X_i\right)\\
&= 1-(1-X_1X_3X_5)(1-X_2X_3X_5)(1-X_4X_5)\\
&= 1-(1-X_2X_3X_5-X_1X_3X_5+X_1X_2X_3^2X_5^2)(1-X_4X_5)\\
&= 1-(1-X_4X_5-X_2X_3X_5+X_2X_3X_4X_5^2-X_1X_3X_5+X_1X_3X_4X_5^2\\
&\quad +X_1X_2X_3^2X_5^2-X_1X_2X_3^2X_4X_5^2)\\
&\stackrel{d}{=} X_4X_5+X_2X_3X_5+X_1X_3X_5-X_2X_3X_4X_5-X_1X_3X_4X_5-X_1X_2X_3X_5\\
&\quad +X_1X_2X_3X_4X_5
\end{aligned}
$$

- **Reliability**
  Taking into account that $X_i \stackrel{indep}{\sim} \text{Bernoulli}(p_i)$, we obtain:

$$
\begin{aligned}
r(\underline{p}) &= E\left[\phi(\underline{X})\right]\\
&= E\left(X_4X_5+X_2X_3X_5+X_1X_3X_5-X_2X_3X_4X_5-X_1X_3X_4X_5\right.\\
&\quad \left.-X_1X_2X_3X_5+X_1X_2X_3X_4X_5\right)\\
&= p_4p_5+p_2p_3p_5+p_1p_3p_5-p_2p_3p_4p_5-p_1p_3p_4p_5-p_1p_2p_3p_5+p_1p_2p_3p_4p_5\\
&\stackrel{p_i=p}{=} p^2+2p^3-3p^5+p^5\\
&\stackrel{p=0.95}{\simeq} 0.947512.
\end{aligned}
$$

(b) *Qual o número mínimo de replicações ao nível do sistema necessário para que a fiabilidade seja de pelo menos 0.995?* (2.0)

- **Reliability of the replicated system (at system level)**
  By replicating the system once, we get:

$$
\begin{aligned}
r_{RSL}(\underline{p},\underline{p}') &\stackrel{(1.26)}{=} 1-\left[1-r(\underline{p})\right]\times\left[1-r(\underline{p}')\right]\\
&\stackrel{\underline{p}=\underline{p}'}{\simeq} 1-(1-0.947512)^2\\
&\simeq 0.997245\\
&\geq 9.995.
\end{aligned}
$$

Therefore on replication at system level is enough to ensure a reliability of at least 0.995.

4. *Identifique e elabore sobre dois dos vários tipos de custos de prevenção associados a programas de garantia de qualidade.* (1.0)

- **[Prevention costs**
  According to Montgomery (1985, p. 5), prevention costs are associated with efforts in design and manufacturing that are directed toward the prevention of nonconformance, that is, they correspond to costs incurred in an effort to *make it right the first time*.]

- **Examples of prevention costs**
  Montgomery (1985, pp. 5–6) and the lecture notes on Quality Control (pp. 6–7) identify several self-explanatory prevention costs. For example:
  - **Product/process design costs**
    These are costs incurred during the design of the product or the selection of production processes that are intended to improve the overall quality of the product (Montgomery, 1985, p. 6).
  - **Quality data acquisition and analysis costs**
    These are the costs of: running the quality data system to acquire data on product and process performance; analyzing these data to identify problems; and summarizing and publishing quality information for management.

5. *O aumento da concentração (em mg/l) de proteínas no líquido cefalorraquidiano (LCR) e o aspecto deste mesmo líquido são utilizados como auxiliares no diagnóstico de patologias.*

Admita-se que tal concentração em certo paciente é uma característica de qualidade com distribuição normal com valor esperado e desvio-padrão iguais a $\mu_0 = 300 mg/l$ e $\sigma_0 = 75 mg/l$, na ausência de patologia.

Uma vez que certas patologias elevam consideravelmente o valor esperado μ de tal concentração, um médico optou por efectuar uma medição de hora a hora e recorrer a carta unilateral superior do tipo Shewhart para μ com número esperado de amostras recolhidas sob controlo igual a 250 horas.

(a) *Obtenha os números esperados de amostras recolhidas até emissão de alarme caso o referido paciente possua patalogias associadas a $\mu = 650mg/l$ (meningite linfocítica benigna) e $\mu = 3000mg/l$ (meningite por cocus). Comente.* (1.5)

- **Quality characteristic**
  X = protein concentration in the cerebrospinal fluid (in mg/l)
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Upper one-sided Shewhart chart for μ**
  - Control statistic
    $\bar{X}_N = X_N$ = mean of the N^{th} random sample of UNITARY size
  - Control limits

$$\begin{aligned} LCL_\mu &= -\infty \text{ or } 0 \\ UCL_\mu &= \mu_0 + \gamma \frac{\sigma_0}{\sqrt{n}} \\ &\stackrel{n=1}{=} \mu_0 + \gamma\sigma_0 \end{aligned}$$

  - Distributions of the control statistic (recall that $n = 1$)
    $X_N \sim \text{Normal}\left(\mu = \mu_0, \sigma^2 = \sigma_0^2\right)$, IN CONTROL, where $\mu_0 = 300mg/l$, $\sigma_0 = 75mg/l$
    $X_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \sigma_0^2, \sigma^2 = (\theta\sigma_0)^2\right)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of the shift (an increase!) in μ and θ ($\theta > 0$) represents a shift in the standard deviation σ
- **Probability of triggering a signal**
  Taking into account the given in-control ARL ($ARL_\mu(0,1) = 250$) of the upper one-sided Shewhart chart for μ described above and the fact that the run length of this chart satisfies $RL_\mu(\delta,\theta) \sim \text{Geometric}(\xi(\delta,\theta))$, we have:

$$\begin{aligned} \xi_\mu(\delta,\theta) &= P\left(X_N \notin [LCL_\mu, UCL_\mu] \mid \delta,\theta\right) \\ &= P\left(X > \mu_0 + \gamma\sigma_0 \mid \delta,\theta\right) \\ &= 1 - \Phi\left(\frac{\gamma-\delta}{\theta}\right), \end{aligned}$$

  where

$$\begin{aligned} \gamma \quad : \quad & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1) \\ & 1 - \Phi\left(\frac{\gamma - 0}{1}\right) = \frac{1}{ARL_\mu(0,1)} \\ & \gamma = \Phi^{-1}\left(1 - \frac{1}{ARL_\mu(0,1)}\right) \\ & \gamma = \Phi^{-1}(0.996) \\ & \gamma \stackrel{table}{=} 2.6521. \end{aligned}$$

- **Two shifts in μ and the expected number of samples until signal**

μ	$\delta \stackrel{n=1}{=} \frac{\mu-\mu_0}{\sigma_0}$	θ	$ARL_\mu(\delta,\theta) = \frac{1}{1-\Phi\left(\frac{\gamma-\delta}{\theta}\right)}$
650	$\frac{650-300}{75} \simeq 4.66$	1	$\frac{1}{1-\Phi\left(\frac{2.6521-4.66}{1}\right)} \simeq \frac{1}{1-\Phi(-2.02)} \simeq 1.022183$
3000	$\frac{3000-300}{75} \simeq 36$	1	$\frac{1}{1-\Phi\left(\frac{2.6521-36}{1}\right)} \simeq \frac{1}{1-\Phi(-33.35)} \simeq 1.000000$

- **Comment**
  The detection of a benign lymphocytic meningitis, corresponding to a shift from $\mu_0 = 300mg/l$ to $\mu = 650mg/l$ requires in average the collection of 1.022183 samples, that is, the detection is practically immediate.
  The detection of a meningitis Cocus, corresponding to a shift from $\mu_0 = 300mg/l$ to $\mu = 3000mg/l$ requires in average the collection of 1 sample, that is, the detection is IMMEDIATE.

(b) *Admita que o valor esperado da concentração de proteínas no LCR está sob controlo e que ocorreu aumento de σ_0 para $\sigma_1 = 100mg/l$.*
Após ter classificado este sinal, obtenha a probabilidade de emissão de sinal pela carta aquando da recolha da $m-$ésima amostra sabendo que esta carta emitiu pelo menos um sinal entre as primeiras m amostras. Comente. (1.5)

- **Magnitude of the shift**
  Since μ is in-control and σ increased from $\sigma_0 = 75mg/l$ to $\sigma_1 = 100mg/l$, we get:

$$\begin{aligned} (\delta,\theta) &= 1 - \Phi\left(\frac{\gamma-\delta}{\theta}\right) \\ &= \left(0, \frac{\sigma_1}{\sigma_0}\right) \\ &= \left(0, \frac{100}{75}\right) \\ &= \left(0, \frac{4}{3}\right). \end{aligned}$$

- **Classification of the signal**
  If the chart described above triggers a signal after a shift with magnitude $(\delta,\theta) = \left(0, \frac{4}{3}\right)$, then the signal can be interpreted as a misleading signal — after all μ is in-control, σ is out-of-control and the only chart we are using is for μ.
- **Run length**
  In the presence of a magnitude $(\delta,\theta) = \left(0, \frac{4}{3}\right)$, this charts triggers a signal with probability:

$$\begin{aligned} \xi_\mu(\delta,\theta) &= 1 - \Phi\left(\frac{2.6521 - 0}{\frac{4}{3}}\right) \\ &\simeq 1 - \Phi(1.99) \\ &\stackrel{table}{=} 1 - 0.9767 \\ &= 0.0233. \end{aligned}$$

  Thus,

$$RL_\mu(\delta,\theta) \sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right).$$

- **Requested probability**

$$\begin{aligned} P\left[RL_\mu(\delta,\theta) = m \mid RL_\mu(\delta,\theta) \le m\right] &= \frac{P\left[RL_\mu(\delta,\theta) = m\right]}{P\left[RL_\mu(\delta,\theta) \le m\right]} \\ &= \frac{\left[1-\xi_\mu(\delta,\theta)\right]^{m-1}\xi_\mu(\delta,\theta)}{1-\left[1-\xi_\mu(\delta,\theta)\right]^m} \\ &= \frac{(1-0.0233)^{m-1}0.0233}{1-(1-0.0233)^m}, \; m \in I\!N \end{aligned}$$

- **Comment**
  The requested probability is NOT CONSTANT, unlike the failure rate function of $RL_\mu(\delta,\theta)$:

$$P\left[RL_\mu(\delta,\theta) = m \mid RL_\mu(\delta,\theta) \ge m\right] = \xi_\mu(\delta,\theta), \; m \in I\!N.$$

(c) *Disserte sobre a pertinência da utilização de uma carta S^2 para controlar a variância da concentração de proteínas no LCR do paciente e procure adiantar uma solução satisfatória para o controlo do mesmo parâmetro.* (1.0)

- **Relevance of the S^2-chart**
  It does not make any sense to use the S^2-chart because its control statistic,
  $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$
  is not defined for $n = 1$.

- **Satisfactory solution**
  We could control the variance of the quality characteristic by using a moving sample range,
  $$R_N = X_N - X_{N-1},$$
  where N represents the measurement number, X_N (resp. X_{N-1}) the N^{th} (resp. $N-1$) measurement of the protein concentration in the cerebrospinal fluid.

6. *Um engenheiro informático decidiu que os aumentos do número esperado de e-mails contaminados com vírus, em blocos de 20 e-mails recebidos, deveriam ser controlados por uma carta CUSUM unilateral superior sem head start, com $UCL = 2$, valor nominal de np e valor de referência iguais a $np_0 = 2$ e $k = 4$ (respectivamente).*

   Após ter enunciado uma hipótese de trabalho que entender razoável, determine a probabilidade de esta carta emitir um falso alarme entre as primeiras 4 amostras. Comente. (2.5)

- **Quality characteristic**
  Y = number of virus contaminated e-mails, in a batch of 20 e-mails

- **Distribution assumptions**
  Y_N = number of virus contaminated e-mails, in the N^{th} batch of 20 e-mails, $N \in I\!N$
  $Y_N \sim \text{Binomial}(n, p = p_0)$, IN CONTROL, where $n = 20$, $p_0 = 0.1$
  $Y_N \sim \text{Binomial}(n, p = p_0 + \theta)$, OUT OF CONTROL, where θ ($\theta > 0$) represents the magnitude of the shift in p.

- **Upper one-sided CUSUM chart for binomial data**
  - Control limits
    $LCL = 0$
    $UCL = x = 2$
  - Reference value
    $k = 4$
  - Initial value of the control statistic
    $u = 0$ (no head-start)
  - Control statistic
    $$Z_N = \begin{cases} 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in I\!N, \end{cases}$$

- **Requested probability**
  According to Table 10.3,
  $$P\left[RL^0(0) \leq m\right] = 1 - \underline{e}_0^\top \times [\mathbf{Q}(0)]^m \times \underline{1},$$
  where:
  $$\underline{e}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$
  and
  $$\begin{aligned} \mathbf{Q}(0) &\overset{(10.9),(10.10)}{=} \begin{bmatrix} F_{Binomial(n,p_0)}(k) & P_{Binomial(n,p_0)}(k+1) & P_{Binomial(n,p_0)}(k+2) \\ F_{Binomial(n,p_0)}(k-1) & P_{Binomial(n,p_0)}(k) & P_{Binomial(n,p_0)}(k+1) \\ F_{Binomial(n,p_0)}(k-2) & P_{Binomial(n,p_0)}(k-1) & P_{Binomial(n,p_0)}(k) \end{bmatrix} \\ &\overset{n=20,\,p_0=0.1,\,table}{=} \begin{bmatrix} 0.9568 & 0.9887-0.9568 & 0.9976-0.9887 \\ 0.8670 & 0.9568-0.8670 & 0.9887-0.9568 \\ 0.6769 & 0.8670-0.6769 & 0.9568-0.8670 \end{bmatrix} \\ &= \begin{bmatrix} 0.9568 & 0.0319 & 0.0089 \\ 0.8670 & 0.0898 & 0.0319 \\ 0.6769 & 0.1901 & 0.0898 \end{bmatrix}. \end{aligned}$$
  Noting that
  $$[\mathbf{Q}(0)]^4 = \left\{[\mathbf{Q}(0)]^2\right\}^2,$$
  we successively obtain by using a calculator
  $$[\mathbf{Q}(0)]^2 \simeq \begin{bmatrix} 0.949148 & 0.035078 & 0.010332 \\ 0.928995 & 0.041786 & 0.13446 \\ 0.873260 & 0.055735 & 0.020153 \end{bmatrix}$$
  $$[\mathbf{Q}(0)]^4 = \begin{bmatrix} 0.942492 & 0.035336 & 0.010487 \\ 0.932314 & 0.035083 & 0.010435 \\ 0.898229 & 0.034085 & 0.010178 \end{bmatrix}.$$
  Thus,
  $$\begin{aligned} P\left[RL^0(0) \leq 4\right] &= 1 - [1\,0\,0] \times \begin{bmatrix} 0.942492 & 0.035336 & 0.010487 \\ 0.932314 & 0.035083 & 0.010435 \\ 0.898229 & 0.034085 & 0.010178 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= 1 - \text{sum of the entries of the 1st. line of } [\mathbf{Q}(0)]^4 \\ &\simeq 1 - 0.988378 \\ &= 0.011624. \end{aligned}$$

- **Comment**
  This rather small probability suggests that false alarms within the first four samples are quite unfrequent.

7. *Um revendedor de material eléctrico e o produtor desse mesmo material comprometeram-se a lidar com lotes de 2000 peças e a recorrer a um plano de amostragem dupla com as seguintes características:*
   - *a primeira amostra será constituída por $n_1 = 25$ peças, o lote é aceite caso não haja peças defeituosas nesta amostra;*
   - *ao recolher-se uma segunda amostra, esta será constituída por $n_2 = 50$ peças e nesta altura aceitar-se-á o lote se o número acumulado de peças defeituosas não exceder dois.*

   (a) *Defina o ponto de risco do produtor deste plano de amostragem dupla ao considerar-se $AQL = 0.02$.* (1.0)

- **Double sampling plan (for attributes)**
  $N = 2000$ (lot size)
  $n_1 = 25$, $n_2 = 50$ (sample sizes)
  $c_1 = 0$, $c_2 = 2$ (acceptance numbers)[1]

- **Producer's risk point**
  $(AQL, \alpha) = (0.02, 1 - P(\text{lot acceptance} \mid p = AQL))$

- **Auxiliary r.v. and their approximate distributions**

$$\begin{aligned} D_i &= \text{number of defective units in the } i^{th} \text{ sample} \\ &\overset{a}{\sim} \text{Binomial}(n_i, p),\ i = 1, 2 \end{aligned}$$

- **Probability of accepting the lot in the first stage of the double sampling plan**

$$\begin{aligned} P_a^I(p) &\overset{(13.16)}{=} P(D_1 \leq c_1) \\ &\simeq F_{Binomial(n_1,p)}(c_1) \\ &\overset{p=AQL}{=} F_{Binomial(25,0.02)}(0) \\ &= \binom{25}{0} \times 0.02^0 \times (1-0.02)^{25-0} \\ &\simeq 0.603465 \end{aligned}$$

- **Probability of accepting the lot in the second stage of the double sampling plan**

$$\begin{aligned} P_a^{II}(p) &\overset{(13.17)}{=} P(c_1 < D_1 \leq c_2,\ D_1 + D_2 \leq c_2) \\ &= \sum_{k=c_1+1}^{c_2} P(D_1 = k) \times P(D_2 \leq c_2 - k) \\ &\simeq \sum_{k=c_1+1}^{c_2} P_{Binomial(n_1,p)}(k) \times F_{Binomial(n_2,p)}(c_2 - k) \\ &= \sum_{k=1}^{2} P_{Binomial(25,0.02)}(k) \times F_{Binomial(50,0.02)}(2 - k) \\ &= \binom{25}{1} \times 0.02^1 \times (1-0.02)^{25-1} \\ &\quad \times \left[\binom{50}{0} \times 0.02^0 \times (1-0.02)^{50-0} + \binom{50}{1} \times 0.02^1 \times (1-0.02)^{50-1}\right] \\ &\quad + \binom{25}{2} \times 0.05^2 \times (1-0.05)^{25-2} \times \binom{50}{0} \times 0.02^0 \times (1-0.02)^{50-0} \\ &\overset{p=AQL}{\simeq} 0.253996 \end{aligned}$$

- **Probability of accepting the lot in the double sampling plan**

$$\begin{aligned} P_a(p) &\overset{(13.18)}{=} P_a^I(p) + P_a^{II}(p) \\ &\simeq 0.603465 + 0.253996 \\ &= 0.857461 \end{aligned}$$

- **Conclusion**

$$\begin{aligned} (AQL, \alpha) &= (0.02, 1 - 0.857461) \\ &= (0.02, 0.142536). \end{aligned}$$

[1] According to the text: the lot is accepted after having collected the first sample if this sample has no defective items; the lot is accepted after having collected the second sample if the total of defective items in the two samples does not exceed 2.

(b) *Admita agora que o revendedor e o produtor passam a efectuar rectificação da inspecção em qualquer plano de amostragem que venham a adoptar.*
Obtenha e compare a qualidade média à saída do plano de amostragem dupla definido anteriormente com o de um plano de amostragem simples caracterizado por $n = 200$ *e* $c = 2$*, quando a fracção de peças defeituosas é igual a* $AQL = 0.02$. Comente. (1.5)

- **Double sampling plan (for attributes)**
  $N = 2000$ (lot size)
  $n_1 = 25$, $n_2 = 50$ (sample sizes)
  $c_1 = 0$, $c_2 = 2$ (acceptance numbers)

- **Average outgoing quality (AOQ) of a double sampling plan with rectifying inspection**

$$\begin{aligned} AOQ_{DSP}(p) &\overset{(13.26)}{=} \frac{p\left[(N - n_1)\, P_a^I(p) + (N - n_1 - n_2)\, P_a^{II}(p)\right]}{N} \\ &\overset{(a)}{\simeq} \frac{0.02 \times [(2000-25) \times 0.603465 + (2000-25-50) \times 0.253996]}{2000} \\ &= 0.016808 \end{aligned}$$

- **Single sampling plan (for attributes)**
  $N = 2000$ (lot size)
  $n = 200$ (sample size)
  $c = 2$ (acceptance number)

- **Auxiliary r.v. and its approximate distribution**

$$\begin{aligned} D &= \text{number of defective units in the sample} \\ &\overset{a}{\sim} \text{Binomial}(n, p) \end{aligned}$$

- **Average outgoing quality (AOQ) of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQ_{SSP}(p) &\overset{(13.14)}{=} \frac{p\,(N-n)\, P_{a,SSP}(p)}{N} \\ &= \frac{p\,(N-n)\, P(D \leq c)}{N} \\ &\simeq \frac{p\,(N-n) \times F_{Binomial(n,p)}(c)}{N} \\ &= \frac{p\,(N-n)}{N} \times \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}) \\ &\overset{c=2}{=} \frac{p\,(N-n)}{N} \times \left[(1-p)^n + np(1-p)^{n-1} + \frac{n(n-1}{2} p^2 (1-p)^{n-2}\right] \\ &\simeq \frac{0.02 \times (2000-200)}{2000} \times (0.017588 + 0.071788 + 0.145773) \\ &= 0.018 \times 0.235148 \\ &\simeq 0.004233 \end{aligned}$$

- **Comment**
  If we decide to adopt the sampling with the smallest AOQ (resp. with larger probability of lot acceptance) when $p = AQL = 0.02$, then the single (resp. double) sampling plan with rectifying inspection is the best choice because
  $$AOQ_{SSP}(0.02) = 0.004233 < AOQ_{DSP}(p) = 0.016808$$
  (resp. $P_{a,DSP}(0.02) = 0.857461 > P_{a,SSP}(0.02) = 0.235148$).