Fiabilidade e Controlo de Qualidade — LMAC

Exame de 2a. Época / 2o. Teste — 2o. Semestre – 2006/07

Duração: 3 horas / 1 hora e 30 minutos — 16/Julho/07 (2a. feira) – V1.08, 13h

1. *Um veículo militar foi sujeito a uma rodagem de* $36000Km$*, tendo sido registadas 12 falhas eléctricas graves ao Km 63, 114, 14820, 16105, 17393, 18707, 19179, 22642, 23128, 24145, 33832 e 34345.*

   (a) *Após ter identificado o teste de vida em questão, averigue a adequação do modelo exponencial a este conjunto de dados censurados, considerando para o efeito um nível de significância de 1%.* (1.5)

- **Life test**
  Since the test was scheduled to end after exactly $t_0 = 36000Km$ and the exercise mentions just one military vehicle submitted to this life test, we are dealing with a
  - Type I/item censored testing with replacement.

- **R.v.**
  $T_{(i)}$ = time (in Km) of the i^{th} serious electrical failure of the military vehicle
  $Z_i = T_{(i)} - T_{(i-1)}$ = run time (in Km) between the i^{th} and $(i-1)^{th}$ serious electrical failure of the military vehicle
  $Z_i \stackrel{i.i.d.}{\sim} Z,\ i \in I\!N$

- **Censored data**
  $n = 1$
  $r = 12$ electrical failures during the life test
  $(t_{(1)}, \ldots, t_{(r)}) = (63, 114, 14820, 16105, 17393, 18707, 19179, 22642, 23128, 24145, 33832, 34345)$

- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= n \times t_0\\ &= 1 \times 3600\\ &= 3600Km\end{aligned}$$

- **Hypotheses**
  $H_0 : Z \sim \text{Exponential}(\lambda)$
  $H_1 : Z \sim \text{Weibull}(\lambda^{-1}, \alpha),\ \alpha \neq 1$

- **Significance level**
  $\alpha_0 = 1\%$

- **Test statistic** (Bartlett's test)

$$\begin{aligned}B_r &\stackrel{(5.19)}{=} \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r Z_i}{r}\right) - \frac{1}{r}\sum_{i=1}^r \ln(Z_i)\right]\\ &\stackrel{a}{\sim}_{H_0} \chi^2_{(r-1)}\end{aligned}$$

- **Rejection region of** H_0

$$\begin{aligned}W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right)\\ &\stackrel{r=12,\,\alpha_0=0.01}{=} (0, 2.603) \cup (26.76, +\infty)\end{aligned}$$

- **Decision**
  The observed value of the test statistic is

$$\begin{aligned}b_r &= \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r z_i}{r}\right) - \frac{1}{r}\sum_{i=1}^r \ln(z_i)\right]\\ &= \frac{2\times 12}{1+\frac{12+1}{6\times 12}} \times \left[\ln\left(\frac{34345}{12}\right) - \frac{1}{12}82.008\right]\\ &= 22.877\\ &\notin (0, 2.603) \cup (26.76, +\infty),\end{aligned}$$

  therefore we should not reject H_0 for any significance level $\alpha \leq 1\%$.

   (b) *Considerando as hipóteses de trabalho que entender convenientes, obtenha uma estimativa pontual e outra intervalar (esta ao nível de confiança de 95%) para o número esperado de km entre duas falhas eléctricas graves consecutivas.* (1.5)

- **Distribution assumption**
  $Z_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda),\ i = 1, \ldots, 25$

- **Unknown parameter**
  λ

- **ML estimate of** λ

$$\begin{aligned}\hat{\lambda} &\stackrel{Table\ 5.13}{=} \frac{r}{\tilde{t}}\\ &= \frac{12}{36000}\\ &= \frac{1}{3000}\end{aligned}$$

- **Confidence interval for** λ

$$\begin{aligned}CI_{(1-\alpha)\times 100\%}(\lambda) &\stackrel{Table\ 5.16}{=} [\lambda_L,\ \lambda_U]\\ &= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}};\ \frac{F^{-1}_{\chi^2_{(2r)}}(1-\alpha/2)}{2\times\tilde{t}}\right]\\ CI_{90\%}(\lambda) &\stackrel{a)}{=} \left[\frac{F^{-1}_{\chi^2_{(24)}}(0.025)}{2\times 36000};\ \frac{F^{-1}_{\chi^2_{(24)}}(0.975)}{2\times 36000}\right]\\ &= \left[\frac{12.40}{2\times 36000}, \frac{41.92}{2\times 36000}\right]\end{aligned}$$

- **Another unknown parameter**
  $E(Z_i) = \frac{1}{\lambda}$, which is a decreasing function of $\lambda > 0$.

- **ML estimate of** $E(Z_i) = \frac{1}{\lambda}$
  Due to the invariance property of the MLE, we get:

$$\begin{aligned}\widehat{E(Z_i)} &= \frac{1}{\hat{\lambda}}\\ &= \frac{36000}{12}\\ &= 3000km\end{aligned}$$

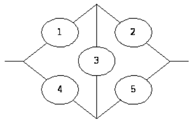
- **Confidence interval for** $E(Z_i) = \frac{1}{\lambda}$

$$\begin{aligned}CI_{95\%}(1/\lambda) &= [1/\lambda_U;\ 1/\lambda_L]\\ &= [2\times 36000/41.92;\ 2\times 36000/12.40]\\ &\simeq [1717.56;\ 5806.45].\end{aligned}$$

2. *Um motor a gasóleo depende criticamente de um sistema em ponte constituído por peças que funcionam de modo independente e possuem durações (em meses) com distribuição comum lognormal($\mu^* = 1, \sigma^2 = 2^2$).*

(a) *Como caracteriza a duração das peças e daquele sistema em ponte quanto ao envelhecimento estocástico?* (1.0)

- **Bridge system**



- **Duration**
  T = duration of the bridge system
- **Durations of the components**
  T_i = duration of the i^{th} component of the bridge system, $i = 1, \ldots, 5$
  $T_i \overset{i.i.d.}{\sim}$ Lognormal($\mu^* = 1, \sigma^2 = 2^2$), $i = 1, \ldots, 5$
- **Stochastic ageing of T**
  According to page 97 of the lecture notes, $T_i \overset{i.i.d.}{\not\sim} IHR$, $i = 1, \ldots, 5$ because the hazard rate function of the lognormal distribution is increasing and then decreasing. Thus, even though T result from a coherent operation on the i.i.d. r.v. T_i, nothing can be said about the stochastic ageing of T.

(b) *Sabendo de antemão que as peças possuem distribuição $NBUE$, obtenha um limite inferior para o valor esperado da duração do sistema em ponte.*
Adiante também um limite inferior o mais estrito possível para a fiabilidade do sistema para um período de 25 meses. (2.5)

- **Stochastic ageing**
  $T_i \overset{i.i.d.}{\sim} NBUE$, $i = 1, \ldots, 5$
- **Lower bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - minimal path sets

$$
\begin{aligned}
\mathcal{P}_1 &= \{1,2\} \\
\mathcal{P}_2 &= \{4,5\} \\
\mathcal{P}_3 &= \{1,3,5\} \\
\mathcal{P}_4 &= \{2,3,4\} \\
p &= 4 \text{ minimal path sets}
\end{aligned}
$$

  - the n components have durations $T_i \overset{i.i.d.}{\sim} NBUE$, $i = 1, \ldots, 5$;
  - the expected value of the duration of each of the n components is equal to

$$
\begin{aligned}
\mu^* &= \mu_i \\
&= E\left[\text{Lognormal}\left(\mu^*, \sigma^2\right)\right] \\
&\overset{p.\ 96)}{=} E\left[e^{-(-1)X}\right], \text{ where } X \sim \text{Normal}(\mu^*, \sigma^2) \\
&\overset{Table\ 4.3}{=} e^{-(-1)\mu^* + \frac{(-1)^2\sigma^2}{2}} \\
&= e^{\mu^* + \frac{\sigma^2}{2}} \\
&= e^{1+\frac{2^2}{2}} \\
&\simeq 20.085537, \ i = 1, \ldots, 5.
\end{aligned}
$$

Now, we can apply Theorem 3.65, and conclude that

$$
\begin{aligned}
\mu &= E(T) \\
&\geq \max_{j=1,\ldots,p}\left\{\left(\sum_{i \in \mathcal{P}_j} \mu_i^{-1}\right)^{-1}\right\} \\
&\overset{\mu_i = \mu^*}{=} \max_{j=1,\ldots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\} \\
&= \max_{j=1,\ldots,p}\left\{\frac{\mu^*}{\#\mathcal{P}_j}\right\} \\
&= \frac{\mu^*}{\min_{j=1,\ldots,p} \#\mathcal{P}_j} \\
&\simeq \frac{20.085537}{2} \\
&\simeq 10.042769.
\end{aligned}
$$

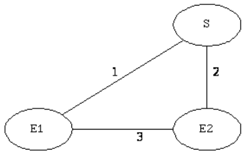
- **Lower bound for $R_T(t)$**
  This coherent system is also characterized by components with independent — and therefore positively associated — durations, with common reliability functions

$$
\begin{aligned}
R_i(t) &= R(t) \\
&= R_{Lognormal(\mu^*,\sigma^2)}(t) \\
&\overset{(4.19)}{=} 1 - \Phi\left[\frac{\ln(t) - \mu^*}{\sigma}\right].
\end{aligned}
$$

As a consequence, we can apply Theorem 2.22, and state that

$$
\begin{aligned}
R_T(t) &\geq \max_{j=1,\ldots,p}\left\{\prod_{i \in \mathcal{P}_j} R_i(t)\right\} \\
&\overset{R_i(t)=R(t)}{=} \max_{j=1,\ldots,p}\left\{[R(t)]^{\#\mathcal{P}_j}\right\} \\
&\overset{R(t)\in[0,1]}{=} [R(t)]^{\min_{j=1,\ldots,p} \#\mathcal{P}_j} \\
&= [R(t)]^2 \\
&= \left\{1 - \Phi\left[\frac{\ln(t) - \mu^*}{\sigma}\right]\right\}^2 \\
&\overset{t=25,\ etc.}{=} \left\{1 - \Phi\left[\frac{\ln(25) - 1}{2}\right]\right\}^2 \\
&\simeq [1 - \Phi(1.11)]^2 \\
&\simeq (1 - 0.8655)^2 \\
&\simeq 0.017822.
\end{aligned}
$$

3. *Uma rede de comunicação é composta por um servidor S e duas estações periféricas (E_1, E_2) dispostos da seguinte forma:*



Saliente-se também que os 3 canais funcionam de modo independente com fiabilidade $p = 0.95$. Refira-se por fim que esta rede de comunicação falha desde que o servidor fique isolado ou pelo menos uma das estações fique isolada do servidor.

(a) *Obtenha a fiabilidade desta rede de comunicação.* (1.5)

- **System**
  This coherent system fails as long as:
  - the server is isolated; or
  - at least one workstation is not connected to the server.

  Moreover, this system is nothing but a $2 - \text{out of} - 3$ system.
- **Structure function**
  According to result (1.6),
  $$\phi(\underline{X}) = \begin{cases} 1, & \sum_{i=1}^{3} X_i \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$
  where $X_i \stackrel{i.i.d.}{\sim} \text{Bernoulli}(p_i = p = 0.95)$, $i = 1, 2, 3$, and, thus, $\sum_{i=1}^{3} X_i \sim \text{Binomial}(3, p)$.
- **Reliability**
  $$\begin{aligned} r(\underline{p}) &= E[\phi(\underline{X})] \\ &= P\left(\sum_{i=1}^{3} X_i \geq 2\right) \\ &= P\left(3 - \sum_{i=1}^{3} X_i \leq 3 - 2\right) \\ &= F_{Binomial(3,1-0.95)}(3-2) \\ &= F_{Binomial(3,0.05)}(1) \\ &\stackrel{table}{=} 0.9927. \end{aligned}$$

(b) *Admita agora que os 3 canais possuem fiabilidades $p_1 \geq p_2 \geq p_3$.*
Compare a importância da fiabilidade dos canais 1 e 2. (2.0)

- **New reliabilities of the components**
  $p_1 \geq p_2 \geq p_3$
- **Minimal path sets**
  $$\begin{aligned} \mathcal{P}_1 &= \{1, 2\} \\ \mathcal{P}_2 &= \{1, 3\} \\ \mathcal{P}_3 &= \{2, 3\} \\ p^* &= 3 \text{ minimal path sets} \end{aligned}$$
- **Structure function**
  Note that, for $X_i \sim \text{Bernoulli}(p_i)$, $X_i^k \stackrel{d}{\sim} X_i$. Thus:
  $$\begin{aligned} \phi(\underline{X}) &\stackrel{(1.13)}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) \\ &= 1 - (1 - X_1X_2)(1 - X_1X_3)(1 - X_2X_3) \\ &= 1 - (1 - X_1X_2 - X_1X_3 - X_1^2X_2X_3)(1 - X_2X_3) \\ &= 1 - (1 - X_1X_2 - X_1X_3 - X_1^2X_2X_3 - X_2X_3 + X_1X_2^2X_3 + X_1X_2X_3^2 \\ &\quad + X_1^2X_2^2X_3^2) \\ &\stackrel{d}{=} X_1X_2 + X_1X_3 + X_1X_2X_3 + X_2X_3 - X_1X_2X_3 - X_1X_2X_3 - X_1X_2X_3 \\ &= X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3 \end{aligned}$$
- **Reliability**
  Taking into account that $X_i \stackrel{indep}{\sim} \text{Bernoulli}(p_i)$, we obtain:
  $$\begin{aligned} r(\underline{p}) &= E[\phi(\underline{X})] \\ &= E(X_1X_2 + X_1X_3 + X_2X_3 - 2X_1X_2X_3) \\ &= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3 \end{aligned}$$
- **Importance of the reliability of channels 1 and 2**
  $$\begin{aligned} I_r(1) &\stackrel{(1.29)}{=} \frac{\partial r(\underline{p})}{\partial p_1} \\ &= p_2 + p_3 - 2p_2p_3 \\ I_r(2) &= \frac{\partial r(\underline{p})}{\partial p_2} \\ &= p_1 + p_3 - 2p_1p_3 \end{aligned}$$
- **Comparing the importance of the reliability of channels 1 and 2**
  Since $p_1 \geq p_2$ we can conclude that:
  $$\begin{aligned} I_r(1) &= p_2(1 - 2p_3) + p_3 \\ &\leq p_1(1 - 2p_3) + p_3 \\ &= I_r(2). \end{aligned}$$

4. *Identifique e disserte sobre uma alteração que tenha ocorrido no início do sec. XIX e que considere relevante para o controlo de qualidade.* (0.5)

- **Relevant change in the (beginning of the) XIX century — Industrial Revolution**
  The *craftsmanship model* was replaced by the *factory system*.[1]
  In the *craftsmanship model*, young boys learned a skilled trade while serving as an apprentice to a master, often for many years. Since most craftsmen sold their goods locally, each had a tremendous personal stake in meeting customers needs for quality. If quality needs werent met, the craftsman ran the risk of losing customers not easily replaced. Therefore, masters maintained a form of quality control by inspecting goods before sale.
  The *factory system* began to divide the craftsmens trades into specialized tasks. This forced craftsmen to become factory workers and forced shop owners to become production supervisors, and marked an initial decline in employees sense of empowerment and autonomy in the workplace. Quality in the factory system was ensured through the skill of laborers supplemented by audits and/or inspections; defective products were either reworked or scrapped.

[1] The following paragraphs are cited from http://asq.org/learn-about-quality/history-of-quality/overview/industrial-revolution.html

5. *A densidade da madeira de certa espécie de árvore quando medida a uma altura fixa pode considerar-se uma característica de qualidade com distribuição normal com valor e desvio-padrão ideiais iguais a* $\mu_0 = 650 Kg/m^3$ *e* $\sigma_0 = 15 Kg/m^3$.

Uma vez que o preço da madeira cresce com a densidade da mesma é fundamental que sejam controladas quer diminuições do valor esperado μ, *quer aumentos da variância* σ^2. *Para o efeito o produtor de madeira (que possui formação estatística) propõe a recolha regular de amostras de dimensão fixa e igual a* $n = 4$ *bem como o recurso a duas cartas individuais do tipo Shewhart para* μ *e para* σ *com números esperados de amostras recolhidas sob controlo iguais a 500 e 1000 (respectivamente).*

(a) *Obtenha a mediana do número de amostras recolhidas até falso alarme deste esquema conjunto.* (1.5)

- **Quality characteristic**
  X = wood density of a tree measured at a certain height
  $X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistics**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  S^2_N = variance of the N^{th} random sample of size n, $N \in I\!N$

- **Distributions**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 650 Kg/m^3$, $\sigma_0 = 15 Kg/m^3$ and $n = 4$
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0^2}{n}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta < 0$) represents the magnitude of the shift (decrease!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ
  $\frac{(n-1)S^2_N}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
  $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL

- **Control limits of the lower one-sided $\bar{X}-$ chart and the upper one-sided S^2- chart**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = +\infty$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probabilities of triggering false alarms**
  Taking into account the given in-control ARL of the individual charts and the geometric character of the associated run lengths, we have:

$$\xi_\mu(0,1) = \frac{1}{ARL_\mu(0,1)} = \frac{1}{500} = 0.002$$

$$\xi_\sigma(\theta) = \frac{1}{ARL_\sigma(0,1)} = \frac{1}{1000} = 0.001.$$

  The joint scheme triggers a signal if either of the individual charts triggers an alarm; moreover, the control statistics of the individual charts are independent given (δ, θ). As a consequence, the joint scheme for μ and σ triggers a false alarm with probability equal to:

$$\begin{aligned}\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } \bar{S}^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \delta, \theta\right)\\ &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta)\\ &\overset{(\delta,\theta)=(0,1)}{=} 0.002 + 0.001 - 0.002 \times 0.001\\ &= 0.002998.\end{aligned}$$

- **Run length**
  The in-control run length of this joint scheme for μ and σ, $RL_{\mu,\sigma}(0,1)$, has the following distribution:
  $RL_{\mu,\sigma}(0,1) \sim \text{Geometric}(\xi_{\mu,\sigma}(0,1))$.

- **Median of $RL_{\mu,\sigma}(0,1)$**

$$\begin{aligned}F^{-1}_{RL_{\mu,\sigma}(0,1)}(p) &\overset{Table\ 9.2}{=} \inf\left\{m \in I\!N : F_{RL_{\mu,\sigma}(0,1)}(m) \geq p\right\}\\ &= 1 - [1 - \xi_{\mu,\sigma}(0,1)]^m \geq p\\ &= [1 - \xi_{\mu,\sigma}(0,1)]^m \leq 1 - p\\ &= m \times \ln[1 - \xi_{\mu,\sigma}(0,1)] \leq \ln(1-p)\\ &\overset{\ln[1-\xi_{\mu,\sigma}(0,1)]<0}{=} m \geq \frac{\ln(1-p)}{\ln[1 - \xi_{\mu,\sigma}(0,1)]}\\ &\overset{p=0.5}{=} m \geq \frac{\ln(1-0.5)}{\ln(1-0.002998)}\\ &= m \geq 230.856\\ &= 231\end{aligned}$$

(b) *Admita que ocorreu diminuição de* μ_0 *para* $\mu_1 = 642.5 Kg/m^3$ *e que a variância está sob controlo. Obtenha a probabilidade de emissão de sinal válido pelo esquema conjunto aquando da recolha da* $m-$*ésima amostra sabendo que qualquer das* $m-1$ *amostras anteriores não foi responsável por um sinal válido. Comente.* (2.0)

- **Shift**
  A shift from $\mu_0 = 650$ to $\mu_1 = 642.5$ has occurred (and the variance is in control), that is:

$$\begin{aligned}(\delta,\theta) &= \left(\frac{\mu_1 - \mu_0}{\frac{\sigma_0}{\sqrt{n}}}, 1\right)\\ &= \left(\frac{642.5 - 650}{\frac{15}{\sqrt{4}}}, 1\right)\\ &= (-1, 1),\end{aligned}$$

- **Probabilities of signal by the individual charts**
  Taking into account the distributions of the control statistics, the individual charts for μ and σ trigger signals with probabilities equal to:

$$\begin{aligned}\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta, \theta\right)\\ &\overset{UCL_\mu=+\infty}{=} \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\\ &= \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right), \ \delta \leq 0, \ \theta \geq 1,\end{aligned}$$

  where $\gamma_\mu : ARL_\mu(0,1) = 500$, that is, $\gamma_\mu = \Phi^{-1}(1 - 0.002) \overset{table}{=} 2.8782$, thus,

$$\begin{aligned}\xi_\mu(\delta,\theta) &\overset{(\delta,\theta)=(-1,1)}{=} \Phi\left[\frac{-2.8782 - (-1)}{1}\right]\\ &\simeq \Phi(-1.88)\\ &\overset{table}{=} 0.0301;\end{aligned}$$

and

$$\xi_\sigma(\theta) \overset{\theta=1}{=} \xi_\sigma(1) = 0.001.$$

- **Probability of a signal by the joint scheme for μ and σ**

$$\begin{aligned}\xi_{\mu,\sigma}(\delta,\theta) &= \xi_\mu(\delta,\theta)+\xi_\sigma(\theta)-\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta)\\ &\overset{table}{=} 0.0301+0.001-0.0301\times 0.001\\ &= 0.0310699.\end{aligned}$$

- **Run length of the joint scheme for μ and σ**
  $RL_{\mu,\sigma}(-1,1) \sim \text{Geometric}(\xi_{\mu,\sigma}(-1,1))$.
- **Requested probability**

$$\begin{aligned}\frac{P\left[RL_{\mu,\sigma}(\delta,\theta)=m\right]}{P\left[RL_{\mu,\sigma}(\delta,\theta)\geq m\right]} &= \lambda_{RL_{\mu,\sigma}(\delta,\theta)}(m)\\ &\overset{Table\ 9.2}{=} \xi_{\mu,\sigma}(\delta,\theta)\\ &\overset{(\delta,\theta)=(-1,1)}{=} 0.0310699.\end{aligned}$$

- **Comment**
  The requested probability os constant because the run length has a distribution with the lack of memory property.

(c) *Admita agora que ocorreu aumento de μ_0 para $\mu_1 = 657.5Kg/m^3$, que a variância continua sob controlo e que a carta para μ emitiu sinal. Após ter classificado este sinal, obtenha a sua probabilidade de ocorrência.* (1.0)

- **Shift**
  A shift from $\mu_0 = 650$ to $\mu_1 = 657.5$ has occurred (and the variance is in control), that is:

$$\begin{aligned}(\delta,\theta) &= \left(\frac{\mu_1-\mu_0}{\frac{\sigma_0}{\sqrt{n}}},1\right)\\ &= \left(\frac{657.5-650}{\frac{15}{\sqrt{4}}},1\right)\\ &= (+1,1),\end{aligned}$$

- **Classifying the signal**
  In the presence of this shift — an increase in μ —, the lower one-sided chart for μ has triggerred a signal. This is:
  - valid signal;
  - but also a misleading signal (because we are dealing with a one-sided chart for μ and this chart has signaled an increase in μ).
- **Requested probability**

$$\begin{aligned}\xi_\mu(\delta,\theta) &\overset{a)}{=} \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\\ &\overset{(\delta,\theta)=(+1,1)}{=} \Phi\left(\frac{-2.8782-1}{1}\right)\\ &\simeq \Phi(-3.88)\\ &\overset{table}{=} 0.0001.\end{aligned}$$

6. *Considere que os aumentos do número esperado de participações diárias de acidentes de viação são controlados por uma carta CUSUM unilateral superior sem head start e com $UCL = 1$.*

   Uma gestora da companhia de seguros que recebe tais participações considerou que o valor nominal de λ e valor de referência eram iguais a $\lambda_0 = 10$ e $k = 15$ (respectivamente).

   Após ter enunciado uma hipótese de trabalho que entender razoável, determine o número esperado de amostras até falso alarme desta carta. Comente. (2.5)

- **Quality characteristic**
  Y = number of notifications of road accidents
- **Distribution assumption**
  Y = number of notifications of road accidents on the N^{th}, $N \in I\!N$
  $Y_N \sim \text{Poisson}(\lambda = \lambda_0)$, IN CONTROL, where $\lambda_0 = 10$
  $Y_N \sim \text{Poisson}(\lambda = \lambda_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of the shift in λ
- **Upper one-sided CUSUM chart for Poisson data**
  $LCL = 0$
  $UCL = x = 1$
  $k = 15$
  $u = 0$ (no head-start)
- **Control statistic**

$$Z_N = \begin{cases} 0, & N=0\\ \max\{0, Z_{N-1}+(Y_N-k)\}, & N\in I\!N,\end{cases}$$

- **In control ARL**
  According to Table 10.3,

$$ARL^0(0) = \underline{e}_0^\top \times [\mathbf{I}-\mathbf{Q}(0)]^{-1}\times \underline{1},$$

where:

$$\begin{aligned}\underline{e}_0 &= \begin{bmatrix}1\\0\end{bmatrix};\\ \underline{1} &= \begin{bmatrix}1\\1\end{bmatrix};\\ \mathbf{I} &= \begin{bmatrix}1&0\\0&1\end{bmatrix};\\ \mathbf{Q}(0) &\overset{(10.8),(10.10)}{=} \begin{bmatrix}F_{Poisson(\lambda_0)}(k) & P_{Poisson(\lambda_0)}(k+1)\\ F_{Poisson(\lambda_0)}(k-1) & P_{Poisson(\lambda_0)}(k)\end{bmatrix}\\ &= \begin{bmatrix}F_{Poisson(10)}(15) & F_{Poisson(15)}(16)-F_{Poisson(10)}(15)\\ F_{Poisson(10)}(14) & F_{Poisson(15)}(15)-F_{Poisson(10)}(14)\end{bmatrix}\\ &\overset{table}{=} \begin{bmatrix}0.9513 & 0.9730-0.9513\\ 0.9165 & 0.9513-0.9165\end{bmatrix}\\ &= \begin{bmatrix}0.9513 & 0.0217\\ 0.9165 & 0.0348\end{bmatrix}\\ \mathbf{I}-\mathbf{Q}(0) &= \begin{bmatrix}0.0487 & -0.0217\\ -0.9165 & 0.9652\end{bmatrix}\end{aligned}$$

$$[\mathbf{I} - \mathbf{Q}(0)]^{-1} = \frac{1}{0.0487 \times 0.9652 - 0.0217 \times 0.9165} \times \begin{bmatrix} 0.9652 & 0.0217 \\ 0.9165 & 0.0487 \end{bmatrix} = \begin{bmatrix} 35.5937 & 0.8002 \\ 33.7977 & 1.7959 \end{bmatrix}.$$

- **Conclusion**

$$\begin{aligned} ARL^0(0) &= \underline{e}_0^\top \times [\mathbf{I} - \mathbf{Q}(0)]^{-1} \times \underline{1} \\ &= [1 \quad 0] \times \begin{bmatrix} 35.5937 & 0.8002 \\ 33.7977 & 1.7959 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 36.3939. \end{aligned}$$

- **Comment**
  $ARL^0(0)$ is rather small,[2] so the value of UCL should be increased in order to achieve an in-control ARL of at least 200 samples.

7. *Uma empresa — que importa certo produto alimentar inspecciona lotes de* $N = 10000$ *latas desse produto — recorre a um plano de amostragem simples com rectificação da inspecção e que faz uso de amostras de dimensão* $n = 100$ *e constante de aceitação* $c = 1$.

(a) *Determine o AOQL (*Average Outgoing Quality Limit*) deste plano de amostragem.* (1.5)

- **Single sampling plan (for attributes)**
  $N = 10000$ (lot size)
  $n = 100$ (sample size)
  $c = 1$ (acceptance number)

- **Auxiliary r.v. and its approximate distribution**
  D = number of defective units in the sample $\stackrel{a}{\sim}$ Binomial(n, p)

- **Average outgoing quality (AOQ) of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQ(p) &\stackrel{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\ &\simeq p \times P_a(p) \\ &= p \times P(D \leq c) \\ &\simeq p \times F_{Binomial(n,p)}(c) \\ &\stackrel{c=1}{=} p \times \left[\binom{100}{0} \times p^0 \times (1-p)^{100-0} + \binom{100}{1} \times p^1 \times (1-p)^{100-1}\right] \\ &= p \times (1-p)^{99} \times [(1-p) + 100p] \\ &= p \times (1-p)^{99} \times (1 + 99p) \end{aligned}$$

- **Average outgoing quality limit (AOQL) of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQL &= \max_{p \in [0,1]} AOQ(p) \\ &= AOQ(p^*), \end{aligned}$$

where

$$p^* \in [0,1] \; : \; \begin{cases} \left.\frac{d\,AOQ(p)}{dp}\right|_{p=p^*} = 0 \\ \left.\frac{d^2\,AOQ(p)}{dp^2}\right|_{p=p^*} < 0 \end{cases}$$

$$\begin{cases} (1-p^*)^{99} \times (1+99p^*) - 99p^* \times (1-p)^{98} \times (1+99p^*) + 99p^* \times (1-p^*)^{99} = \\ --- \end{cases}$$

$$\ldots$$

$$\begin{cases} 9999(p^*)^2 - 98p^* - 1 = 0 \\ --- \end{cases}$$

$$\begin{cases} p^* = \frac{98 \pm \sqrt{98^2 + 4 \times 9999}}{2 \times 9999} \\ --- \end{cases}$$

- **Conclusion**

$$\begin{aligned} p^* &\simeq 0.016037 \\ AOQL &= AOQ(p^*) \\ &\simeq p^* \times (1-p^*)^{99} \times (1+99p^*) \\ &\simeq 0.008290. \end{aligned}$$

(b) *Obtenha o número esperado de unidades inspeccionadas quando a fracção de peças defeituosas é igual a* $p = 0.01$. (1.0)

- **Average Total Inspection (ATI) of the single sampling plan with rectifying inspection**

$$\begin{aligned} ATI(p) &\stackrel{(13.15)}{=} n \times P_a(p) + N \times [1 - P_a(p)] \\ &= N - (N-n) \times P_a(p) \\ &\simeq N - (N-n) \times F_{Binomial(n,p)}(c) \\ &\stackrel{a)}{=} N - (N-n) \times (1-p)^{99} \times (1+99p) \\ &\stackrel{p=0.01,\,etc.}{=} 1000 - (10000-100) \times (1-0.01)^{99} \times (1 + 99 \times 0.01) \\ &\simeq 2715.96. \end{aligned}$$

[2] Thus, false alarms are very frequent.