Fiabilidade e Controlo de Qualidade LMAC

Exame de 1a. Época / 2o. Teste — 2o. Semestre – 2006/07

Duração: 3 horas / 1 hora e 30 minutos — 25/Junho/07 (2a. feira) – P12, 13h

1. *Dez juntas de borracha foram sujeitas simultaneamente a um teste de vida acelerado. O registo do número de ciclos até falha conduziu aos seguintes resultados: 20400, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000.*

   (a) *Exemplifique a utilização de um gráfico TTT com alguns cálculos (por exemplo três pontos).*
   Que aspecto esperaria para este gráfico, caso se desconfiasse da adequação de um modelo com taxa de falha crescente a este conjunto de dados? (1.5)

   - **Failure times**
     T_i = time (in cycles) to failure of specimen i, $i = 1, \dots, 10$
   - **Complete data**
     $\underline{t} = (20400, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000)$
   - **Total time on test up to time $t_{(i)}$**

$$\begin{aligned}\tau\left(t_{(i)}\right) &= \sum_{j=1}^{i}(n-j+1)\left[t_{(j)}-t_{(j-1)}\right]\\ &= \tau\left(t_{(i-1)}\right)+(n-i+1)\left[t_{(i)}-t_{(i-1)}\right]\end{aligned}$$

   - **Abcissae of the TTT plot**
     $\frac{i}{n}$, $i = 0, 1, \dots, n$
   - **Ordinates of the TTT plot**
     $\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$, $i = 1, \dots, n$; equal to 0, for $i = 0$.
   - **Four points of the TTT plot**

i	$\tau\left(t_{(i)}\right)=\tau\left(t_{(i-1)}\right)+(n-i+1)\left[t_{(i)}-t_{(i-1)}\right]$	$\frac{i}{n}$	$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$
0	0	0	0
1	$0+(10-1+1)\times 20400 = 204000$	$\frac{1}{10}=0.1$	$\frac{20400}{835050}\simeq 0.244$
2	$204000+(10-2+1)\times(30000-20400) = 290400$	$\frac{2}{10}=0.2$	$\frac{290400}{835050}\simeq 0.348$
3	$290400+(10-3+1)\times(50700-30000) = 456000$	$\frac{3}{10}=0.3$	$\frac{456000}{835050}\simeq 0.546$
4	$456000+(10-4+1)\times(57750-50700) = 505350$	...	...
5	$505350+(10-5+1)\times(60300-57750) = 520650$	...	...
6	$520650+(10-6+1)\times(74100-60300) = 589650$	...	...
7	$589650+(10-7+1)\times(78300-74100) = 606450$	...	...
8	$606450+(10-8+1)\times(144000-78300) = 803550$	...	...
9	$803550+(10-9+1)\times(153500-144000) = 822550$	...	...
10	$822550+(10-10+1)\times(166000-153500) = 835050$	...	...

   - **Aspect of the TTT plot**
     According to Remark 5.5, the TTT plot should be concave in case $T_i \sim IHR$, $i = 1, \dots, 10$.

   (b) *Admita agora que estas 10 observações correspondem aos registos de ciclos até falha num período fixo de observação de 175000 ciclos e dizem respeito a 20 juntas de borracha que foram sujeitas a tal teste de vida.*
   Considerando as hipóteses de trabalho que entender convenientes, obtenha estimativas de MV para o número esperado de ciclos até falha e para a função de fiabilidade para um período de 30000 ciclos. (1.5)

   - **Life test**
     Since we now admit that the test was scheduled to end after exactly $t_0 = 175000$ cycles and nothing in this exercise suggests that the $n = 20$ specimens were replaced during this life test, we are dealing with a
     - Type I/item censored testing without replacement.
   - **Distribution assumption**
     $T_i \overset{i.i.d.}{\sim}$ Exponential(λ), $i = 1, \dots, n$
   - **Unknown parameter**
     λ
   - **Censored data**
     $n = 20$ specimens
     $r = 10$ failures occurred until the end of the test after exactly $t_0 = 175000$ cycles
     $(t_{(1)}, \dots, t_{(r)}) = (20400, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000)$
   - **Cumulative total time in test**
     According to Definition 5.17, the cumulative total time in test equals

$$\begin{aligned}\tilde{t} &= \sum_{i=1}^{r} t_{(i)} + (n-r)\times t_0\\ &= 835050 + (20-10)\times 175000\\ &= 1010050.\end{aligned}$$

   - **ML estimate of λ**

$$\begin{aligned}\hat{\lambda} &\overset{Table\ 5.13}{=} \begin{cases}\frac{1}{\tilde{t}}, & r=0\\ \frac{r}{\tilde{t}}, & r=1,\dots,n\end{cases}\\ &= \frac{10}{1010050}\\ &\simeq 0.000010.\end{aligned}$$

   - **Other unknown parameters**
     $E(T_i) = \frac{1}{\lambda}$
     $R_{T_i}(30000) = e^{-30000\lambda}$
   - **ML estimates of $E(T_i) = \frac{1}{\lambda}$ and $R_{T_i}(30000) = e^{-30000\lambda}$**
     Due to the invariance property of the MLE, we get:

$$\begin{aligned}\widehat{E(T_i)} &= \frac{1}{\hat{\lambda}}\\ &= \frac{1010050}{10}\\ &\simeq 101005 \text{ cycles;}\\ \widehat{R_{T_i}}(30000) &= e^{-30000\hat{\lambda}}\\ &= e^{-30000\times\frac{10}{1010050}}\end{aligned}$$

$\simeq \quad 0.970735.$

2. *Um equipamento electrónico obsoleto possui 6 válvulas. Estas funcionam de modo independente e possuem durações (em anos) com f.d.p. comum $f(t) = 50\,t\,e^{-25t^2}, t > 0$.*

   (a) *Qual será a probabilidade de não ocorrer qualquer substituição de válvulas durante os dois primeiros meses de serviço?* (1.5)

   - **Individual durations and common p.d.f., d.f. and distribution**
     $T_i =$ duration of valve i, $i = 1,\ldots,6$
     $T_i \stackrel{i.i.d.}{\sim} f(t)$, $i = 1,\ldots,6$
     $f(t) = 50\,t\,e^{-25t^2} \stackrel{(4.21)}{=} \frac{2}{1/5}\left(\frac{t}{1/5}\right)^{2-1} \exp\left[-\left(\frac{t}{1/5}\right)^2\right]$, $t > 0$
     $R(t) \stackrel{(4.22)}{=} \exp\left[-\left(\frac{t}{1/5}\right)^2\right]$, $t > 0$
     $T_i \stackrel{i.i.d.}{\sim} \text{Weibull}\,(\delta = 1/5, \alpha = 2)$, $i = 1,\ldots,6$
   - **Time until the first replacement of a valve**
     $T = \min_{i=1,\ldots,n} T_i$
   - **Distribution of T**
     According to Exercise 4.25,
     $$T \sim \text{Weibull}\left(\delta' = \frac{\delta}{n^{1/\alpha}}, \alpha' = \alpha\right),$$
     where $\delta = \frac{1}{5}$, $n = 6$ and $\alpha = 2$.
   - **Requested probability**
     $$\begin{aligned} P\,(T > 1/6 = 2 \text{ months}) &= R_T\,(1/6) \\ &\stackrel{(4.22)}{=} \exp\left[-\left(\frac{t}{\delta'}\right)^{\alpha'}\right] \\ &= \exp\left[-\left(\frac{1/6}{\frac{1/5}{6^{1/2}}}\right)^2\right] \\ &= e^{-\frac{25}{6}} \\ &\simeq 0.015504. \end{aligned}$$

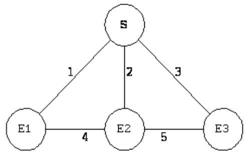
   (b) *Sabendo agora que as válvulas estão dispostas em paralelo caracterize a duração deste equipamento quanto ao envelhecimento estocástico e obtenha a fiabilidade do mesmo para um período de 2 meses.*
   Compare este valor com um limite inferior para a função de fiabilidade para esse mesmo período de serviço tirando partido do facto de $\Gamma(1.5) \simeq 0.886227$. (2.5)

   - **Duration of the parallel system**
     $T_{(n)} = \max_{i=1,\ldots,n} T_i$
   - **Stochastic ageing of $T = T_{(n)}$**
     First note that $\alpha = 2 > 1$. Therefore, according to subsection 4.3.4 (see table in page 100),
     $$T_i \stackrel{i.i.d.}{\sim} IHR,\ i = 1,\ldots,n.$$
     Now, if we apply Proposition 3.23, namely result (3.14), we can conclude that
     $$T = T_{(n)} \sim IHR.$$

   - **Requested probability**
     $$\begin{aligned} P\left(T_{(n)} > 1/6\right) &= 1 - F_{T_{(n)}}\,(1/6) \\ &\stackrel{(2.3)}{=} 1 - [1 - R\,(1/6)]^n \\ &= 1 - \left[1 - \exp\left[-\left(\frac{1/6}{1/5}\right)^2\right]\right]^6 \\ &\simeq 1 - (1 - 0.499352)^6 \\ &\simeq 0.984253. \end{aligned}$$
   - **Lower bound for the reliability function $R_{T_{(n)}}(t)$**
     We are dealing with a coherent system characterized as follows:
     - the system's reliability is given by
       $$r(\underline{p}) \stackrel{(1.21)}{=} 1 - \prod_{i=1}^{n}(1 - p_i);$$
     - the n components have durations $T_i \stackrel{i.i.d.}{\sim} IHR$, $i = 1,\ldots,n$;
     - the expected value of the duration of each of the n components is equal to
       $$\begin{aligned} \mu &= E(T_i) \\ &= E\,[\text{Weibull}\,(\delta = 1/5, \alpha = 2)] \\ &\stackrel{Exer\,4.22b)}{=} \delta \times \Gamma\left(\frac{1}{\alpha} + 1\right) \\ &= \frac{1}{5} \times \Gamma(1.5) \\ &\simeq \frac{1}{5} \times 0.886227 \\ &\simeq 0.177246,\ i = 1,\ldots,6. \end{aligned}$$

     Now, we can apply Theorem 3.58, and conclude that
     $$\begin{aligned} R_{T_{(n)}}(t) &\geq r\left(e^{-\frac{t}{\mu_1}},\ldots,e^{-\frac{t}{\mu_n}}\right) \\ &= 1 - \prod_{i=1}^{n}\left(1 - e^{-\frac{t}{\mu_i}}\right) \\ &\stackrel{\mu_i=\mu}{=} 1 - \left(1 - e^{-\frac{t}{\mu}}\right)^n \\ &= 1 - \left(1 - e^{-\frac{\frac{1}{6}}{0.177246}}\right)^6 \\ &\simeq 1 - (1 - 0.390505)^6 \\ &\simeq 0.948735. \end{aligned}$$
   - **Comment**
     The lower bound we just determined is associated to a small relative error,
     $$\left(1 - \frac{0.948735}{0.984253}\right) \times 100\% = 3.608652\%,$$
     and, thus, serves as clear evidence of the quality of such a bound for $t = \frac{1}{6}$.

3. *Uma rede de comunicação é composta por um servidor S e três estações periféricas (E_1, E_2, E_3) dispostos da seguinte forma:*



Com efeito, o servidor S comunica com a estação periférica E_i $(i = 1, 2, 3)$ através de um canal com fiabilidade $p_i = p = 0.95$, ao passo que as estações periféricas comunicam entre si graças a dois canais com a mesma fiabilidade $\tilde{\pi} = 0.9$. Saliente-se também que todos os canais funcionam de modo independente. Refira-se por fim que esta rede de comunicação falha desde que o servidor fique isolado ou pelo menos uma das estações fique isolada do servidor.

(a) *Comece por listar todos os cortes mínimos e de seguida determine a função estrutura desta rede de comunicação.* (1.5)

- **System**
  This coherent system fails as long as:
  - the server is isolated; or
  - at least one workstation is not connected to the server.

- **Minimal cut sets**

$$\begin{aligned}
\mathcal{K}_1 &= \{1,2,3\}\\
\mathcal{K}_2 &= \{1,4\}\\
\mathcal{K}_3 &= \{2,4,5\}\\
\mathcal{K}_4 &= \{3,5\}\\
\mathcal{K}_5 &= \{1,2,5\}\\
\mathcal{K}_6 &= \{2,3,4\}\\
q &= 6 \text{ minimal cut sets}
\end{aligned}$$

- **Structure function**
  Since we are dealing with a coherent system, we can apply Theorem 1.30 and state that:

$$\begin{aligned}
\phi(\underline{X}) &= \prod_{j=1}^{q}\left[(1 - \prod_{i\in\mathcal{K}_j}(1 - X_i)\right]\\
&= [1-(1-X_1)(1-X_2)(1-X_3)] \times [1-(1-X_1)(1-X_4)]\\
&\quad \times[1-(1-X_2)(1-X_4)(1-X_5)] \times [1-(1-X_3)(1-X_5)]\\
&\quad \times[1-(1-X_1)(1-X_2)(1-X_5)] \times [1-(1-X_2)(1-X_3)(1-X_4)].
\end{aligned}$$

(b) *Obtenha um limite inferior e outro superior o mais estritos possível para a fiabilidade desta rede de comunicação.* (1.5)

- **Reliability of the components**
  $p_i = p = 0.95$, $i = 1, 2, 3$
  $p_i = \tilde{\pi} = 0.9$, $i = 4, 5$
  Since the 5 components form a coherent system and operate independently, we can apply Theorem 1.68. Before doing so, we have to identify the minimal path sets.

- **Minimal path sets**

$$\begin{aligned}
\mathcal{P}_1 &= \{1,2,3\}\\
\mathcal{P}_2 &= \{1,4,5\}\\
\mathcal{P}_3 &= \{2,4,5\}\\
\mathcal{P}_4 &= \{3,4,5\}\\
\mathcal{P}_5 &= \{2,3,4\}\\
\mathcal{P}_6 &= \{1,2,5\}\\
\mathcal{P}_7 &= \{1,3,5\}\\
\mathcal{P}_8 &= \{1,3,4\}\\
p^* &= 8 \text{ minimal path sets}
\end{aligned}$$

- **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned}
r(\underline{p}) &\overset{T1.68}{\geq} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathcal{K}_j}(1-p_i)\right]\\
&= \left[1-(1-p)^3\right] \times [1-(1-p)(1-\tilde{\pi})] \times \left[1-(1-p)(1-\tilde{\pi})^2\right]\\
&\quad \times [1-(1-p)(1-\tilde{\pi})] \times \left[1-(1-p)^2(1-\tilde{\pi})\right] \times \left[1-(1-p)^2(1-\tilde{\pi})\right]\\
&\overset{p=0.95,\,\tilde{\pi}=0.9}{=} 0.988912
\end{aligned}$$

- **Upper bound for the reliability**

$$\begin{aligned}
r(\underline{p}) &\overset{T1.68}{\leq} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathcal{P}_j} p_i\right)\\
&= 1-(1-p^3)\times(1-p^3)\times(1-p\tilde{\pi})\times(1-p\tilde{\pi}^2)\\
&\quad \times(1-p\tilde{\pi}^2)\times(1-p^2\tilde{\pi})\times(1-p^2\tilde{\pi})\times(1-p^2\tilde{\pi})\\
&\overset{p=0.95,\,\tilde{\pi}=0.9}{=} 0.999998.
\end{aligned}$$

- **Obs.**
  The communication network is very reliable.

4. *Uma das definições mais gerais de qualidade deve-se a Juran (1974): Quality is fitness for use.*[1] *Disserte um pouco sobre esta definição.* (0.5)

- **Comment**
  This definition by Juran emphasizes that quality begins with who, how, and why the customers will use a product.[2] Moreover, fitness for use of a product should be translated into numerical requisite/specifications.

5. *Considere que os aumentos no número esperado de notificações de casos de assédio sexual no local de trabalho — em amostras de 20 queixas de carácter diverso apresentadas a um sindicato de trabalhadores/as — são controlados por uma carta np unilateral superior com limites $LCL = 0$ e $UCL = np_0 + \gamma\sqrt{np_0(1-p_0)}$, onde o valor alvo é igual $np_0 = 1$.*

[1] *Qualidade é adequação ao uso.*
[2] Without this information any product improvement will be guesswork, i.e., all improvement activities should be customer focused.

(a) *Identifique a gama de valores de γ de modo a lidar com uma carta com número esperado de amostras recolhidas até sinal inferior a 100 quando a magnitude do* shift *é igual a* $\delta = p - p_0 = 0.05$. (1.0)

- **Control statistic**
  Y_N = number of cases of sexual harassment in the N^{th} sample of 20 complaints (of various types), $N \in \mathbb{N}$
- **Distributions**
  $Y_N \sim \text{Binomial}(n, p = p_0)$, IN CONTROL, where $n = 20$, $p_0 = 0.05$
  $Y_N \sim \text{Binomial}(n, p = p_0 + \delta)$, OUT OF CONTROL, where δ $(0 < \delta < 1 - p_0)$ represents the magnitude of the shift in p
- **Control limits of the $np-$chart**
  $LCL = 0$
  $UCL = np_0 + \gamma\sqrt{np_0(1-p_0)}$
- **Target of the $np-$chart**
  $CL = np_0 = 1$
- **Probability of triggering a signal**

$$\begin{aligned}\xi(\delta) &= P(Y_N \notin [LCL, UCL] \mid \delta)\\ &\overset{Y_N\geq 0,\, LCL=0}{=} P(Y_N > UCL \mid \delta)\\ &= 1 - F_{Binomial(n,p=p_0+\delta)}(UCL),\ 0 < \delta < 1 - p_0\end{aligned}$$

- **Run length**
  We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:

$$RL(\delta) \sim \text{Geometric}(\xi(\delta)).$$

- **Obtaining γ**

$$\begin{aligned}\gamma\ :\ & ARL(\delta) < 100 \text{ samples}\\ & \frac{1}{\xi(\delta)} < 100\\ & 1 - F_{Binomial(n,p=p_0+\delta)}(UCL) > \frac{1}{100}\\ & UCL = np_0 + \gamma\sqrt{np_0(1-p_0)} < F^{-1}_{Binomial(n,p=p_0+\delta)}\left(1 - \frac{1}{100}\right) \overset{table,\, \delta=0.05,\, etc.}{=} 6\\ & \gamma < \frac{F^{-1}_{Binomial(n,p=p_0+\delta)}\left(1-\frac{1}{100}\right) - np_0}{\sqrt{np_0(1-p_0)}}\\ & \gamma < \frac{6 - 20\times 0.05}{\sqrt{20\times 0.05\times(1-0.05)}}\\ & \gamma < 5.129892.\end{aligned}$$

- **[Obs.**
  However, note that γ should be chosen in such way that $ARL(0)$ is as large as possible. If in addition we remind the reader that Y_N is an integer r.v., and γ should be taken such that $UCL < 6$ (see the calculations above) and be also an integer, we conclude that:

$$\begin{aligned}\gamma\ :\ & UCL = 5\\ & \gamma = \frac{5 - np_0}{\sqrt{np_0(1-p_0)}}\end{aligned}$$

$$\begin{aligned}& \gamma = \frac{5 - 20\times 0.05}{\sqrt{20\times 0.05\times(1-0.05)}}\\ & \gamma = 4.103913.]\end{aligned}$$

(b) *Admita que o valor esperado np sofreu um* shift *para 2. Obtenha e comente o valor da mediana do número de amostras recolhidas até à emissão de sinal válido.* (1.0)

- **Shift**
  A shift from the target value $np_0 = 20 \times 0.05 = 1$ to $n(p_0 + \delta) = 2$ has occurred, that is, $\delta = 0.05$.
- **Probability of triggering a valid signal with $UCL = 5$**

$$\begin{aligned}\xi(\delta) &= 1 - F_{Binomial(n,p=p_0+\delta)}(UCL)\\ &= 1 - F_{Binomial(20,0.05+0.05)}(5)\\ &\overset{table}{=} 1 - 0.9887\\ &= 0.0113\end{aligned}$$

- **Median of $RL(\delta)$**

$$\begin{aligned}F^{-1}_{RL(\delta)}(p) &\overset{Table\ 9.2}{=} \inf\left\{m \in \mathbb{N} : F_{RL(\delta)}(m) \geq p\right\}\\ &= 1 - [1-\xi(\delta)]^m \geq p\\ &= [1-\xi(\delta)]^m \leq 1 - p\\ &= m \times \ln[1-\xi(\delta)] \leq \ln(1-p)\\ &\overset{\ln[1-\xi(\delta)]<0}{=} m \geq \frac{\ln(1-p)}{\ln[1-\xi(\delta)]}\\ &\overset{p=0.5}{=} m \geq \frac{\ln(1-0.5)}{\ln(1-0.0113)}\\ &= m \geq 60.993229\\ &= 61\end{aligned}$$

- **Comment**
  When the expected value of the number of sexual harassment cases duplicates, in samples of $n = 20$ complaints (of various types), the probability of triggering a valid signal within the first 61 samples is at least 50%.

(c) *Pretende utilizar-se um esquema* CUSUM *unilateral superior para a detecção de aumento do número esperado de tais notificações em amostras sucessivas de 20 queixas do valor nominal* $np_0 = 1$ *para* $np_1 = 2$.
Após ter obtido (e arredondado convenientemente) o valor de referência óptimo para o esquema acima referido, calcule as duas primeiras entradas da 1a. linha da matriz de probabilidades de transição associada ao esquema na ausência de causas assinaláveis.
Que utilidade terá aquela matriz? (1.5)

- **Reference value, k, of the upper one-sided CUSUM chart for binomial data**
  It should be the closest integer to

$$\begin{aligned}n \times \frac{\ln\left[\frac{1-p_0}{1-p_1}\right]}{\ln\left[\frac{(1-p_0)\times p_1}{(1-p_1)\times p_0}\right]} &= 20 \times \frac{\ln\left[\frac{1-0.05}{1-0.1}\right]}{\ln\left[\frac{(1-0.05)\times 0.1}{(1-0.1)\times 0.05}\right]}\\ &= 1.447168\end{aligned}$$

(see result (10.14)), that is, $k = 1$.

- **Control statistic**

$$Z_N = \begin{cases} 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in I\!N, \end{cases}$$

(see Example 10.9 and (10.4)).

- **Requested entries**

The 1st. and 2nd. entries of the 1st. row of the in control transition probability matrix, $\tilde{P}(0)$, follows from Example 10.9, namely result (10.5), and are equal to

$$\begin{aligned} F_{Binomial(n,p_0)}(k) &= F_{Binomial(20,0.05)}(1) \\ &\overset{table}{=} 0.7358 \\ P_{Binomial(n,p_0)}(k+1) &= F_{Binomial(n,p_0)}(k+1) - F_{Binomial(n,p_0)}(k) \\ &= F_{Binomial(20,0.05)}(1+1) - F_{Binomial(20,0.05)}(1) \\ &\overset{table}{=} 0.9245 - 0.7358 \\ &= 0.1887, \end{aligned}$$

respectively.

- **Use of the in-control transition probability matrix**

The matrix $\tilde{P}(0)$ governs the Markov chain $\{Z_N = Z_N, N \in I\!N\}$ and, thus, determines the distribution of the in-control RL, $RL(0)$, and can be used to derive all the RL-related quantities such as the in-control ARL, $ARL(0)$.

6. *O desvio (em graus) que a trajectória de voo de um avião faz após a descolagem em relação ao alinhamento da pista é uma característica de qualidade com distribuição sob controlo normal com valor e desvio-padrão iguais a $\mu_0 = 0^o$ e $\sigma_0 = 5^o$.*

De forma a controlar quer o valor esperado μ, quer aumentos da variância σ^2, a torre de controlo decidiu recolher com regularidade amostras de 4 descolagens e transmitir ou não alerta aos 4 pilotos após cada recolha. Mais, a torre faz uso de duas cartas individuais do tipo Shewhart com número esperado de descolagens até falso alerta igual a 4000.

(a) *Determine a probabilidade de ser emitido alerta pelo esquema conjunto o mais tardar ao fim de 40 descolagens, caso tenha ocorrido "shift" de $\mu_0 = 0^o$ para $\mu_1 = 3^o$ e de $\sigma_0 = 5^o$ para $\sigma_1 = 6.25^o$.*[3] (2.0)

- **Quality characteristic**

X = deviation between the flight trajectory of an airplane after take-off and the runway
$X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistics**

$\bar{X}_N$ = mean of the N^{th} random sample of size n
S^2_N = variance of the N^{th} random sample of size n, $N \in I\!N$

- **Distributions**

$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 0$, $\sigma_0 = 5$ and $n = 4$

$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or an increase!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ

$\frac{(n-1)S^2_N}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL

$\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL

- **Control limits of the standard $\bar{X}-$ chart and the upper one-sided S^2- chart**

$LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
$UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
$LCL_\sigma = 0$
$UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probabilities of triggering signals**

Taking into account the distributions of the control statistics, the individual charts for μ and σ trigger signals with probabilities equal to:

$$\begin{aligned} \xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \not\in [LCL_\mu, UCL_\mu] \mid \delta, \theta\right) \\ &= 1 - \left[\Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \ \delta \in I\!R, \ \theta \geq 1, \\ \xi_\sigma(\theta) &= P\left(\bar{S}^2_N \not\in [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{(n-1)UCL_\sigma}{\sigma^2}\right) \\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right), \ \theta \geq 1, \end{aligned}$$

respectively.

- **Run length of the individual charts**

We are dealing with Shewhart charts, thus, the number of samples collected until each chart triggers a signal given (δ, θ), $RL_\mu(\delta, \theta)$ and $RL_\sigma(\theta)$, have the following distribution:

$$\begin{aligned} RL_\mu(\delta,\theta) &\sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right) \\ RL_\sigma(\theta) &\sim \text{Geometric}\left(\xi_\sigma(\theta)\right). \end{aligned}$$

- **Obtaining γ_μ and γ_σ**

γ_μ : $ARL_\mu(0,1) = 1000$ samples of 4 take-offs each

$$\frac{1}{\xi_\mu(0,1)} = 1000$$

$$1 - [\Phi(\gamma_\mu) - \Phi(-\gamma_\mu)] = 0.001$$

$$\gamma_\mu = \Phi^{-1}\left(1 - \frac{0.001}{2}\right)$$

$$\gamma_\mu \overset{table}{=} 3.29$$

γ_σ : $ARL_\sigma(1) = 1000$ samples of 4 take-offs each

$$\frac{1}{\xi_\sigma(1)} = 1000$$

$$1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = 0.001$$

$$\gamma_\sigma = F^{-1}_{\chi^2_{(4-1)}}(1 - 0.001)$$

$$\gamma_\sigma \overset{table}{=} 16.27$$

[3] *Na impossibilidade de obter valor exacto da probabilidade pedida, obtenha um intervalo de valores para esta probabilidade e para as que se seguirem neste exercício.*

- **Probability of a signal by the joint scheme for μ and σ**

The joint scheme triggers a signal if either of the individual charts triggers an alarm. Moreover, the control statistics of the individual charts are independent given (δ,θ). As a consequence, the joint scheme for μ and σ triggers a signal with probability equal to:

$$\begin{aligned}\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } \bar{S}^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \delta,\theta\right)\\ &= \xi_\mu(\delta,\theta)+\xi_\sigma(\theta)-\xi_\mu(\delta,\theta)\times\xi_\sigma(\theta).\end{aligned}$$

When, a shift from $\mu_0 = 0^o$ to $\mu_1 = 3^o$ and from $\sigma_0 = 5^o$ to $\sigma_1 = 6.25^o$ have occurred, that is, when

$$\begin{aligned}(\delta,\theta) &= \left(\frac{\mu_1-\mu_0}{\frac{\sigma_0}{\sqrt{n}}}, \frac{\sigma_1}{\sigma_0}\right)\\ &= \left(\frac{3-0}{\frac{5}{\sqrt{4}}}, \frac{6.25}{5}\right)\\ &= (1.2, 1.25),\end{aligned}$$

we get:

$$\begin{aligned}\xi_\mu(\delta,\theta) &= 1-\left[\Phi\left(\frac{\gamma_\mu-\delta}{\theta}\right)-\Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\right]\\ &\overset{(\delta,\theta)=(1.2,1.25)}{=} 1-\left[\Phi\left(\frac{3.29-1.2}{1.25}\right)-\Phi\left(\frac{-3.29-1.2}{1.25}\right)\right]\\ &\simeq 1-[\Phi(1.67)-\Phi(-3.59)]\\ &\overset{table}{=} 1-(0.9525-0.0002)\\ &= 0.0477\\ \xi_\sigma(\theta) &= 1-F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right)\\ &\overset{\theta=1.25}{=} 1-F_{\chi^2_{(4-1)}}\left(\frac{16.27}{1.25^2}\right)\\ &\simeq 1-F_{\chi^2_{(3)}}(10.41)\\ &\in (0.01, 0.025)\end{aligned}$$

because

$$\begin{aligned}F^{-1}_{\chi^2_{(3)}}(0.975)=9.348 &< 10.41 < 11.34 = F^{-1}_{\chi^2_{(3)}}(0.99)\\ 1-0.99 &< \xi_\sigma(1.25) < 1-0.975.\end{aligned}$$

Then a signal is triggered by the joint scheme, when $(\delta,\theta)=(1.2,1.25)$, with probability:

$$\begin{aligned}\xi_{\mu,\sigma}(1.2,1.25) &= \xi_\mu(1.2,1.25)+[1-\xi_\mu(1.2,1.25)]\times\xi_\sigma(1.25)\\ &\in (0.0477+(1-0.0477)\times 0.01,\ 0.0477+(1-0.0477)\times 0.025)\\ &= (0.057223,\ 0.071598).\end{aligned}$$

- **Run length**

The run length of this joint scheme for μ and σ, $RL_{\mu,\sigma}(\delta,\theta)$, has the following distribution:

$$RL_{\mu,\sigma}(\delta,\theta) \sim \text{Geometric}(\xi_{\mu,\sigma}(\delta,\theta)).$$

- **Requested probability**

$$\begin{aligned}P\left[RL_{\mu,\sigma}(\delta,\theta)\le 10\right] &= 1-[1-\xi_{\mu,\sigma}(\delta,\theta)]^{10}\\ &\in \left(1-(1-0.057223)^{10}, 1-(1-0.071598)^{10}\right)\\ &= (0.445260, 0.523806).\end{aligned}$$

(b) *Obtenha a probabilidade de ocorrência de sinal erróneo de Tipo III (IV) quando $\theta = 1.25$ ($\delta = 0.25$).* (1.5)

- **Probability of a misleading signal of type III**

$$\begin{aligned}PMS_{III}(\theta) &\overset{Table\ 10.12}{=} \frac{1-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]}{[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)]^{-1}-[\Phi(\gamma_\mu/\theta)-\Phi(-\gamma_\mu/\theta)]}\\ &\overset{\theta=1.25}{=} \frac{1-[\Phi(3.29/1.25)-\Phi(-3.29/1.25)]}{[F_{\chi^2_{(3)}}(16.27/1.25^2)]^{-1}-[\Phi(3.29/1.25)-\Phi(-3.29/1.25)]}\\ &\simeq \frac{1-[\Phi(2.63)-\Phi(-2.63)]}{[F_{\chi^2_3}(10.41)]^{-1}-[\Phi(2.63)-\Phi(-2.63)]}\\ &\overset{table,\ a)}{=} \left(\frac{1-(0.9957-0.0043)}{0.975^{-1}-(0.9957-0.0043)}, \frac{1-(0.9957-0.0043)}{0.99^{-1}-(0.9957-0.0043)}\right)\\ &\simeq (0.251161, 0.459868)\end{aligned}$$

- **Probability of a misleading signal of type IV**

$$\begin{aligned}PMS_{IV}(\delta) &\overset{Table\ 10.12}{=} \frac{1-F_{\chi^2_{(n-1)}}(\gamma_\sigma)}{[\Phi(\gamma_\mu-\delta)-\Phi(-\gamma_\mu-\delta)]^{-1}-F_{\chi^2_{(n-1)}}(\gamma_\sigma)}\\ &\overset{\delta=0.25}{=} \frac{1-F_{\chi^2_{(3)}}(16.27)}{[\Phi(3.29-0.25)-\Phi(-3.29-0.25)]^{-1}-F_{\chi^2_{(3)}}(16.27)}\\ &\overset{a)}{=} \frac{0.001}{[\Phi(3.04)-\Phi(-3.54)]^{-1}-(1-0.001)}\\ &\overset{table}{=} \frac{0.001}{(0.998817-0.0002)^{-1}-0.999}\\ &\simeq 0.419302.\end{aligned}$$

7. *Uma companhia inspecciona lotes fazendo uso de plano de amostragem dupla com $n_1 = 100$, $c_1 = 0$, $n_2 = 100$ e $c_2 = 2$.*

(a) *Confronte este plano de amostragem dupla com um plano de amostragem simples com $n = 100$ e $c = 0$, no que diz respeito à dimensão média da amostra (*average sample number — *ASN) e à probabilidade de aceitação do lote, caso a fracção de peças defeituosas seja igual a $p = 0.01$.* (1.5)

- **Double sampling plan (for attributes)**

$n_1 = 100$, $n_2 = 100$ (sample sizes)

$c_1 = 0$, $c_2 = 2$ (acceptance numbers)

- **Auxiliary r.v. and their approximate distributions**

D_i = number of defective units in the i^{th} sample $\overset{a}{\sim}$ Binomial(n_i, p), $i = 1, 2$

- **Probability of accepting the lot in the first stage of the plan**

$$\begin{aligned}P_a^I(p) &\overset{(13.16)}{=} P(D_1\le c_1)\\ &\simeq F_{Binomial(n_1,p)}(c_1)\\ &\overset{p=0.01}{=} F_{Binomial(100,0.01)}(0)\\ &= \binom{100}{0}\times 0.01^0\times(1-0.01)^{100-0}\\ &= 0.99^{100}\\ &\simeq 0.366032\end{aligned}$$

- **Probability of accepting the lot in the second stage of the plan**

$$\begin{aligned}
P_a^{II}(p) &\overset{(13.17)}{=} P(c_1 < D_1 \leq c_2,\ D_1 + D_2 \leq c_2) \\
&= \sum_{k=c_1+1}^{c_2} P(D_1 = k) \times P(D_2 \leq c_2 - k) \\
&\simeq \sum_{k=c_1+1}^{c_2} P_{Binomial(n_1,p)}(k) \times F_{Binomial(n_2,p)}(c_2 - k) \\
&\overset{p=0.01}{=} \sum_{k=1}^{2} P_{Binomial(100,0.01)}(k) \times F_{Binomial(100,0.01)}(2 - k) \\
&= \binom{100}{1} \times 0.01^1 \times (1 - 0.01)^{100-1} \\
&\quad \times \left[\binom{100}{0} \times 0.01^0 \times (1 - 0.01)^{100-0} + \binom{100}{1} \times 0.01^1 \times (1 - 0.01)^{100-1}\right] \\
&\quad + \binom{100}{2} \times 0.01^2 \times (1 - 0.01)^{100-2} \times \binom{100}{0} \times 0.01^0 \times (1 - 0.01)^{100-0} \\
&\simeq 0.339707
\end{aligned}$$

- **Probability of accepting the lot in the double sampling plan**

$$\begin{aligned}
P_a(p) &\overset{(13.18)}{=} P_a^I(p) + P_a^{II}(p) \\
&\simeq 0.366032 + 0.339707 \\
&= 0.705739.
\end{aligned}$$

- **Average sample number of the double sampling plan**

$$\begin{aligned}
ASN(p) &\overset{(13.21)}{=} n_1 + n_2 \times P(c_1 < D_1 \leq c_2) \\
&\simeq n_1 + n_2 \times \left[F_{Binomial(n_1,p)}(c_2) - F_{Binomial(n_1,p)}(c_1)\right] \\
&\overset{p=0.01}{=} 100 + 100 \times \left[F_{Binomial(100,0.01)}(2) - F_{Binomial(100,0.01)}(0)\right] \\
&= 100 + 100 \\
&\quad \times \left[\binom{100}{1} \times 0.01^1 \times (1 - 0.01)^{100-1} + \binom{100}{2} \times 0.01^2 \times (1 - 0.01)^{100-2}\right] \\
&\simeq 155.459446
\end{aligned}$$

- **Single sampling plan**
  $n = 100$
  $c = 0$

- **Auxiliary r.v. and its approximate distribution**
  $D =$ number of defective units in the sample $\overset{a}{\sim}$ Binomial(n, p)

- **Probability of accepting the lot in the single sampling plan**

$$\begin{aligned}
P'_a(p) &= P(D \leq c) \\
&\simeq F_{Binomial(n,p)}(c) \\
&\overset{p=0.01}{=} F_{Binomial(100,0.01)}(0) \\
&= \binom{100}{0} \times 0.01^0 \times (1 - 0.01)^{100-0} \\
&= 0.99^{100} \\
&\simeq 0.366032 \\
&< P_a(0.01) = 0.705739
\end{aligned}$$

- **Average sample number of the single sampling plan**

$$\begin{aligned}
ASN'(p) &= n \\
&= 100,\ \forall p \in (0, 1) \\
&< ASN(0.01) \simeq 155.459446
\end{aligned}$$

(b) *Reavalie o desempenho deste plano de amostragem simples à luz da rectificação da inspecção, em particular, determine a redução relativa na fracção de unidades defeituosas nos lotes considerando para o efeito o valor de p da alínea anterior. Comente.* (1.0)

- **Average outgoing quality of a single sampling plan with rectifying inspection**
  By admitting a large value for the lot size N, we get:

$$\begin{aligned}
AOQ'(p) &\overset{(13.14)}{=} \frac{p\,(N - n)\,P'_a(p)}{N} \\
&\simeq p\,P'_a(p) \\
&\overset{p=0.01}{\simeq} 0.01 \times 0.366032 \\
&\simeq 0.003660.
\end{aligned}$$

- **Relative reduction in the percentage defective**

$$\begin{aligned}
\left[1 - \frac{AOQ'(p)}{p}\right] \times 100\% &\simeq \left[1 - \frac{p\,P'_a(p)}{p}\right] \times 100\% \\
&= \left[1 - P'_a(p)\right] \times 100\% \\
&\overset{p=0.01}{\simeq} (1 - 0.366032) \times 100\% \\
&= 63.3968\%
\end{aligned}$$

- **Comment**
  The relative reduction in the percentage defective is rather large, thus, rectifying inspection is well worth doing when $p = 0.01$.