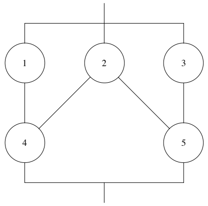
Fiabilidade e Controlo de Qualidade — LMAC

1o. Teste — 2o. Semestre — 2006/07

Duração: 1 hora e 30 minutos — 05/Maio/2007 (Sab.) — 9h — V1.08

1. *A figura abaixo descreve uma peça de um aparelho de radar, peça essa constituída por 5 componentes independentes com fiabilidades* $p_i = p = 0.975$, $i = 1, \ldots, 5$.



(a) *Após ter argumentado que a componente 2 não é irrelevante, determine a função estrutura desta peça por decomposição fulcral em torno da componente 2.* (2.5)

- **Relevance of component 2**

  Component 2 is not irrelevant because there is at least a state vector $\underline{x}$ such that:

  $\phi(0_2, \underline{x}) \neq \phi(1_2, \underline{x})$.

  In fact, for

  $\underline{x} = (0, 1, 0, 0, 1)$,

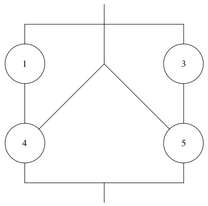
  we get

$$\begin{aligned} \phi(0_2, \underline{x}) &= \phi(0, 0, 0, 0, 1) = 0 \\ &\neq \phi(0_2, \underline{x}) = \phi(0, 1, 0, 0, 1) = 1, \end{aligned}$$

  because the system does not function when only component 5 operates, whereas it operates with components 2 and 5 functioning.

- **Structure function by pivotal decomposition around component 2**

  – **Sub-system associated to** $(1_2, \underline{X})$



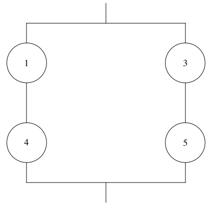
Minimal path sets

$$\begin{aligned} \mathcal{P}_1 &= \{4\} \\ \mathcal{P}_2 &= \{5\} \\ p^* &= 2 \text{ minimal path sets} \end{aligned}$$

Structure function

$$\begin{aligned} \phi(1_2, \underline{X}) &\stackrel{(1.13)}{=} 1 - \prod_{j=1}^{p^*} \left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) \\ &= 1 - (1 - X_4)(1 - X_5). \end{aligned}$$

  – **Sub-system associated to** $(0_2, \underline{X})$



Minimal path sets

$$\begin{aligned} \mathcal{P}_1 &= \{1, 4\} \\ \mathcal{P}_2 &= \{3, 5\} \\ p^* &= 2 \text{ minimal path sets} \end{aligned}$$

Structure function

$$\begin{aligned} \phi(0_2, \underline{X}) &\stackrel{(1.13)}{=} 1 - \prod_{j=1}^{p^*} \left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) \\ &= 1 - (1 - X_1 X_4)(1 - X_3 X_5). \end{aligned}$$

- **Structure function of the original system**

$$\begin{aligned} \phi(\underline{X}) &\stackrel{(1.7)}{=} X_2 \times \phi(1_2, \underline{X}) + (1 - X_2) \times \phi(0_2, \underline{X}) \\ &= X_2 \times [1 - (1 - X_4)(1 - X_5)] + (1 - X_2) \times [1 - (1 - X_1 X_4)(1 - X_3 X_5)]. \end{aligned}$$

(b) *Obtenha agora a fiabilidade da peça, represente-a após uma replicação ao nível do sistema e avalie o impacto desta replicação na fiabilidade da peça.* (3.0)

- **Reliability**

  By capitalizing on the structure function and on the fact that $X_i \stackrel{i.i.d.}{\sim} \text{Bernoulli}(p_i = p)$, $i = 1, \ldots, 5$, we successively get:

$$\begin{aligned} r(\underline{p}) &= E[\phi(\underline{X})] \\ &\stackrel{X_i \, indep}{=} E(X_2) \times \{1 - [1 - E(X_4)]\,[1 - E(X_5)]\} \\ &\quad + [1 - E(X_2)] \times \{1 - [1 - E(X_1)E(X_4)] \times [1 - E(X_3)E(X_5)]\} \\ &\stackrel{X_i \sim Ber(p_i)}{=} p_2 \times [1 - (1 - p_4)\,(1 - p_5)] \\ &\quad + (1 - p_2) \times [1 - (1 - p_1 p_4)]\,[1 - (1 - p_3 p_5)] \\ &\stackrel{p_i = p}{=} p \times \left[1 - (1 - p)^2\right] + (1 - p) \times \left[1 - \left(1 - p^2\right)^2\right] \\ &\stackrel{p = 0.975}{=} 0.999330. \end{aligned}$$

- **Reliability of the replicated system (at system level)**

$$r_{RSL}(\underline{p}, \underline{p}') \stackrel{(1.26)}{=} 1 - \left[1 - r(\underline{p})\right] \times \left[1 - r(\underline{p}')\right]$$

$$\begin{aligned} &\overset{p=p'}{=} 1-(1-0.999330)^2 \\ &\simeq 1.000000. \end{aligned}$$

- **Comment**
  Since the relative increase in the reliability of the system due to its replication is equal to
  $$\left|1-\frac{1.000000}{0.999330}\right| \times 100\% = 0.067045\%,$$
  such a replication is pointless.

(c) *Calcule um par de limites inferior e superior o mais estritos possível para a fiabilidade da peça original agora com componentes associadas (positivamente). Comente.* (2.0)

- **Components**
  $p_i = p = 0.975$, $i = 1, \ldots, 5$
  Since the 5 components form a coherent system and operate in a positively associated fashion, we can apply Theorem 1.70, namely result (1.42).

- **Minimal path sets**
  $$\begin{aligned} \mathcal{P}_1 &= \{1,4\} \\ \mathcal{P}_2 &= \{2,4\} \\ \mathcal{P}_3 &= \{2,5\} \\ \mathcal{P}_4 &= \{3,5\} \\ p^* &= 4 \text{ minimal path sets} \end{aligned}$$

- **Minimal cut sets**
  $$\begin{aligned} \mathcal{K}_1 &= \{1,2,3\} \\ \mathcal{K}_2 &= \{4,5\} \\ \mathcal{K}_3 &= \{1,2,5\} \\ \mathcal{K}_4 &= \{2,3,4\} \\ q &= 4 \text{ minimal cut sets} \end{aligned}$$

- **Lower bound for the reliability** $r(\underline{p})$
  $$\begin{aligned} r(\underline{p}) &\overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i\in\mathcal{P}_j} p_i \\ &\overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\ &\overset{\#\mathcal{P}_j=2,\forall j}{=} p^2 \\ &\overset{p=0.975}{=} 0.975^2 \\ &= 0.950625. \end{aligned}$$

- **Upper bound for the reliability**
  $$\begin{aligned} r(\underline{p}) &\overset{(1.42)}{\leq} \min_{j=1,\ldots,q} \left[1-\prod_{i\in\mathcal{K}_j}(1-p_i)\right] \\ &\overset{p_i=p}{=} \min_{j=1,\ldots,q} \left[1-(1-p)^{\#\mathcal{K}_j}\right] \\ &\overset{\#\mathcal{K}_j=2,3}{=} \min\left\{1-(1-p)^2,\, 1-(1-p)^3\right\} \\ &= 1-(1-p)^2 \\ &\overset{p=0.975}{=} 1-(1-0.975)^2 \\ &= 0.999375. \end{aligned}$$

- **Comment**
  If the components are independent (and therefore positively associated) the reliability of the system belongs to the interval [0.950625, 0.999375].

2. *O sistema de propulsão de um satélite é constituído por 4 motores que possuem durações (em anos) i.i.d. com função taxa de falha* $\lambda(t) = 2.5 \times t^{1.5}, t \geq 0$. *Considere ainda que o satélite só se mantém operacional caso 3 dos 4 motores funcionem.*

   (a) *Depois de ter identificado a distribuição do tempo de vida das componentes, caracterize o tempo de vida do satélite quanto ao envelhecimento estocástico e obtenha uma expressão para um limite inferior para a duração esperada da vida do mesmo.* (3.5)

- **System**
  $3 - \text{out} - \text{of} - 4$ system

- **Individual durations and common hazard rate function**
  T_i, $i = 1, \ldots, 4$, i.i.d. r.v. with common hazard rate function $\lambda(t) = 2.5 \times t^{1.5}$, $t \geq 0$.

- **Distribution of the individual durations**
  According to Definition 4.21, the hazard rate function of a Weibull distribution with scale parameter δ and shape parameter α, Weibull(δ, α), is equal to
  $$\lambda(t) = \frac{\alpha}{\delta}\left(\frac{t}{\delta}\right)^{\alpha-1}, \; t \geq 0.$$
  Thus,
  $$\begin{aligned} T_i &\overset{i.i.d.}{\sim} \text{Weibull}(\delta = 1, \alpha = 2.5),\; i = 1.\ldots,4 \\ R_i(t) &= R(t) \\ &= \begin{cases} e^{-t^{2.5}}, & t \geq 0 \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

- **Duration of the system**
  $$\begin{aligned} T &\overset{(2.6)}{=} T_{(n-k+1)} \\ &\overset{n=4,k=3}{=} T_{(2)} \end{aligned}$$

- **Stochastic ageing of** $T = T_{(2)}$
  First note that $\alpha = 2.5 > 1$. Therefore, according to subsection 4.3.4 (see table in page 100),
  $$T_i \overset{i.i.d.}{\sim} IHR,\; i = 1, \ldots, 4.$$
  Now, if we apply Proposition 3.25, namely result (3.15), we can conclude that
  $$T = T_{(2)} \sim IHR.$$

- **Expression for the lower bound for** $E(T)$
  We are dealing with a coherent system characterized as follows:
  - the 4 components have durations $T_i \overset{i.i.d.}{\sim} IHR$, $i = 1, \ldots, 4$ and, thus, according to Proposition 3.36,
    $$T_i \overset{i.i.d.}{\sim} NBUE,\; i = 1, \ldots, 4;$$

– the expected value of the duration of each of the 4 components is equal to

$$\begin{aligned}
\mu^* &= \mu_i \\
&= E(T_i) \\
&= E\left[\text{Weibull}(\delta = 1, \alpha = 2.5)\right] \\
&\stackrel{Exer\,4.22b)}{=} \delta\Gamma\left(\frac{1}{\alpha}+1\right) \\
&= \Gamma(1.4),\ i = 1,\ldots,4;
\end{aligned}$$

– the minimal path sets are

$$\begin{aligned}
\mathcal{P}_1 &= \{1,2,3\} \\
\mathcal{P}_2 &= \{1,2,4\} \\
\mathcal{P}_3 &= \{1,3,4\} \\
\mathcal{P}_4 &= \{2,3,4\} \\
p &= 4 \text{ minimal path sets.}
\end{aligned}$$

Now, we can apply Theorem 3.65, and conclude that

$$\begin{aligned}
\mu &= E(T) \\
&\geq \max_{j=1,\ldots,p}\left\{\left(\sum_{i\in\mathcal{P}_j}\mu_i^{-1}\right)^{-1}\right\} \\
&\stackrel{\mu_i=\mu^*}{=} \max_{j=1,\ldots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\} \\
&\stackrel{\#\mathcal{P}_j=3}{=} \frac{\mu^*}{3} \\
&= \frac{\Gamma(1.4)}{3}.
\end{aligned}$$

(b) *É de longe mais realista assumir que os motores funcionam de modo associado (positivo) já que a falha de um motor poderá resultar na sobrecarga dos restantes.*

Calcule limites inferior e superior o mais estritos possível para a função de fiabilidade do tempo de vida do satélite para um período de 1 ano. Comente. (2.5)

- **Components**
  Since the 4 components form a coherent system and operate in a positively associated fashion, we can apply Theorem 2.22, namely result (2.18), to provide bounds for the reliability function.

- **Minimal path sets**

$$\begin{aligned}
\mathcal{P}_1 &= \{1,2,3\} \\
\mathcal{P}_2 &= \{1,2,4\} \\
\mathcal{P}_3 &= \{1,3,4\} \\
\mathcal{P}_4 &= \{2,3,4\} \\
p &= 4 \text{ minimal path sets.}
\end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned}
\mathcal{K}_1 &= \{1,2\} \\
\mathcal{K}_2 &= \{1,3\} \\
\mathcal{K}_3 &= \{1,4\} \\
\mathcal{K}_4 &= \{2,3\} \\
\mathcal{K}_5 &= \{2,4\} \\
\mathcal{K}_6 &= \{3,4\} \\
q &= 6 \text{ minimal cut sets}
\end{aligned}$$

- **Lower bound for the reliability function of T for a period of one year**

$$\begin{aligned}
r_T(t) &\stackrel{(2.18)}{\geq} \max_{j=1,\ldots,p}\prod_{i\in\mathcal{P}_j}R_i(t) \\
&\stackrel{R_i(t)=R(t)}{=} \max_{j=1,\ldots,p}\left\{R(t)^{\#\mathcal{P}_j}\right\} \\
&\stackrel{\#\mathcal{P}_j=3,\forall j}{=} R(t)^3 \\
&= \left(e^{-t^{2.5}}\right)^3 \\
&\stackrel{t=1}{=} e^{-3} \\
&\simeq 0.049787.
\end{aligned}$$

- **Upper bound for the reliability**

$$\begin{aligned}
r_T(t) &\stackrel{(2.18)}{\leq} \min_{j=1,\ldots,q}\left\{1-\prod_{i\in\mathcal{K}_j}\left[1-R_i(t)\right]\right\} \\
&\stackrel{R_i(t)=R(t)}{=} \max_{j=1,\ldots,q}\left\{1-\left[1-R(t)\right]^{\#\mathcal{K}_j}\right\} \\
&\stackrel{\#\mathcal{K}_j=2,\forall j}{=} 1-\left[1-R(t)\right]^2 \\
&= 1-\left(1-e^{-t^{2.5}}\right)^2 \\
&\stackrel{t=1}{=} 1-\left(1-e^{-1}\right)^2 \\
&\simeq 0.600424.
\end{aligned}$$

- **Comment**
  If the components have independent durations (and therefore positively associated) the reliability function of the system for a period of one year belongs to the interval [0.049787, 0.600424].

- **Obs.**
  The reliability of $T = T_{(n-k+1)} = T_{(2)}$ for a period of one year is equal to

$$\begin{aligned}
R_T(t) &\stackrel{(2.8)}{=} 1 - F_{binomial(n,R(t))}(k-1) \\
&\stackrel{n=4,\,k=3,etc.}{=} 1 - F_{binomial(4,e^{-t^{2.5}})}(3-1) \\
&= 1 - \sum_{j=0}^{2}\binom{4}{j}\left(e^{-t^{2.5}}\right)^j\left(1-e^{-t^{2.5}}\right)^{4-j} \\
&\stackrel{t=1}{=} 1-(1-e^{-1})^4 - 4e^{-1}(1-e^{-1})^3 - 6e^{-2}(1-e^{-1})^2 \\
&\simeq 0.144201,
\end{aligned}$$

and indeed belongs to the interval [0.049787, 0.600424].

3. *Os registos seguintes dizem respeito ao número de ciclos até falha de 8 de entre 15 bombas de extracção de petróleo, colocadas sob teste em condições muito similares às da prática:*

i	1	2	3	4	5	6	7	8
$t_{(i)}$	118141	227628	310981	328661	388670	456801	1279943	1383253
$\ln[t_{(i)} - t_{(i-1)}]$	11.680	11.604	11.331	9.7802	11.002	11.129	13.621	11.545

(a) *Averigue a adequação do modelo* exponencial *a este conjunto de dados censurados, considerando para o efeito um nível de significância de* 5%. (2.5)

- **Life test**
  Since the end of the test was determined by the $r = 8^{th}$ failure and nothing in this exercise suggests that the $n = 15$ oil pumps were replaced during the life test, we are dealing with a
  - Type II/item censored testing without replacement.

- **R.v.**
  $T_i =$ duration (in cycles) of the i^{th} oil pump
  $T_i \stackrel{i.i.d.}{\sim} T$, $i = 1, \dots, n$

- **Censored data**
  $n = 15$
  $r = 8$
  $(t_{(1)}, \dots, t_{(r)}) = (118141, \dots, 1383253)$

- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= \sum_{i=1}^{r} t_{(i)} + (n-r) \times t_{(r)} \\ &= 4494078 + (15-8) \times 1383253 \\ &= 14176849\end{aligned}$$

- **Hypotheses**
  $H_0 : T \sim \text{Exponential}(\lambda)$
  $H_1 : T \sim \text{Weibull}(\lambda^{-1}, \alpha)$, $\alpha \neq 1$

- **Significance level**
  $\alpha_0 = 5\%$

- **Test statistic** (Bartlett's test)

$$\begin{aligned}B_r &\stackrel{(5.18)}{=} \frac{2r}{1+\frac{r+1}{6r}} \left(\ln\left(\frac{\mathcal{T}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln\left\{(n-i+1)[T_{(i)} - T_{(i-1)}]\right\} \right) \\ &\stackrel{a}{\sim}_{H_0} \chi^2_{(r-1)}\end{aligned}$$

  where $T_{(i)} - T_{(i-1)}$ represents the times between consecutive failure times.

- **Rejection region of H_0**

$$\begin{aligned}W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right) \\ &\stackrel{r=8,\,\alpha_0=0.05}{=} (0, 1.690) \cup (16.01, +\infty)\end{aligned}$$

- **Decision**
  The observed value of the test statistic is

$$\begin{aligned}b_r &= \frac{2r}{1+\frac{r+1}{6r}} \left(\ln\left(\frac{\tilde{t}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln\left\{(n-i+1)[t_{(i)} - t_{(i-1)}]\right\} \right) \\ &= \frac{2r}{1+\frac{r+1}{6r}} \left(\ln\left(\frac{\tilde{t}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln[t_{(i)} - t_{(i-1)}] - \frac{1}{r}\sum_{i=1}^{r} \ln(n-i+1) \right) \\ &= 13.472684 \\ &\notin W = (0, 1.690) \cup (16.01, +\infty),\end{aligned}$$

  therefore we should not reject H_0 for any significance level $\alpha \leq 5\%$.

(b) *Após ter enunciado as hipóteses de trabalho que entender convenientes, obtenha um intervalo de confiança aproximado a* 90% *para o número de ciclos associado a uma fiabilidade de* 95%. (2.5)

- **Distribution assumption**
  $T_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \dots, 15$

- **Confidence interval for the unknown parameter λ**

$$\begin{aligned}CI_{(1-\alpha)\times 100\%}(\lambda) &\stackrel{Table\ 5.16}{=} [\lambda_L; \lambda_U] \\ &= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r)}}(1-\alpha/2)}{2 \times \tilde{t}}\right] \\ CI_{90\%}(\lambda) &\stackrel{a)}{=} \left[\frac{F^{-1}_{\chi^2_{(16)}}(0.05)}{2 \times 14176849}; \frac{F^{-1}_{\chi^2_{(16)}}(0.95)}{2 \times 14176849}\right] \\ &= \left[\frac{7.692}{2 \times 14176849}; \frac{26.30}{2 \times 14176849}\right] \\ &\simeq [2.8081 \times 10^{-7}; 9.2757 \times 10^{-6}]\end{aligned}$$

- **Another unknown parameter**
  $F_T^{-1}(0.95) = -\frac{1}{\lambda}\ln(1-0.95)$, which is a decreasing function of $\lambda > 0$.

- **Confidence interval for $F_T^{-1}(0.95)$**

$$\begin{aligned}CI_{95\%}\left(F_T^{-1}(0.95)\right) &= \left[\frac{1}{\lambda_U}\ln(1-0.95); \frac{1}{\lambda_L}\ln(1-095)\right] \\ &\simeq [55298.653175; 182661.966653].\end{aligned}$$

(c) *Determine a redução relativa percentual na duração esperada deste teste, tendo como referência a duração esperada do teste envolvendo à partida somente 8 bombas de extracção. Comente.* (1.5)

- **Relative reduction of the test duration**
  Considering $T_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \dots, n$ and using result (5.26), the relative reduction of the test duration is

$$\begin{aligned}\left[1 - \frac{E(T_{r:n})}{E(T_{r:r})}\right] \times 100\% &= \left[1 - \frac{\sum_{i=1}^{r} \frac{1}{n-i+1}}{\sum_{i=1}^{r} \frac{1}{r-i+1}}\right] \times 100\% \\ &\simeq 73.310891\%,\end{aligned}$$

  when Type II/item censored testing without replacement has been adopted.

- **Comment**
  This is a very substantial reduction in the test duration, a somewhat expected result because the end of the Type II/item censored testing (without replacement) is determined by approximately 50% of the possible failures.