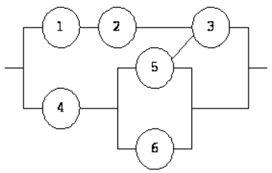
Fiabilidade e Controlo de Qualidade — LMAC

Exame de Época Especial — 2o. Semestre – 2004/05

Duração: 3 horas — 09/Setembro/2005 (6a. feira) – 17h, Q4.1

- Justifique convenientemente **todas as respostas** às 6 questões que se seguem.

1. *Uma bomba de extracção é constituída por 6 componentes que funcionam de modo independente, possuem fiabilidade comum e igual a $p = 0.9$ e estão dispostas de acordo com o esquema abaixo.*



(a) *Comece por listar todos os cortes mínimos para de seguida obter a função estrutura deste sistema.* (1.5)

- **State vector**
  $\underline{X} = (X_1, \dots, X_n)$
  $n = 6$ components
  $X_i \stackrel{i.i.d.}{\sim}$ Bernouilli$(p_i = p = 0.9)$, $i = 1, \dots, n$
- **Minimal cut sets**

$$\begin{aligned} \mathcal{K}_1 &= \{1,4\} \\ \mathcal{K}_2 &= \{2,4\} \\ \mathcal{K}_3 &= \{3,4\} \\ \mathcal{K}_4 &= \{1,5,6\} \\ \mathcal{K}_5 &= \{2,5,6\} \\ \mathcal{K}_6 &= \{3,5,6\} \\ q &= 6 \text{ minimal cut sets} \end{aligned}$$

- **Structure function**
  Since we are dealing with a coherent system, we can apply Theorem 1.30 and state that:

$$\begin{aligned} \phi(\underline{X}) &\stackrel{(1.14)}{=} \prod_{j=1}^{q} \left[(1 - \prod_{i \in \mathcal{K}_j} (1 - X_i) \right] \\ &= [1 - (1 - X_1)(1 - X_4)] \times [1 - (1 - X_2)(1 - X_4)] \\ &\quad \times [1 - (1 - X_3)(1 - X_4)] \times [1 - (1 - X_1)(1 - X_5)(1 - X_6)] \\ &\quad \times [1 - (1 - X_2)(1 - X_5)(1 - X_6)] \times [1 - (1 - X_3)(1 - X_5)(1 - X_6)]. \end{aligned}$$

(b) *Obtenha um limite inferior e outro superior o mais estritos possível para a fiabilidade da bomba de extração.* (1.5)

- **Components**
  Since the 6 components form a coherent system and operate independently, we can apply Theorem 1.68.
- **Minimal path sets**

$$\begin{aligned} \mathcal{P}_1 &= \{1,2,3\} \\ \mathcal{P}_2 &= \{4,5\} \\ \mathcal{P}_3 &= \{4,6\} \\ p^* &= 3 \text{ minimal path sets} \end{aligned}$$

- **Minimal cut sets**
  See line (a).
- **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned} r(\underline{p}) &\stackrel{T1.68}{\geq} \prod_{j=1}^{q} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - p_i) \right] \\ &\stackrel{p_i = p}{=} \prod_{j=1}^{q} \left[1 - (1-p)^{\#\mathcal{K}_j} \right] \\ &= \left[1 - (1-p)^2 \right]^3 \times \left[1 - (1-p)^3 \right]^3 \\ &\stackrel{p=0.9}{=} 0.967391. \end{aligned}$$

- **Upper bound for the reliability**

$$\begin{aligned} r(\underline{p}) &\stackrel{T1.68}{\leq} 1 - \prod_{j=1}^{p^*} \left(1 - \prod_{i \in \mathcal{P}_j} p_i \right) \\ &\stackrel{p_i = p}{=} 1 - \prod_{j=1}^{p^*} \left(1 - p^{\#\mathcal{P}_j} \right) \\ &= 1 - (1 - p^3) \times (1 - p^2)^2 \\ &\stackrel{p=0.9}{=} 0.990217. \end{aligned}$$

2. *Um sistema de arrefecimento é constituído por 3 componentes colocadas em série.*

(a) *O que pode adiantar sobre o envelhecimento estocástico da duração do sistema, caso admita que as durações (em centenas de horas) das componentes são i.i.d. com função de fiabilidade comum $R(t) = e^{-t^{0.5}}, t \geq 0$. Justifique.* (1.5)

- **System**
  Series system
- **Individual durations and common reliability function**
  T_i, $i = 1, 2, 3$, i.i.d. r.v. with common hazard rate function $R(t) = e^{-t^{0.5}}$, $t \geq 0$, $i = 1, 2, 3$.
- **Distribution of the individual durations**
  According to Definition 4.21, the reliability function of a Weibull distribution with scale parameter δ and shape parameter α, Weibull(δ, α), is equal to

$$R(t) = \exp\left[- \left(\frac{t}{\delta} \right)^{\alpha} \right], \; t \geq 0.$$

Thus,

$$T_i \overset{i.i.d.}{\sim} \text{Weibull}(\delta = 1, \alpha = 0.5),\ i = 1, 2, 3.$$

- **Stochastic ageing of T_i**
  Note that $\alpha = 0.5 > 1$, thus, according to subsection 4.3.4 (see table in page 100),
  $$T_i \overset{i.i.d.}{\sim} DHR,\ i = 1, 2, 3.$$
- **Duration of the system**
  $T \overset{(2.2)}{=} T_{(1)}$
- **Stochastic ageing of $T = T_{(1)}$**
  Now, if we apply Proposition 3.23, namely result (3.12), we can conclude that
  $$T = T_{(1)} \sim DHR.$$

(b) *Obtenha o valor esperado e a variância do tempo até falha do sistema de arrefecimento. Confronte tal variância com um seu limite inferior.* (1.5)

- **Distribution of T**
  Taking into account the fact that $T = T_{(1)}$ and the result of line (c) of Example 4.25, we get
  $$T \sim \text{Weibull}\left(\delta' = \frac{\delta}{n^{1/\alpha}} = \frac{1}{3^{1/0.5}} = \frac{1}{9},\ \alpha' = \alpha = 0.5\right)$$
- **Expected value and variance of T**
  Capitalizing the result of line (b) of Example 4.22, we have
  $$\begin{aligned} E(T) &= \delta' \times \Gamma\left(\frac{1}{\alpha'} + 1\right) \\ &= \frac{1}{9} \times \Gamma\left(\frac{1}{0.5} + 1\right) \\ &= \frac{1}{9} \times \Gamma(3) \\ &= \frac{1}{9} \times (3-1)! \\ &= \frac{2}{9} \\ V(T) &= (\delta')^2 \times \left[\Gamma\left(\frac{2}{\alpha'} + 1\right) - \Gamma^2\left(\frac{1}{\alpha'} + 1\right)\right] \\ &= \left(\frac{1}{9}\right)^2 \times \left[\Gamma(5) - \Gamma^2(3)\right] \\ &= \left(\frac{1}{9}\right)^2 \times \left[(5-1)! - (3-1)!^2\right] \\ &= \frac{20}{81}. \end{aligned}$$
- **Lower bound for $V(T)$**
  $$\begin{aligned} T \sim DHR &\overset{Prop.\,3.36}{\Rightarrow} T \in DHRA \\ &\overset{Cor.\,3.54}{\Rightarrow} \frac{\sqrt{V(T)}}{E(T)} \geq 1 \\ & \qquad V(T) \geq E^2(T) = \frac{4}{81}. \end{aligned}$$
- **Confronting $V(T)$ and its lower bound**
  The corollary we just apply requires that $T \in DHRA$, a more loose condition than the one that reflects the true stochastic behavior of T, $T \in DHR$. Unsurprisingly, $V(T) = \frac{20}{81} >> \frac{4}{81}$ and a more accurate lower bound would be more useful.

3. *Os registos que se seguem reportam-se aos instantes de falha (em horas) devidamente ordenados (e arredondados) de 10 geradores eléctricos sujeitos a um teste de vida acelerado: 121, 279, 592, 848, 1425, 1657, 1883, 5296, 6303, 7201.*

(a) *Exemplifique a utilização do papel de probabilidade exponencial com alguns cálculos (por exemplo três pontos).*
Que aspecto esperaria para este gráfico, caso o modelo exponencial fosse adequado a este conjunto de dados? (1.0)

- **Times to failure**
  T_i = time to failure of the i^{th}electrical generator (a device that converts mechanical energy to electrical energy), $i = 1, \ldots, n$
  $T_i \overset{i.i.d.}{\sim} T,\ i = 1, \ldots, n$
- **Complete and ordered data**
  $n = 10$
  $(t_{(1)}, \ldots, t_{(n)}) = (121, 279, 592, 848, 1425, 1657, 1883, 5296, 6303, 7201)$
- **Postulated model**
  $\{\text{Exponential}(\lambda),\ \lambda > 0\}$
  $F_\lambda(t) = 1 - e^{-\lambda t},\ t \geq 0$
- **Probability paper**
  It is well known that, for any absolutely continuous model, we get
  $$F_T(T_{(i)}) \sim \text{Beta}(i, n - i + 1).$$
  Thus, if consider as an estimate of
  $$p_i = F_T(t_{(i)}) = 1 - \exp\left[-\lambda\, t_{(i)}\right]$$
  the expected value
  $$\hat{p}_i = E[F_T(X_{(i)})] = E[\text{Beta}(i, n - i + 1)] = \frac{i}{n+1},$$
  we shall confront directly (or indirectly) $\hat{p}_i$ and $F_T(t_{(i)})$, that is,
  $$\begin{aligned} \frac{i}{n+1} &\rightarrow 1 - e^{-\lambda\, t_{(i)}} \\ \ln\left(1 - \frac{i}{n+1}\right) &\rightarrow -\lambda\, t_{(i)} \\ \frac{1}{\lambda} \ln\left(\frac{n+1}{n-i+1}\right) &\rightarrow t_{(i)}. \end{aligned}$$
  Therefore we are supposed to plot the points
  $$\left(\ln\left(\frac{n+1}{n-i+1}\right), t_{(i)}\right)$$
  in the exponential probability paper.

- **Example (with 3 points)**

i	Abcissa $\ln\left(\frac{n+1}{n-i+1}\right)$	Ordinate $t_{(i)}$
1	$\ln\left(\frac{10+1}{10-1+1}\right) \simeq 0.095$	121
2	$\ln\left(\frac{10+1}{10-2+1}\right) \simeq 0.201$	279
...	...	...
10	$\ln\left(\frac{10+1}{10-10+1}\right) \simeq 2.398$	7201

- **Comment**
  If the points are set in an approximately linear fashion in the probability paper, we can say that the exponential model fits the data set.

(b) *Admita agora que as 10 observações dizem respeito a um teste de vida envolvendo 20 geradores eléctricos e concluído à décima falha.*
Após ter identificado o tipo de teste de vida a que se recorreu e ter enunciado as hipóteses de trabalho que entender mais convenientes, averigue o ajustamento do modelo exponencial ao nível de significância de 5%. (1.5)

- **Life test**
  Since the end of the test was determined by the $r = 10^{th}$ failure and nothing in this exercise suggests that the $n = 20$ electrical generatos were replaced during the life test, we are dealing with a
  - Type II/item censored testing without replacement.

- **Assumption**
  T_i = time to failure of the i^{th} electrical generator
  $T_i \overset{i.i.d.}{\sim} T,\ i = 1, \ldots, n$

- **Censored data**
  $n = 20$
  $r = 10$
  $(t_{(1)}, \ldots, t_{(r)}) = (121, 279, 592, 848, 1425, 1657, 1883, 5296, 6303, 7201)$

- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= \sum_{i=1}^{r} t_{(i)} + (n-r)\times t_{(r)} \\ &= (121 + \ldots + 7201) + (20-10)\times 7201 \\ &= 97615\end{aligned}$$

- **Hypotheses**
  $H_0 : T \sim \text{Exponential}(\lambda)$
  $H_1 : T \sim \text{Weibull}(\lambda^{-1}, \alpha),\ \alpha \neq 1$

- **Significance level**
  $\alpha_0 = 5\%$

- **Test statistic** (Bartlett's test)

$$\begin{aligned}B_r &\overset{(5.18)}{=} \frac{2r}{1+\frac{r+1}{6r}}\left(\ln\left(\frac{\mathcal{T}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r}\ln\left\{(n-i+1)[T_{(i)} - T_{(i-1)}]\right\}\right) \\ &\overset{a}{\sim}_{H_0} \chi^2_{(r-1)}\end{aligned}$$

  where the $T_{(i)} - T_{(i-1)}$s represent the times between consecutive failure times.

- **Rejection region of H_0**

$$\begin{aligned}W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right) \\ &\overset{r=10,\alpha_0=0.1}{=} (0, 2.700) \cup (19.02, +\infty)\end{aligned}$$

- **Decision**
  The observed value of the test statistic is

$$\begin{aligned}b_r &= \frac{2r}{1+\frac{r+1}{6r}}\left(\ln\left(\frac{\tilde{t}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r}\ln\left\{(n-i+1)[t_{(i)} - t_{(i-1)}]\right\}\right) \\ &= \frac{2r}{1+\frac{r+1}{6r}}\left(\ln\left(\frac{\tilde{t}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r}\ln[t_{(i)} - t_{(i-1)}] - \frac{1}{r}\sum_{i=1}^{r}\ln(n-i+1)\right) \\ &= \frac{1200}{71}\times\left(9.1862 - \frac{1}{10}\times 87.4563\right) \\ &= 7.446220 \\ &\notin W = (0, 2.700) \cup (19.02, +\infty),\end{aligned}$$

  therefore we should not reject H_0 for any significance level $\alpha \leq 10\%$.

(c) *Calcule uma estimativa pontual centrada e outra intervalar da função de fiabilidade de um gerador eléctrico para um período de 6 meses à luz do teste de vida descrito em (b). Comente.* (1.5)

- **Distribution assumption**
  $T_i \overset{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda),\ i = 1, \ldots, n$, which quite reasonable considering the result in (a).

- **Unknown parameters**
  λ
  $R_T(t) = e^{-\lambda t}$, where $t = 100000$ cycles.

- **Censored data**
  $n = 20$
  $r = 10$
  $(t_{(1)}, \ldots, t_{(r)}) = (121, 279, 592, 848, 1425, 1657, 1883, 5296, 6303, 7201)$
  $\tilde{t} = 97615$

- **Unbiased estimate of $R_T(t)$**
  According to Table 5.14, the UMVUE of $R_T(t)$ is, for $t = 6\times 30\times 24 = 4320 < \tilde{t} = 97615$ and $r > 0$, equal to

$$\begin{aligned}\tilde{R}_T(t) &= \left(1 - \tilde{t}^{-1}\times t\right)^{r-1} \\ &= \left(1 - \frac{1}{97615}\times 4320\right)^{10-1} \\ &\simeq 0.665390.\end{aligned}$$

- **Confidence interval for λ**

According to Table 5.16 of the lecture notes,

$$\begin{aligned} CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_l; \lambda_U] \\ &= \left[\frac{F_{\chi^2_{(2r)}}(\alpha/2)}{2\times \tilde{t}}; \frac{F_{\chi^2_{(2r)}}(1-\alpha/2)}{2\times \tilde{t}}\right] \\ CI_{95\%}(\lambda) &= [\lambda_l; \lambda_U] \\ &= \left[\frac{F_{\chi^2_{(20)}}(0.025)}{2\times 2495050}; \frac{F_{\chi^2_{(20)}}(0.975)}{2\times 2495050}\right] \\ &= \left[\frac{9.591}{2\times 97615}; \frac{34.17}{2\times 97615}\right] \\ &\simeq [0.0000491267; 0.000175024] \end{aligned}$$

- **Confidence interval for $R_T(t)$**
  Since $R_T(t) = e^{-\lambda t}$ is a decreasing function of $\lambda > 0$, we can state that
  $$\begin{aligned} CI_{95\%}\left(e^{-\lambda t}\right) &= \left[e^{-\lambda_U\, t}; e^{-\lambda_L\, t}\right] \\ &\overset{t=4320}{\simeq} [0.469491; 0.808781]. \end{aligned}$$

- **Comment**
  The point and interval estimates seem to be very reasonable because they were obtained considering a distributional assumption that has been tested and not rejected previously in (a).

4. *Um serviço de manutenção de equipamento informático pretende melhorar a qualidade do seu trabalho. Para o efeito acompanhar-se-á o número de pedidos de manutenção aceites que requiseram mais de uma intervenção, tendo-se registado os seguintes dados durante 10 semanas.*

Semana	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Pedidos aceites	*200*	*250*	*250*	*250*	*200*	*200*	*150*	*150*	*150*	*200*
Ped. aceit. c/ mais de uma interv.	*6*	*7*	*8*	*7*	*3*	*4*	*2*	*1*	*0*	*2*

(a) *Que tipo de carta de controlo lhe parece mais razoável utilizar? Justifique.*
Determine o respectivo alvo e os seus limites de controlo, caso se decida por aceitar somente 200 pedidos por semana. (1.5)

- **Control statistic / quality characteristic**
  Y_N = number of maintenance requests requiring more than one intervention in the N^{th} sample of size n_N, $N \in \mathbb{N}$

- **Distributions**
  $Y_N \sim \text{Binomial}(n_N, p_0)$, IN CONTROL
  $Y_N \sim \text{Binomial}(n_N, p = p_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of an upward shift in p

- **Obtaining the target value of λ**
  Taking into account the data $\underline{y} = (y_1, \ldots, y_{10}) = (6, 7, 8, 7, 3, 4, 2, 1, 0, 2)$, associated to sample sizes $\underline{n} = (n_1, \ldots, n_{10}) = (200, 250, 250, 250, 200, 200, 150, 150, 150, 200)$, and the fact that the likelihood function is
  $$L(p|\underline{y}) = \prod_{N=1}^{10} \binom{n_i}{y_i} p^{y_i}(1-p)^{n_i - x_i},$$
  we obtain the following maximum likelihood estimate for p:
  $$\begin{aligned} \hat{p} &= \frac{\sum_{N=1}^{10} x_i}{\sum_{N=1}^{10} y_i} \\ &= \frac{6+7+8+7+3+4+2+1+0+2}{200+250+250+250+200+200+150+150+150+200} \\ &= 0.02. \end{aligned}$$
  Please note that we shall use $\hat{p}$ to estimate/obtain the target and the upper control limit of chart described below.

- **Suitable chart**
  An UPPER ONE-SIDED np chart should be adopted because we plan to collect data concerning the r.v. that represents the number of such requests per sample of fixed size n, and it is very important to detect increases in np, the expected value of this r.v.

- **(Estimate of the) target of the upper one-sided np chart**
  $CL = np_0 = 200 \times 0.02 = 4$

- **(Estimate of the upper) control limit of the 3 *sigma* upper one-sided np chart**
  $$\begin{aligned} LCL &= 0 \text{ (because we are dealing with an upper one-sided chart)} \\ UCL &= np_0 + \gamma\sqrt{np_0(1-p_0)} \\ &= 200\times 0.02 + 3\sqrt{200\times 0.02\times(1-0.02)} \\ &= 9.939697 \end{aligned}$$

- **Comment**
  Please note that for $n = 150$, $np_0 + \gamma\sqrt{np_0(1-p_0)} = 8.143929$ and all the 10 initial observations above do not exceed this limit or any other upper control limit with $n = 200, 250$. Thus, there is no need to omit any of the 10 initial observations and reestimate p.
  Moreover, from now on we are going to assume that the estimate of UCL is the true value of the upper control limit of the np chart.

(b) *Obtenha o valor esperado e o primeiro quartil do número de amostras recolhidas até à detecção, caso ocorra um aumento no valor esperado da percentagem de pedidos que necessitam mais de uma intervenção para 3%. Comente.* (1.5)

- **Run length**
  We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:
  $$RL(\delta) \sim \text{Geometric}\,(\xi(\delta))\,.$$

- **Shift**
  From $p_0 = 0.02$ to $p = p_0 + \delta = 0.03$ (i.e., $\delta = p - p_0 = 0.01$).

- **Probability of triggering a signal**
  By using the Central Limit Theorem (CLT), we can obtain an approximate value of the probability that this charts triggers a signal:
  $$\begin{aligned} \xi(\delta) &= P(Y_N \notin [LCL, UCL] \mid \delta) \\ &\overset{Y_N\geq 0,\, LCL=0}{=} P(Y_N > UCL \mid \delta) \\ &= 1 - F_{Binomal(n_N, p=p_0+\delta)}(UCL) \end{aligned}$$

$$
\begin{aligned}
&\overset{n_N=200}{=} && 1-F_{Binomial(200,0.03)}(9.939697)\\
&= && 1-F_{Binomial(200,0.03)}(9)\\
&\overset{CLT}{\simeq} && 1-\Phi\left(\frac{2-200\times 0.03}{\sqrt{200\times 0.03\times(1-0.03)}}\right)\\
&\simeq && 1-\Phi(1.24)\\
&\overset{table}{=} && 1-0.8925\\
&= && 0.1075
\end{aligned}
$$

- **Average run length**

$$
\begin{aligned}
ARL(\delta) &= \frac{1}{\xi(\delta)}\\
&\simeq \frac{1}{0.1075}\\
&\simeq 9.30236
\end{aligned}
$$

- **First quartile of $RL(\delta)$**

$$
\begin{aligned}
F^{-1}_{RL(\delta)}(p) &\overset{Table\,9.2}{=} \inf\left\{m\in I\!N: F_{RL(\delta)}(m)\geq\alpha\right\}\\
&= 1-[1-\xi(\delta)]^m\geq\alpha\\
&= [1-\xi(\delta)]^m\leq 1-\alpha\\
&= m\times\ln[1-\xi(\delta)]\leq\ln(1-\alpha)\\
&\overset{\ln[1-\xi(\delta)]<0}{=} m\geq\frac{\ln(1-p)}{\ln[1-\xi(\delta)]}\\
&\overset{\alpha=0.25}{=} m\geq\frac{\ln(1-0.25)}{\ln(1-0.1075)}\\
&= m\geq 2.529545\\
&= 3
\end{aligned}
$$

- **Comment**

  When the expected value of the number of special requests, per sample of size $n=200$, increases 50% — from its target value $np_0=4$ to $np=6$—, the probability of triggering a valid signal within the first 3 samples is at least 25%, which is quite reassuring.

5. *A qualidade do enchimento de garrafas de refrigerante é controlada recolhendo observações respeitantes ao desvio entre a altura do líquido em cada garrafa e uma marca-chave no gargalo da mesma. Admita que o referido desvio possui, sob controlo, distribuição normal com valor esperado $\mu_0=0cm$ e desvio-padrão $\sigma_0=0.1cm$.*

   Na tabela seguinte foram registadas as médias e as variâncias de 10 amostras de 5 garrafas cada:

N	1	2	3	4	5	6	7	8	9	10
$\bar{x}_N$	0.108	-0.074	-0.248	0.539	0.144	0.497	0.206	1.152	0.560	0.235
s^2_N	0.236	1.364	0.552	1.823	2.504	0.504	0.923	1.354	0.898	3.723

   (a) *Considere-se que o controlo de σ é feito à custa de uma carta* EWMA *unilateral superior, caracterizada por $\lambda_\sigma=0.043$ e $\gamma_\sigma=1.2198$, que possui $ARL_\sigma(1)=500.027$ e $ARL_\sigma(1.9)=4.120$. Averigue se alguma das três primeiras observações apontam para a alteração de σ.* (1.0)

- **Quality characteristic**

  $X=$ reading (difference between the height of the liquid and the gauge)

  $X\sim\text{Normal}(\mu,\sigma^2)$

- **Nominal values of μ and σ**

  $\mu_0=0$

  $\sigma_0=0.1$

- **Estimator of σ^2**

  $S^2_N=$ variance of the N^{th} random sample of size n, $N\in I\!N$

- **Control limits of the upper one-sided EWMA chart for $sigma^2$**

  To obtain the control limits of this chart, recall that $\lambda_\sigma=0.043$, $\gamma_\sigma=1.2198$, $n=5$ and $\psi'\left(\frac{n-1}{2}\right)=\psi'(2)\overset{(10.31)}{=}\psi'(1)-\frac{1}{1^2}=1.6449340668-\frac{1}{1^2}=0.6449340668$. Thus, according to Table 10.10, we get:

$$
\begin{aligned}
LCL_\sigma &= \ln\left(\sigma^2_0\right)\\
&= \ln(0.1^2)\\
&\simeq -4.605170\\
UCL_\sigma &= \ln\left(\sigma^2_0\right)+\gamma_\sigma\times\sqrt{\psi'\left(\frac{n-1}{2}\right)\times\frac{\lambda_\sigma}{2-\lambda_\sigma}}\\
&= \ln\left(0.1^2\right)+1.2198\times\sqrt{0.6449340668\times\frac{0.043}{2-0.043}}\\
&= -4.459960
\end{aligned}
$$

- **A few observed values of the control statistic**

  According to Table 10.10, the control statistic is given by

$$
V_N = \begin{cases} v_0=\ln\left(\sigma^2_0\right), & N=0\\ \max\{\ln\left(\sigma^2_0\right),(1-\lambda_\sigma)\times V_{N-1}+\lambda_\sigma\times\ln\left(S^2_N\right)\}, & N\in I\!N.\end{cases}
$$

  Thus, the first 3 observed values of the control statistic are

$$
\begin{aligned}
v_0 &= \ln\left(\sigma^2_0\right)=\ln\left(0.1^2\right)\\
&\simeq -4.605170\\
v_1 &= \max\{-4.605170,\,(1-0.043)\times(-4.605170)+0.043\times\ln(0.236)\}\\
&\simeq -4.469237\\
v_2 &= \max\{-4.605170,\,(1-0.043)\times(-4.469237)+0.043\times\ln(1.364)\}\\
&\simeq -4.263711\\
(v_3 &= \max\{-4.605170,\,(1-0.043)\times(-4.263711)+0.043\times\ln(0.552)\}\\
&\simeq 4.105923)
\end{aligned}
$$

  Since $v_2\notin[LCL_\sigma,UCL_\sigma]$ we should suspect that the process is out-of-control.

   (b) *Admita agora que para o controlo de μ se toma uma carta padrão do tipo* Shewhart *cujos limites de controlo são tais que*

   - *o número esperado de amostras recolhidas até à emissão de falso alarme por parte desta carta é de 370.4.*

   Determine a probabilidade de esta carta emitir um sinal quando ocorre um aumento de 81% na variância σ^2. Comente. (1.5)

- **Control statistic**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n

- **Distribution**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 0$, $\sigma_0 = 0.1$ and $n = 5$
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0^2}{n}, \frac{\sigma_0^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or an increase!) in μ.

- **Control limits of the standard $\bar{X}$ control chart**
  $LCL_{EWMA} = \mu_0 - \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$
  $UCL_{EWMA} = \mu_0 + \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$

- **Probability of triggering a signal**
  Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta, \theta\right) \\
&= 1 - \left[\Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \ \delta \in \mathbb{R}, \ \theta \geq 1.
\end{aligned}$$

- **Run length**
  We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given (δ,θ), $RL_\mu(\delta,\theta)$, has the following distribution:

$$RL_\mu(\delta,\theta) \sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right)$$

- **Obtaining γ_μ**
  The constant γ_μ is such that $ARL_\mu(0,1) = 370.4$, that is,

$$\begin{aligned}
\gamma_\mu \ : \ & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1) \\
& 1 - \left[\Phi(\gamma_\mu) - \Phi(-\gamma_\mu)\right] = \frac{1}{ARL_\mu(0,1)} \\
& 2 \times \Phi(\gamma_\mu) - 1 = \frac{1}{ARL_\mu(0,1)} \\
& \gamma_\mu = \Phi^{-1}\left(1 - \frac{1}{2 \times ARL_\mu(0,1)}\right) \\
& \gamma_\mu = \Phi^{-1}(0.99865) \\
& \gamma_\mu \overset{table}{=} 3
\end{aligned}$$

- **Requested probability**
  An increase of 81% of the processs variance corresponds to a shift from σ_0^2 to $\sigma^2 = (1 + 0.81) \times \sigma_0^2$, that is, $\theta = \frac{\sigma}{\sigma_0} = \sqrt{1 + 0.81}$. Thus, the probability that the standard $\bar{X}$ chart triggers a signal in this situation (assuming that μ is in-control, i.e., $\delta = 0$) is equal to:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{3 - 0}{\sqrt{1 + 0.81}}\right) - \Phi\left(\frac{-3\mu - 0}{\sqrt{1 + 0.81}}\right)\right] \\
&= 2 \times \left[1 - \Phi\left(\frac{3}{\sqrt{1 + 0.81}}\right)\right] \\
&\simeq 2 \times \left[1 - \Phi(2.23)\right] \\
&\overset{table}{=} 2 \times (1 - 0.9871) \\
&= 0.0258.
\end{aligned}$$

- **Comment**
  This probability is most likely smaller than the one associated to the S^2 chart because the $\bar{X}$ is not tailored to detect shifts in σ^2.

(c) *Ao utilizar-se a carta descrita em (b) e simultaneamente uma carta unilateral superior do tipo* Shewhart *para σ, obtém-se o que se designa por esquema conjunto para μ e σ.*
Determine a probabilidade de ocorrência de sinal erróneo de Tipo III (IV) quando $\theta = 1.9$ ($\delta = 0.1$), caso a carta para σ possua $ARL_\sigma(1) = 200$. Comente estes resultados. (2.0)
Nota: Na impossibilidade de obter valores exactos obtenha intervalos de valores para estas duas probabilidades.

- **Control statistic of the upper one-sided S^2 chart**
  S_N^2 = variance of the N^{th} random sample of size n, $N \in \mathbb{N}$

- **Distribution**
  $\frac{(n-1)S_N^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
  $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL

- **Control limits of the upper one-sided S^2- chart**
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probability of triggering a signal**

$$\begin{aligned}
\xi_\sigma(\theta) &= P\left(\bar{S}_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\
&= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] \\
&= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right), \ \theta \geq 1.
\end{aligned}$$

- **Run length**
  We are dealing once again with a Shewhart chart, thus,

$$RL_\sigma(\theta) \sim \text{Geometric}\left(\xi_\sigma(\theta)\right).$$

- **Obtaining γ_σ**

$$\begin{aligned}
\gamma_\sigma \ : \ & \frac{1}{\xi_\sigma(1)} = ARL_\sigma(1) \\
& 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = \frac{1}{ARL_\sigma(1)} \\
& \gamma_\sigma = F^{-1}_{\chi^2_{(n-1)}}\left(1 - \frac{1}{ARL_\sigma(1)}\right) \\
& \gamma_\sigma = F^{-1}_{\chi^2_{(5-1)}}(0.995) \\
& \gamma_\sigma \overset{table}{=} 14.86
\end{aligned}$$

- **Probability of a misleading signal of type III**

$$PMS_{III}(\theta) \overset{Table\ 10.12}{=} \frac{1 - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}{[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)]^{-1} - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}$$

$$\begin{aligned}
&\stackrel{\theta=1.9}{=} \frac{1-[\Phi(3/1.9)-\Phi(-3/1.9)]}{[F_{\chi^2_{(5-1)}}(14.86/1.9^2)]^{-1}-[\Phi(3/1.9)-\Phi(-3/1.9)]}\\
&\stackrel{table,(a)}{=} \frac{2\times(1-0.9429)}{[F_{\chi^2_4}(4.12)]^{-1}-(2\times 0.9429-1)}\\
&\in \left(\frac{2\times(1-0.9429)}{0.600^{-1}-(2\times 0.9429-1)};\frac{2\times(1-0.9429)}{0.700^{-1}-(2\times 0.9429-1)}\right)\\
&\simeq (0.146248; 0.210402)
\end{aligned}$$

because $F^{-1}_{\chi^2_{(4)}}(0.600) = 4.045 < 4.12 < 4.878 = F^{-1}_{\chi^2_{(4)}}(0.700)$.

- **Probability of a misleading signal of type IV**

$$\begin{aligned}
PMS_{IV}(\delta) &\stackrel{Table\ 10.12}{=} \frac{1-F_{\chi^2_{(n-1)}}(\gamma_\sigma)}{[\Phi(\gamma_\mu-\delta)-\Phi(-\gamma_\mu-\delta)]^{-1}-F_{\chi^2_{(n-1)}}(\gamma_\sigma)}\\
&\stackrel{\delta=0.1}{=} \frac{1-F_{\chi^2_{(5-1)}}(14.86)}{[\Phi(3-0.1)-\Phi(-3-0.1)]^{-1}-F_{\chi^2_{(5-1)}}(14.86)}\\
&\stackrel{table}{=} \frac{1-0.995}{(0.9981-0.0010)^{-1}-0.995}\\
&\simeq 0.632236.
\end{aligned}$$

6. *Um revendedor recebe lotes de $N = 5000$ componentes electrónicas usadas na reparação de automóveis e pretende recorrer a um plano de amostragem simples por atributos.*

(a) *Averigue se o plano de amostragem, caracterizado por $n = 125$ e $c = 2$, está associado aos pontos do risco do revendedor ($AQL = 0.01, 1-\alpha = 0.95$) e do risco do produtor ($LTPD = 0.10$, $\beta = 0.10$).* (1.5)

- **Single acceptance plan for attributes**
  $N = 5000$
  $n = 125$
  $c = 2$
- **Producer's and consumer's risk points**
  $(AQL = p_1, 1-\alpha) = (1\%, 0.95)$
  $(LTPD = p_2, \beta) = (10\%, 0.10)$
- **Requested check**
  According to Wetherill and Brown (1991), the acceptance number c and sample size n of a sampling plan for attributes, associated to risk points $(p_1, 1-\alpha)$ and (p_2, β), can be approximately obtained:[1]
  - c should be taken as the smallest integer satisfying
    $$r(c) \leq \frac{p_2}{p_1},$$
    where $r(c) = \frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{F^{-1}_{\chi^2_{2(c+1)}}(\alpha)}$;
  - n should be taken as the smallest integer satisfying
    $$\frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{2p_2} \leq n \leq \frac{F^{-1}_{\chi^2_{2(c+1)}}(\alpha)}{2p_1}.$$

[1]See page 129 of the lecture notes, in particular, formulae (13.11), (13.10) and (13.12).

For $c = 2$, we get

$$\begin{aligned}
r(2) &= \frac{F^{-1}_{\chi^2_{2\times(2+1)}}(1-0.10)}{F^{-1}_{\chi^2_{2\times(2+1)}}(0.05)}\\
&= \frac{F^{-1}_{\chi^2_{(6)}}(0.90)}{F^{-1}_{\chi^2_{(6)}}(0.05)}\\
&\stackrel{table}{=} \frac{10.64}{1.635}\\
&\simeq 6.507645
\end{aligned}$$

which is indeed smaller than or equal to $\frac{p_2}{p_1} = 10$. However,

$$\begin{aligned}
\frac{F^{-1}_{\chi^2_{2(c+1)}}(1-\beta)}{2p_2} &\leq n \leq \frac{F^{-1}_{\chi^2_{2(c+1)}}(\alpha)}{2p_1}\\
\frac{F^{-1}_{\chi^2_{(6)}}(0.90)}{0.20} &\leq n \leq \frac{F^{-1}_{\chi^2_{(6)}}(0.05)}{2\times 0.01}\\
\frac{10.64}{2\times 0.10} &\leq n \leq \frac{1.635}{0.02}\\
53.2 &\leq n = 125 \leq 81.75
\end{aligned}$$

is FALSE proposition.

- **[Obs.** Alternatively, we could have checked if the conditions
  $$\begin{cases} F_{Binomial(n,AQL)}(c) \geq 1-\alpha \\ F_{Binomial(n,LTPD)}(c) \leq \beta \end{cases}$$
  hold for $n = 125$ and $c = 2$.]

(b) *Determine a redução relativa da fracção de unidades defeituosas nos lotes devido à rectificação da inspecção, quando o verdadeiro valor da fracção de peças defeituosas é igual a $p = 0.04$. Comente.* (1.0)

- **Auxiliary r.v. and its approximate distribution**
  D = number of defective units in the sample
  $D \stackrel{a}{\sim} \text{Binomial}(n, p)$
- **Average outgoing quality (AOQ) of a single sampling plan with rectifying inspection**

$$\begin{aligned}
AOQ(p) &\stackrel{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N}\\
&= \frac{p\,(N-n)}{N} \times P(D \leq c)\\
&= \frac{p\,(N-n)}{N} \times \sum_{d=0}^{c} \binom{n}{d} p^d (1-p)^{n-d}\\
&= \frac{0.04\times(5000-125)}{5000} \times \sum_{d=0}^{2} \binom{125}{d} 0.04^d (1-0.04)^{125-d}\\
&\simeq \frac{0.04\times(5000-125)}{5000} \times 0.119552\\
&\simeq 0.004663
\end{aligned}$$

- **Relative reduction in the fraction defective**
  Rectifying inspection leads to a relative reduction in the fraction defective equal to:

$$\begin{aligned}\left[1-\frac{AOQ(p)}{p}\right]\times 100\% &= \left[1-\frac{AOQ(0.04)}{0.04}\right]\times 100\% \\ &\simeq \left[1-\frac{0.004663}{0.04}\right]\times 100\% \\ &\simeq 88.3436\%.\end{aligned}$$

- **Comment**
  Rectifying inspection is worth being done because it is responsible for a substantial relative decrease of 88% in the fraction defective.