Fiabilidade e Controlo de Qualidade — LMAC

Exame de 2a. Época / 2o. Teste — 2o. Semestre – 2004/05

Duração: 3 horas / 1 hora e 45 minutos — 08/Julho/05 (6a. feira) – 17h, P12

1. *Considere os seguintes registos dos instantes de falha graves (em milhas percorridas) de 9 autocarros de carreira: 162, 200, 271, 302, 393, 508, 539, 629, 706.*

   (a) *Exemplifique a utilização de um gráfico TTT com alguns cálculos (por exemplo três pontos).*
   Que aspecto esperaria para este gráfico, caso se desconfiasse da adequação do modelo exponencial a este conjunto de dados? (1.5)

   - **Failure times**
     T_i = time (in miles) to a serious failure of the i^{th}bus, $i = 1, \ldots, 9$
   - **Complete data**
     $\underline{t} = (162, 200, 271, 302, 393, 508, 539, 629, 706)$
   - **Total time on test up to time $t_{(i)}$**

$$\begin{aligned}\tau\left(t_{(i)}\right) &= \sum_{j=1}^{i}(n-j+1)\left[t_{(j)} - t_{(j-1)}\right] \\ &= \tau\left(t_{(i-1)}\right) + (n-i+1)\left[t_{(i)} - t_{(i-1)}\right]\end{aligned}$$

   - **Abcissae of the TTT plot**
     $\frac{i}{n}$, $i = 0, 1, \ldots, n$
   - **Ordinates of the TTT plot**
     $\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$, $i = 1, \ldots, n$, where $\tau\left(t_{(n)}\right) = 3710$; equal to 0, for $i = 0$.
   - **Four points of the TTT plot**

i	$\tau\left(t_{(i)}\right) = \tau\left(t_{(i-1)}\right) + (n-i+1)\left[t_{(i)} - t_{(i-1)}\right]$	$\frac{i}{n}$	$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$
0	0	0	0
1	$0 + (9-1+1) \times 162 = 1458$	$\frac{1}{9} = 0.(1)$	$\frac{1458}{3710} \simeq 0.393$
2	$1458 + (9-2+1) \times (200-162) = 1762$	$\frac{2}{9} = 0.(2)$	$\frac{1762}{3710} \simeq 0.475$
3	$1762 + (9-3+1) \times (271-200) = 2259$	$\frac{3}{9} = 0.(3)$	$\frac{2259}{3710} \simeq 0.609$
4	$2259 + (9-4+1) \times (302-271) = 2445$	...	...
5	$2445 + (9-5+1) \times (393-302) = 2900$	...	...
6	$2900 + (9-6+1) \times (508-393) = 3360$	...	...
7	$3360 + (9-7+1) \times (539-508) = 3453$	...	...
8	$3453 + (9-8+1) \times (629-539) = 3633$	...	...
9	$3633 + (9-9+1) \times (706-629) = 3710$	...	...

   - **Aspect of the TTT plot**
     According to Remark 5.5, the TTT plot should be a line in case $T_i \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$. [Judging by the three points we just obtained that does not seem to be the case.]

   (b) *Calcule uma estimativa da função de fiabilidade de um autocarro para a distância de 200 milhas.*
   Admita agora que as 9 observações dizem respeito a 9 autocarros de carreira colocados a circular simultaneamente, de entre um grupo de 20 de uma frota. Considerando as hipóteses de trabalho que entender convenientes, reavalie a estimativa que obteve. Comente. (2.0)

   - **Distribution assumption**
     $T_i \stackrel{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$.
     To know if this assumption is reasonable the Bartlett's goodness of fit test should be done.]
   - **Unknown parameter**
     $R_T(t) = e^{-\lambda t}$, where $t = 200$ miles.
   - **Censored data**
     $n = 20$
     $r = 9$
     $(t_{(1)}, \ldots, t_{(r)}) = (162, 200, 271, 302, 393, 508, 539, 629, 706)$
   - **Cumulative total time in test**
     According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= \sum_{i=1}^{r} t_{(i)} + (n-r) \times t_{(r)} \\ &= (162 + \ldots + 706) + (20-9) \times 760 \\ &= 11476.\end{aligned}$$

   - **Unbiased estimate of $R_T(t)$**
     According to Table 5.14, the UMVUE of $R_T(t)$ is, for $t = 200 < \tilde{t} = 11476$ and $r > 0$,

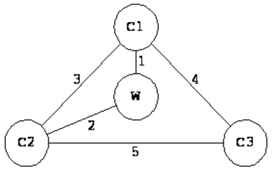
$$\begin{aligned}\tilde{R}_T(t) &= \left(1 - \tilde{t}^{-1} \times t\right)^{r-1} \\ &= \left(1 - \frac{1}{11476} \times 200\right)^{9-1} \\ &\simeq 0.868793.\end{aligned}$$

   - **Comment**
     According to Table 5.1, the nonparametric estimate of $R_T\left(t_{(i)}\right)$ proposed by Blom is equal to

$$\begin{aligned}\hat{R}_T\left(t_{(i)}\right) &= \frac{n-i+0.625}{n+0.25} \\ &\stackrel{i=2}{=} 0.824324.\end{aligned}$$

     The relative difference between the UMVUE of $R_T(200)$, $\tilde{R}_T(200) = 0.868793$, and the nonparametric estimate of $R_T(200)$, $\hat{R}_T(200) = 0.824324$ is approximately equal to 5%. This suggest that censoring the data had a mild impact on this particular point estimate.

2. *A figura abaixo descreve um sistema de (re)distribuição de água a três cidades C_1, C_2 e C_3 a partir de uma central de fornecimento de água W. Diz-se que o sistema de (re)distribuição de água está operacional se as três cidades receberem água.*



(a) *Admita que os tempos até entupimento das condutas de água (em centenas de horas) são v.a.s i.i.d., IHRA, com valor esperado igual a 750h. Obtenha um limite inferior para o valor esperado do tempo até que o sistema de (re)distribuição de água deixe de estar operacional, limite esse que determinará o intervalo entre operações de manutenção/desobstrução das condutas de água.* (1.5)

- **Individual durations (in 10^2h), common stochastic aging character and expected value**
  $T_i \stackrel{i.i.d.}{\sim} IHRA$, $i = 1, \ldots, 5$, with common expected value $\mu_i = E(T_i) = \mu^* = 7.5$.
- **Duration of the system (in 10^2h)**
  T
- **Lower bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - the 5 components have durations $T_i \stackrel{i.i.d.}{\sim} IHRA$, $i = 1, \ldots, 5$ and, thus, according to Proposition 3.36,
    $T_i \stackrel{i.i.d.}{\sim} NBUE$, $i = 1, \ldots, 5$;
  - the expected value of the duration of each of the 5 components is equal to $\mu_i = E(T_i) = \mu^* = 7.5$;
  - the minimal path sets are

$$
\begin{aligned}
\mathcal{P}_1 &= \{1,2,4\} \\
\mathcal{P}_2 &= \{1,2,5\} \\
\mathcal{P}_3 &= \{1,4,5\} \\
\mathcal{P}_4 &= \{1,3,4\} \\
\mathcal{P}_5 &= \{1,3,5\} \\
\mathcal{P}_6 &= \{2,3,4\} \\
\mathcal{P}_7 &= \{2,3,5\} \\
\mathcal{P}_8 &= \{2,4,5\} \\
p &= 8 \text{ minimal path sets.}
\end{aligned}
$$

  Now, we can apply Theorem 3.65, and conclude that

$$
\begin{aligned}
\mu &= E(T) \\
&\geq \max_{j=1,\ldots,p} \left\{ \left(\sum_{i \in \mathcal{P}_j} \mu_i^{-1} \right)^{-1} \right\} \\
&\stackrel{\mu_i=\mu^*}{=} \max_{j=1,\ldots,p} \left\{ \left(\frac{\#\mathcal{P}_j}{\mu^*} \right)^{-1} \right\} \\
&\stackrel{\#\mathcal{P}_j=3}{=} \frac{\mu^*}{3} \\
&= \frac{7.5}{3} \\
&= 2.5.
\end{aligned}
$$

(b) *Como caracterizaria o tempo até que o sistema deixa de estar operacional (T) quanto ao envelhecimento estocástico? Justifique.*
Obtenha a expressão de um limite superior para $E(T^2)$. (2.0)

- **Stochastic ageing of T**
  We are dealing with a coherent system with 5 components whose durations $T_i \stackrel{i.i.d.}{\sim} IHRA$, $i = 1, \ldots, 5$ and, thus, according to Table 3.2, we can conclude that
  $T \sim IHRA$.
- **Upper bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - $T_i \stackrel{i.i.d.}{\sim} IHRA$, $i = 1, \ldots, 5$;
  - $\mu_i = E(T_i) = \mu^* = 7.5$;
  - the minimal cut sets are

$$
\begin{aligned}
\mathcal{K}_1 &= \{1,2\} \\
\mathcal{K}_2 &= \{4,5\} \\
\mathcal{K}_3 &= \{1,3,4\} \\
\mathcal{K}_4 &= \{2,3,5\} \\
\mathcal{K}_5 &= \{1,3,5\} \\
\mathcal{K}_6 &= \{2,3,4\} \\
q &= 6 \text{ minimal cut sets.}
\end{aligned}
$$

  Now, we can apply Theorem 3.69, and conclude that

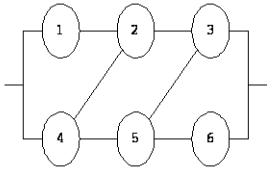
$$
\begin{aligned}
\mu &= E(T) \\
&\leq \min_{j=1,\ldots,q} \int_0^{+\infty} \left[1 - \prod_{i \in \mathcal{K}_j} \left(1 - e^{-t/\mu_i} \right) \right] dt \\
&\stackrel{\mu_i=\mu^*}{=} \min_{j=1,\ldots,q} \int_0^{+\infty} \left[1 - \left(1 - e^{-t/\mu^*} \right)^{\#\mathcal{K}_j} \right] dt \\
&= \int_0^{+\infty} \left[1 - \left(1 - e^{-t/\mu^*} \right)^{\min_{j=1,\ldots,q} \#\mathcal{K}_j} \right] dt \\
&= \int_0^{+\infty} \left[1 - \left(1 - e^{-t/\mu^*} \right)^2 \right] dt \\
&= \int_0^{+\infty} \left(2e^{-t/\mu^*} - e^{-2t/\mu^*} \right) dt \\
&= \left(-2\mu^* e^{-t/\mu^*} + \frac{\mu^*}{2} e^{-2t/\mu^*} \right) \Big|_0^{+\infty} \\
&= 2\mu^* - \frac{\mu^*}{2}
\end{aligned}
$$

$$= \frac{3\mu^*}{2}$$
$$= 11.25.$$

- **Upper bound for $E(T^2)$**
  Since $T \in IHRA$ we can apply Theorem 3.53, capitalize on the previous upper bound for $E(T)$ and conclude that

$$\begin{aligned} E(T^2) &\leq \Gamma(2+1) \times [E(T)]^2 \\ &\leq 2! \times \left(\frac{3\mu^*}{2}\right)^2 \\ &= 253.125. \end{aligned}$$

3. *Considere que as componentes do sistema* double crosslinked *abaixo são independentes e possuem fiabilidade comum e igual a* $p = 0.95$.



(a) *Determine a função de estrutura usando para o efeito os cortes mínimos.* (1.5)

- **Minimal cut sets**

$$\begin{aligned} \mathcal{K}_1 &= \{1,4\} \\ \mathcal{K}_2 &= \{2,4\} \\ \mathcal{K}_3 &= \{2,5\} \\ \mathcal{K}_4 &= \{3,5\} \\ \mathcal{K}_5 &= \{3,6\} \\ \mathcal{K}_6 &= \{3,4\} \\ q &= 6 \text{ minimal cut sets} \end{aligned}$$

- **Structure function**
  By considering $X_i \sim \text{Bernoulli}(p_i)$, $i = 1, \ldots, 6$ and applying result (1.14), we can conclude that the structure function of this system equals

$$\begin{aligned} \phi(\underline{X}) &\stackrel{(1.14)}{=} \prod_{j=1}^{q} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - X_i)\right] \\ &= [1-(1-X_1)(1-X_4)] \times [1-(1-X_2)(1-X_4)] \\ &\quad \times [1-(1-X_2)(1-X_5)] \times [1-(1-X_3)(1-X_5)] \\ &\quad \times [1-(1-X_3)(1-X_6)] \times [1-(1-X_3)(1-X_4)]. \end{aligned}$$

(b) *Obtenha limites inferior e superior o mais estritos possível para a fiabilidade do sistema, ao admitir que as componentes estão associadas (positivamente) ao invés de serem independentes.* (1.5)

- **Components**
  $p_i = p = 0.95$, $i = 1, \ldots, 7$

Since the 5 components form a coherent system and operate in a positively associated fashion, we can apply Theorem 1.70, namely result (1.42).

- **Minimal path sets**

$$\begin{aligned} \mathcal{P}_1 &= \{1,2,3\} \\ \mathcal{P}_2 &= \{4,5,6\} \\ \mathcal{P}_3 &= \{4,2,3\} \\ \mathcal{P}_4 &= \{4,5,3\} \\ p^* &= 4 \text{ minimal path sets} \end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned} \mathcal{K}_1 &= \{1,4\} \\ \mathcal{K}_2 &= \{2,4\} \\ \mathcal{K}_3 &= \{2,5\} \\ \mathcal{K}_4 &= \{3,5\} \\ \mathcal{K}_5 &= \{3,6\} \\ \mathcal{K}_6 &= \{3,4\} \\ q &= 6 \text{ minimal cut sets} \end{aligned}$$

- **Lower bound for the reliability $r(\underline{p})$**

$$\begin{aligned} r(\underline{p}) &\stackrel{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i \in \mathcal{P}_j} p_i \\ &\stackrel{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\ &\stackrel{\#\mathcal{P}_j=3,\forall j}{=} p^3 \\ &\stackrel{p=0.975}{=} 0.95^3 \\ &= 0.857375. \end{aligned}$$

- **Upper bound for the reliability**

$$\begin{aligned} r(\underline{p}) &\stackrel{(1.42)}{\leq} \min_{j=1,\ldots,q} \left[1 - \prod_{i \in \mathcal{K}_j} (1-p_i)\right] \\ &\stackrel{p_i=p}{=} \min_{j=1,\ldots,q} \left[1-(1-p)^{\#\mathcal{K}_j}\right] \\ &\stackrel{\#\mathcal{K}_j=2,\forall j}{=} 1-(1-p)^2 \\ &\stackrel{p=0.95}{=} 1-(1-0.95)^2 \\ &= 0.9975. \end{aligned}$$

4. *O gestor de uma grande tinturaria responsável pelo tingimento de peças de* $100m$ *de tecido acredita que perderá um importante contrato se o número de imperfeições por peça exceder 4 em exactamente 1.86% dos casos.*

(a) *Que tipo de carta lhe parece mais razoável utilizar?*
Determine o respectivo alvo e os seus limites de controlo. (1.0)

- **Control statistic**
  Y_N = number of imperfections in the N^{th} piece of 100m of fabric, $N \in \mathbb{N}$
- **Distributions**
  $Y_N \sim \text{Poisson}(\lambda_0)$, IN CONTROL
  $Y_N \sim \text{Poisson}(\lambda = \lambda_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of the shift in λ
- **Obtaining the target value of λ**
  According to the text and looking at the table of the c.d.f. of some Poisson r.v., we get
  $$\begin{aligned} \lambda_0 \quad : \quad & P(Y_N > 4|\lambda = \lambda_0) = 0.0186 \\ & F_{Poisson(\lambda_0)}(4) = 0.9814 \\ & \lambda_0 = 1.5. \end{aligned}$$
- **Suitable chart**
  An UPPER ONE-SIDED u chart should be adopted because we are collecting the number of imperfections per piece of 100m of fabric, and it is very important to detect increases in λ, the expected value of this r.v.
- **Target of the upper one-sided u chart**
  $CL = \lambda_0$
- **Control limits of the $3\,sigma$ upper one-sided u chart**
  $$\begin{aligned} LCL &= 0 \text{ (because we are dealing with an upper one-sided chart)} \\ UCL &= \lambda_0 + \gamma\sqrt{\lambda_0} \\ &= 1.5 + 3\sqrt{1.5} \\ &= 5.174235 \end{aligned}$$

(b) *Caso tivesse ocorrido uma alteração no valor esperado do número de imperfeições por peça para 2, qual seria a mediana do número de amostras recolhidas até à detecção de tal alteração por parte da carta que escolheu? Comente o seu significado.* (1.5)

- **Run length**
  We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:
  $$RL(\delta) \sim \text{Geometric}\left(\xi(\delta)\right).$$
- **Shift**
  $\delta = \lambda - \lambda_0 = 2 - 1.5 = 0.5$
- **Probability of triggering a signal**
  $$\begin{aligned} \xi(\delta) &= P(Y_N \notin [LCL, UCL] \mid \delta) \\ &\overset{Y_N \geq 0, LCL=0}{=} P\left(Y_N > UCL \mid \delta\right) \\ &= 1 - F_{Poisson(\lambda=\lambda_0+\delta)}(UCL) \\ &= 1 - F_{Poisson(2)}(5.174235) \\ &= 1 - F_{Poisson(2)}(5) \\ &\overset{table}{=} 1 - 0.9834 \\ &= 0.0166 \end{aligned}$$

- **Median of $RL(\delta)$**
  $$\begin{aligned} F^{-1}_{RL(\delta)}(p) &\overset{Table\,9.2}{=} \inf\left\{m \in \mathbb{N} : F_{RL(\delta)}(m) \geq p\right\} \\ &= 1 - [1 - \xi(\delta)]^m \geq p \\ &= [1 - \xi(\delta)]^m \leq 1 - p \\ &= m \times \ln\left[1 - \xi(\delta)\right] \leq \ln(1 - p) \\ &\overset{\ln[1-\xi(\delta)]<0}{=} m \geq \frac{\ln(1-p)}{\ln\left[1 - \xi(\delta)\right]} \\ &\overset{p=0.5}{=} m \geq \frac{\ln(1-0.5)}{\ln(1-0.0166)} \\ &= m \geq 41.408314 \\ &= 42 \end{aligned}$$
- **Comment**
  When the expected value of the number of imperfections per piece of 100m of fabric increases from its target value $\lambda_0 = 1.5$ to $\lambda = 2$, the probability of triggering a valid signal within the first 42 samples is at least 50%.

5. *A grossura da parede é uma característica importante na produção de garrafas de vidro para bebidas gaseificadas. Admita que esta v.a. possui distribuição normal com valor esperado sob controlo $\mu_0 =$ 0.6cm e desvio-padrão constante e igual a $\sigma_0 = 0.1cm$.*

   O controlo de μ é feito à custa de uma carta EWMA *padrão com limites assintóticos, caracterizada por $n = 5$, $\lambda = 0.134$ e $\gamma_{EWMA} = 2.8891$, que possui $ARL_{EWMA}(0) = 500$ e $ARL_{EWMA}(1.5) = 5.789$.*

   (a) *Após ter preenchido a tabela abaixo, o que poderá afirmar acerca do estado do processo de produção de garrafas?* (1.0)

N	1	2	3	4	5
$\bar{x}_N$	0.71	0.69	0.72	0.59	0.59
w_N		0.624825	0.637578	0.631203	

- **Quality characteristic**
  X = thickness of the glass bottle
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Estimator of μ**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
- **Distribution**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 0.6$, $\sigma_0 = 0.1$ and $n = 5$
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0^2}{n}, \frac{\sigma_0^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or an increase!) in μ.
- **(Asymptotic) control limits of the standard EWMA control chart for μ**
  Since $n = 5$, $\lambda = 0.134$ and $\gamma_{EWMA} = 2.8891$, they are equal to
  $$\begin{aligned} LCL_{EWMA} &= \mu_0 - \gamma_{EWMA} \times \sqrt{\frac{\lambda}{2-\lambda}\frac{\sigma_0^2}{n}} \\ &= 0.6 - 2.8891 \times \sqrt{\frac{0.134}{2-0.134}\frac{0.1^2}{5}} \end{aligned}$$

$$
\begin{aligned}
&= 0.565376 \\
UCL_{EWMA} &= \mu_0 + \gamma_{EWMA} \times \sqrt{\frac{\lambda}{2-\lambda}\frac{\sigma_0^2}{n}} \\
&= 0.6 + 2.8891 \times \sqrt{\frac{0.134}{2-0.134}\frac{0.1^2}{5}} \\
&= 0.634624
\end{aligned}
$$

- **EWMA control statistic**

$$
W_N = \begin{cases} \mu_0, & N = 0 \\ (1-\lambda)W_{N-1} + \lambda \bar{X}_N, & N \in \mathbb{N} \end{cases}
$$

- **Observed values of the EWMA control statistic**

N	$\bar{x}_N$	$w_N = \begin{cases} \mu_0, & N=0 \\ (1-\lambda)w_{N-1} + \lambda \bar{x}_N, & N \in \mathbb{N} \end{cases}$
0	—	0.6
1	0.71	$(1-0.134) \times 0.6 + 0.134 \times 0.71 = 0.61474$
2	0.69	0.624825
3	0.72	0.637578
4	0.59	0.631203
5	0.59	$(1-0.134) \times 0.631203 + 0.134 \times 0.59 = 0.625682$

- **Comment**
  The fact that $w_3 = 0.637578 \notin [LCL_{EWMA}, UCL_{EWMA}] = [0.565376, 0.634624]$ suggests that the process is out-of-control.

(b) *Descreva uma carta $\bar{X}$ padrão com o número esperado de amostras recolhidas até falso alarme igual a 500 amostras.*
Compare a velocidade média de detecção desta carta com a da carta EWMA *padrão descrita em (a) na detecção de um* shift *de magnitude* $\delta = 1.5$. Comente. (1.5)

- **Control statistic of the standard $\bar{X}$ chart**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n, previously described.

- **Control limits of the standard $\bar{X}$ chart**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$

- **Probability of triggering a signal**
  Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to:

$$
\begin{aligned}
\xi_\mu(\delta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta\right) \\
&= 1 - \left\{\Phi\left[\frac{UCL_\mu - (\mu_0 + \delta\sigma_0/\sqrt{n})}{\frac{\sigma_0}{\sqrt{n}}}\right] - \Phi\left[\frac{LCL_\mu - (\mu_0 + \delta\sigma_0/\sqrt{n})}{\frac{\sigma_0}{\sqrt{n}}}\right]\right\} \\
&= 1 - \left[\Phi\left(\gamma_\mu - \delta\right) - \Phi\left(-\gamma_\mu - \delta\right)\right], \ \delta \in \mathbb{R}.
\end{aligned}
$$

- **Run length**
  The number of samples collected until this Shewhart chart triggers a signal given δ, $RL_\mu(\delta)$, has the following distribution:

$$
RL_\mu(\delta) \sim \text{Geometric}\left(\xi_\mu(\delta)\right)
$$

- **Obtaining γ_μ**
  The constant γ_μ is such that $ARL_\mu(0) = 500$, that is,

$$
\begin{aligned}
\gamma_\mu \ : \ \frac{1}{\xi_\mu(0)} &= ARL_\mu(0) \\
1 - \left[\Phi\left(\gamma_\mu\right) - \Phi\left(-\gamma_\mu\right)\right] &= \frac{1}{ARL_\mu(0,1)} \\
\gamma_\mu &= \Phi^{-1}\left(1 - \frac{1}{2 \times ARL_\mu(0,1)}\right) \\
\gamma_\mu &= \Phi^{-1}(0.999) \\
\gamma_\mu &\stackrel{table}{=} 3.0902
\end{aligned}
$$

- **Probability of detecting a shift of magnitude $\delta = 1.5$**

$$
\begin{aligned}
\xi_\mu(1.5) &= 1 - \left[\Phi\left(3.0902 - 1.5\right) - \Phi\left(-3.0902 - 1.5\right)\right] \\
&\simeq 1 - [\Phi(1.59) - \Phi(-4.59)] \\
&\stackrel{table}{=} 1 - (0.9441 - 0.0000) \\
&= 0.0559
\end{aligned}
$$

- **Requested out-of-control ARL**

$$
\begin{aligned}
ARL_\mu(1.5) &= \frac{1}{\xi_\mu(1.5)} \\
&\simeq \frac{1}{0.0559} \\
&\simeq 17.889.
\end{aligned}
$$

- **Comment**
  $ARL_\mu(1.5) = 17.889 > ARL_{EWMA}(1.5) = 5.789$.
  This is an expected result because the $\bar{X}$ chart tends to be, in average, slower than the EWMA chart in the detection of small and medium size shifts like $\delta = 1.5$.

6. *O fenómeno dos sinais erróneos não é exclusivo dos esquemas conjuntos para μ e σ.*

(a) *Qual a probabilidade de ser emitido um sinal erróneo pelo esquema S^2 unilateral superior com número esperado de amostras até falso alarme igual a 100, quando $n = 10$ e há uma redução de 10% no desvio-padrão?* (1.0)

- **Quality characteristic**
  $X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistic**
  S_N^2 = variance of the N^{th} random sample of size n, $N \in \mathbb{N}$

- **Distributions**
  $\frac{(n-1)S_N^2}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL, where $n = 10$
  $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL, where θ ($\theta > 0$) represents a shift (a decrease or an increase!) in the standard deviation σ

- **Control limits of the upper one-sided S^2- chart**
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probability of triggering a signal**

$$\begin{aligned}\xi_\sigma(\theta) &= P\left(S_N^2 \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right)\\ &= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right]\\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right),\ \theta > 0\end{aligned}$$

- **Run length**

$$RL_\sigma(\theta) \sim \text{Geometric}\,(\xi_\sigma(\theta)).$$

- **Obtaining γ_σ**

The constant γ_σ is such that $ARL_\sigma(1) = 100$, i.e.

$$\begin{aligned}\gamma_\sigma \quad &: \quad \frac{1}{\xi_\sigma(1)} = ARL_\sigma(1)\\ &1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = \frac{1}{ARL_\sigma(1)}\\ &\gamma_\sigma = F^{-1}_{\chi^2_{(n-1)}}\left(1 - \frac{1}{ARL_\sigma(1)}\right)\\ &\gamma_\sigma = F^{-1}_{\chi^2_{(10-1)}}(0.990)\\ &\gamma_\sigma \overset{table}{=} 21.67\end{aligned}$$

- **Requested probability**

A reduction of 90% of σ corresponds to $\theta = 0.9$ and the probability that such a decrease is detected by this chart is equal to

$$\begin{aligned}\xi_\sigma(0.9) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right)\\ &= 1 - F_{\chi^2_{(10-1)}}\left(\frac{21.67}{0.9^2}\right)\\ &\simeq 1 - F_{\chi^2_{(9)}}(26.75)\\ &\in (0.001, 0.005)\end{aligned}$$

because $F^{-1}_{\chi^2_{(9)}}(0.995) = 23.59 < 26.75 < 27.88 = F^{-1}_{\chi^2_{(9)}}(0.999)$.

(b) *Compare-a com a correspondente probabilidade de emissão de sinal válido por parte de um esquema S^2 padrão com ARL sob controlo também igual a 100.*

Confronte também as probabilidades de emissão de sinal entre as primeiras 100 amostras destas duas cartas, mais uma vez quando $\theta = 0.9$. Comente estes resultados. (1.5)

- **Control limits of the standard S^2- chart**

According to Table 9.2,

$$\begin{aligned}LCL'_\sigma &= \frac{\sigma_0^2}{n-1}\gamma_\sigma^-\\ &= \frac{\sigma_0^2}{n-1} \times F^{-1}_{\chi^2_{(n-1)}}\left[\frac{1}{2 \times ARL_\mu(1)}\right]\\ &= \frac{\sigma_0^2}{9} \times F^{-1}_{\chi^2_{(9)}}(0.005)\\ &\overset{table}{=} \frac{\sigma_0^2}{9} \times 1.735\end{aligned}$$

$$\begin{aligned}UCL'_\sigma &= \frac{\sigma_0^2}{n-1}\gamma_\sigma^+\\ &= \frac{\sigma_0^2}{n-1} \times F^{-1}_{\chi^2_{(n-1)}}\left[1 - \frac{1}{2 \times ARL_\mu(1)}\right]\\ &= \frac{\sigma_0^2}{9} \times F^{-1}_{\chi^2_{(9)}}(0.995)\\ &\overset{table}{=} \frac{\sigma_0^2}{9} \times 23.59\end{aligned}$$

- **Requested probability**

A shift of magnitude $\theta = 0.9$ is detected by this standard chart with probability

$$\begin{aligned}\xi'_\sigma(\theta) &= P\left(S_N^2 \notin [LCL'_\sigma, UCL'_\sigma] \mid \theta\right)\\ &\overset{(9.8)}{=} 1 - \left[F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma^+}{\theta^2}\right) - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma^-}{\theta^2}\right)\right]\\ &= 1 - \left[F_{\chi^2_{(10-1)}}\left(\frac{23.59}{0.9^2}\right) - F_{\chi^2_{(10-1)}}\left(\frac{1.735}{0.9^2}\right)\right]\\ &\simeq 1 - \left[F_{\chi^2_{(9)}}(29.12) - F_{\chi^2_{(9)}}(2.14)\right]\\ &\in (1 - (0.9995 - 0.01),\ 1 - (0.999 - 0.025))\\ &\in (0.0105, 0.026)\end{aligned}$$

because $F^{-1}_{\chi^2_{(9)}}(0.999) = 27.88 < 29.12 < 29.67 = F^{-1}_{\chi^2_{(9)}}(0.9995)$ and $F^{-1}_{\chi^2_{(9)}}(0.01) = 2.088 < 2.14 < 2.700 = F^{-1}_{\chi^2_{(9)}}(0.025)$.

- **Comment**

Since the upper one-sided S^2 chart was not designed to detect decreases in σ^2, it is no surprise we have

$$\xi_\sigma(0.9) < \xi'_\sigma(0.9).$$

- **Run lengths**

$RL_\sigma(\theta) \sim \text{Geometric}\,(\xi_\sigma(\theta))$
$RL'_\sigma(\theta) \sim \text{Geometric}\,(\xi'_\sigma(\theta))$

- **Requested probabilities**

$$\begin{aligned}p &= P[RL_\sigma(\theta) \leq 100]\\ &= 1 - [1 - \xi_\sigma(\theta)]^{100}\\ &\in (1 - (1 - 0.001)^{100},\ 1 - (1 - 0.005)^{100})\\ &= (0.095208,\ 0.394230)\\ p' &= P[RL'_\sigma(\theta) \leq 100]\\ &= 1 - [1 - \xi'_\sigma(\theta)]^{100}\\ &\in (1 - (1 - 0.0105)^{100},\ 1 - (1 - 0.026)^{100})\\ &= (0.652000,\ 0.928278).\end{aligned}$$

- **Comment**

Unsurprisingly, $p << p'$ (for the same reason we pointed before).

7. *Uma gestora de uma companhia (que recebe lotes de dimensão $N = 2000$) adoptou um plano de amostragem simples com $n = 65$ e $c = 2$, tendo decidido pela rectificação da inspecção.*

(a) *Determine a redução relativa da fracção de unidades defeituosas nos lotes devida à rectificação da inspecção, quando o verdadeiro valor da fracção de peças defeituosas é igual a $p = 0.3\%$.* (1.5)

- **Single sampling plan (for attributes)**
  $N = 2000$ (lot size)
  $n = 65$ (sample size)
  $c = 2$ (acceptance number)

- **Auxiliary r.v. and its approximate distribution**
  $D =$ number of defective units in the sample $\stackrel{a}{\sim}$ Binomial(n, p)

- **Probability of accepting the lot**

$$\begin{aligned} P_a(p) &= P(D \leq c) \\ &\simeq F_{Binomial(n,p)}(c) \\ &\stackrel{p=0.003}{=} F_{Binomial(65,0.003)}(2) \\ &= \sum_{i=0}^{2} \binom{65}{i} \times 0.003^i \times (1-0.003)^{65-i} \\ &= 0.997^{65} + 65 \times 0.003 \times 0.997^{64} + 65 \times 32 \times 0.003^2 \times 0.997^{63} \\ &\simeq 0.998974 \end{aligned}$$

- **Average outgoing quality of a single sampling plan with rectifying inspection**

$$\begin{aligned} AOQ(p) &\stackrel{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\ &\simeq \frac{0.003 \times (2000-65) \times 0.998974}{2000} \\ &\simeq 0.002900. \end{aligned}$$

- **Relative reduction in the percentage defective**

$$\begin{aligned} \left[1 - \frac{AOQ(p)}{p}\right] \times 100\% &\simeq \left[1 - \frac{0.002900}{0.003}\right] \times 100\% \\ &\simeq 3.349\% \end{aligned}$$

(b) *Obtenha o número médio de unidades inspeccionadas deste plano de amostragem, considerando para o efeito o valor de p da alínea anterior. Comente.* (1.0)

- **Average total inspection of the single sampling plan with rectifying inspection**

$$\begin{aligned} ATI(p) &\stackrel{(13.15)}{=} n \times P_a(p) + N \times [1 - P_a(p)] \\ &\simeq 65 \times 0.998974 + 2000 \times [1 - 0.998974] \\ &\simeq 66.985310 \end{aligned}$$

- **Comment**
  $ATI(0.03) \simeq 66.985310$ is slightly larger than the sample size $n = 65$. Unsurprisingly, the relative reduction in the percentage defective is insignificant (approx. 3%).