Fiabilidade e Controlo de Qualidade — LMAC

Exame de 1a. Época / 2o. Teste — 2o. Semestre – 2004/05

Duração: 3 horas / 1 hora e 45 minutos — 24/Junho/05 (6a. feira) – 17h

1. *Vinte juntas de borracha foram sujeitas simultaneamente a um teste de vida acelerado que se deu por concluído aquando da ocorrência da 10a. falha. O registo do número de ciclos até falha conduziu aos seguintes resultados: 20400, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000.*

   (a) *Após ter identificado o tipo de teste de vida a que se recorreu, efectue um teste de ajustamento que lhe pareça adequado, considerando para o efeito um nível de significância de 10%.* (2.0)

- **Life test**
  Since the end of the test was determined by the $r = 10^{th}$ failure and nothing in this exercise suggests that the $n = 20$ rubber gaskets were replaced during the life test, we are dealing with a
  - Type II/item censored testing without replacement.
- **R.v.**
  T_i = time (in cycles) until failure of the i^{th} rubber gasket
  $T_i \stackrel{i.i.d.}{\sim} T$, $i = 1, \ldots, n$
- **Censored data**
  $n = 20$
  $r = 10$
  $(t_{(1)}, \ldots, t_{(r)}) = (20400, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000)$
- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= \sum_{i=1}^{r} t_{(i)} + (n-r) \times t_{(r)} \\ &= 20400 + \ldots + 166000 + (20-10) \times 166000 \\ &= 2495050\end{aligned}$$

- **Hypotheses**
  $H_0 : T \sim \text{Exponential}(\lambda)$
  $H_1 : T \sim \text{Weibull}(\lambda^{-1}, \alpha)$, $\alpha \neq 1$
- **Significance level**
  $\alpha_0 = 10\%$
- **Test statistic** (Bartlett's test)

$$\begin{aligned}B_r &\stackrel{(5.18)}{=} \frac{2r}{1+\frac{r+1}{6r}} \left(\ln\left(\frac{\mathcal{T}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln\left\{(n-i+1)[T_{(i)} - T_{(i-1)}]\right\}\right) \\ &\stackrel{a}{\sim}_{H_0} \chi^2_{(r-1)}\end{aligned}$$

  where the $T_{(i)} - T_{(i-1)}$s represent the times between consecutive failure times.

- **Rejection region of H_0**

$$\begin{aligned}W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right) \\ &\stackrel{r=10, \alpha_0=0.1}{=} (0, 3.325) \cup (16.92, +\infty)\end{aligned}$$

- **Decision**
  The observed value of the test statistic is

$$\begin{aligned}b_r &= \frac{2r}{1+\frac{r+1}{6r}} \left(\ln\left(\frac{\tilde{t}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln\left\{(n-i+1)[t_{(i)} - t_{(i-1)}]\right\}\right) \\ &= \frac{2r}{1+\frac{r+1}{6r}} \left(\ln\left(\frac{\tilde{t}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln[t_{(i)} - t_{(i-1)}] - \frac{1}{r}\sum_{i=1}^{r}\ln(n-i+1)\right) \\ &= \frac{1200}{71} \times \left(12.427234 - \frac{1}{10} \times 120.527182\right) \\ &= 6.329845 \\ &\notin W = (0, 3.325) \cup (16.92, +\infty),\end{aligned}$$

  therefore we should not reject H_0 for any significance level $\alpha \leq 10\%$.

   (b) *Considerando as hipóteses de trabalho que entender convenientes, obtenha uma estimativa pontual e outra intervalar para a função de fiabilidade de tal v.a. para 100000 ciclos. Comente os resultados obtidos.* (1.5)

- **Distribution assumption**
  $T_i \stackrel{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$, which quite reasonable considering the result in (a).
- **Unknown parameters**
  λ
  $R_T(t) = e^{-\lambda t}$, where $t = 100000$ cycles.
- **Censored data**
  $n = 20$
  $r = 10$
  $(t_{(1)}, \ldots, t_{(r)}) = (20400, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000)$
  $\tilde{t} = 2495050$
- **Unbiased estimate of $R_T(t)$**
  According to Table 5.14, the UMVUE of $R_T(t)$ is, for $t = 100000 < \tilde{t} = 2495050$ and $r > 0$,

$$\begin{aligned}\tilde{R}_T(t) &= \left(1 - \tilde{t}^{-1} \times t\right)^{r-1} \\ &= \left(1 - \frac{1}{2495050} \times 100000\right)^{10-1} \\ &\simeq 0.692019.\end{aligned}$$

- **Confidence interval for λ**
  According to Table 5.16 of the lecture notes,

$$\begin{aligned}CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_l; \lambda_U] \\ &= \left[\frac{F_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F_{\chi^2_{(2r)}}(1-\alpha/2)}{2 \times \tilde{t}}\right] \\ CI_{95\%}(\lambda) &= \left[\frac{F_{\chi^2_{(20)}}(0.025)}{2 \times 2495050}; \frac{F_{\chi^2_{(20)}}(0.975)}{2 \times 2495050}\right]\end{aligned}$$

$$= \left[\frac{9.591}{2\times 2495050}; \frac{34.17}{2\times 2495050}\right]$$
$$\simeq [0.00000192205571; 0.000006847558165]$$

- **Confidence interval for $R_T(t)$**
  Since $R_T(t) = e^{-\lambda t}$ is a decreasing function of $\lambda > 0$, we can state that
$$CI_{95\%}\left(e^{-\lambda t}\right) = \left[e^{-\lambda_U t}; e^{-\lambda_L t}\right] \overset{t=100000}{\simeq} [0.504213; 0.825141].$$

- **Comment**
  The point and interval estimates are very reasonable because they were obtained considering a distributional assumption that has been tested and not rejected previously in (a).

2. *Um armazém recebe motores de três fábricas distintas —* A, B *e* C *— responsáveis por* 20%, 30% *e* 50% *dos motores recebidos, respectivamente. Admita que os tempos até falha dos motores (em milhares de horas) são independentes com funções taxa de falha* $\lambda_A(t) = 0.25t^{-0.75}$, $\lambda_B(t) = \frac{1}{3}t^{-2/3}$, $\lambda_C(t) = 0.5t^{-0.5}$, *para* $t > 0$.

   (a) Obtenha o valor esperado da duração de um motor seleccionado ao acaso do *stock* do armazém. Determine a função de fiabilidade para um período de 1000 h desse mesmo motor. (1.5)

- **Events and probabilities**

A = selecting an engine produced by factory A	$P(A) = 0.20$
B = selecting an engine produced by factory B	$P(B) = 0.30$
C = selecting an engine produced by factory C	$P(C) = 0.50$

- **Conditional durations**
  $(T \mid i)$ = time to failure (in $10^3 h$) of an engine randomly selected from the warehouse, given that it has been produced by factory i, $i = A, B, C$

- **Hazard rate functions of the conditional durations**
  For $t > 0$,
$$(T \mid i) \sim \lambda_i(t) = \begin{cases} 0.25t^{-0.75}, & i = A \\ \frac{1}{3}t^{-2/3}, & i = B \\ 0.5t^{-0.5}, & i = C. \end{cases}$$

- **Distribution of the conditional durations**
  According to Definition 4.21, the hazard rate function of a Weibull distribution with scale parameter δ and shape parameter α, Weibull(δ, α), is equal to
$$\lambda(t) = \frac{\alpha}{\delta}\left(\frac{t}{\delta}\right)^{\alpha-1}, \; t \geq 0.$$
  Thus,
$$(T \mid i) \sim \begin{cases} \text{Weibull}(\delta = 1, \alpha_A = -0.75 + 1 = 0.25), & i = A \\ \text{Weibull}(\delta = 1, \alpha_B = -2/3 + 1 = 1/3), & i = B \\ \text{Weibull}(\delta = 1, \alpha_C = -0.5 + 1 = 0.5), & i = C. \end{cases}$$

- **Reliability functions of the conditional durations**
$$R_{T|i}(t) \overset{Def. 4.21}{=} \begin{cases} e^{-t^{0.25}}, & i = A \\ e^{-t^{1/3}}, & i = B \\ e^{-t^{0.5}}, & i = C \end{cases}$$

- **Another duration**
  T = time to failure (in $10^3 h$) of an engine randomly selected from the warehouse

- **Reliability function of T**
  Since T is a mixture of the three r.v. $(T|i)$, $i = A, B, C$, we successively get
$$F_T(t) \overset{Prop. 3.29}{=} \sum_{i=A}^{C} P(i) \times F_{T|i}(t)$$
$$R_T(t) = \sum_{i=A}^{C} P(i) \times R_{T|i}(t)$$
$$= 0.20 \times e^{-t^{0.25}} + 0.30 \times e^{-t^{1/3}} + 0.50 \times e^{-t^{0.5}}$$
$$\overset{t=1}{=} e^{-1}$$
$$\simeq 0.367879.$$

- **Expected value of T**
  Let us remind the reader that, for a nonnegative r.v. X, $E(X) = \int_0^{+\infty} R_X(t)\, dt$. Therefore,
$$E(T) = \int_0^{+\infty} R_T(t)\, dt$$
$$= \int_0^{+\infty} \left[\sum_{i=A}^{C} P(i) \times R_{T|i}(t)\right] dt$$
$$= \sum_{i=A}^{C} P(i) \times \left[\int_0^{+\infty} R_{T|i}(t)\, dt\right]$$
$$\overset{(T|i)\geq 0}{=} \sum_{i=A}^{C} P(i) \times E(T|i)$$
$$\overset{Ex. 4.22}{=} \sum_{i=A}^{C} P(i) \times \Gamma\left(\frac{1}{\alpha_i} + 1\right)$$
$$= 0.20 \times \Gamma\left(\frac{1}{\frac{1}{4}} + 1\right) + 0.30 \times \Gamma\left(\frac{1}{\frac{1}{3}} + 1\right) + 0.50 \times \Gamma\left(\frac{1}{\frac{1}{2}} + 1\right)$$
$$= 0.20 \times 4! + 0.30 \times 3! + 0.50 \times 2!$$
$$= 7.6.$$

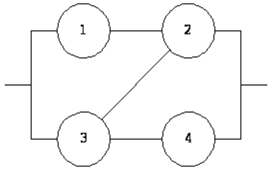
   (b) *Como caracterizaria a duração de um motor proveniente da fábrica* A *e a de um motor escolhido ao acaso no armazém quanto ao envelhecimento estocástico? Justifique.*
   Obtenha um limite inferior para o desvio-padrão da duração deste último motor. (2.0)

- **Stochastic ageing of T**
  First note that $\alpha_i < 1$, $i = A, B, C$. Therefore
$$(T \mid i) \sim DHR,\; i = A, B, C,$$
  according to subsection 4.3.4 (see table in page 100). Now, if capitalize on the fact that T is a mixture of independent DHR r.v. and on Proposition 3.29, namely result (3.22), we can conclude that
$$T \sim DHR.$$

- **Lower bound for $\sqrt{V(T)}$**
$$T \in DHR \overset{Prop. 3.36}{\Rightarrow} T \in DHRA$$

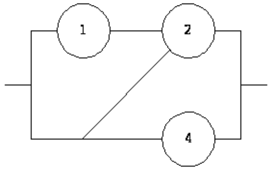
$$\overset{Cor. 3.54}{\Rightarrow} \quad \frac{\sqrt{V(T)}}{E(T)} \geq 1$$
$$\sqrt{V(T)} \geq E(T) = 7.6.$$

3. *O esquema que se segue descreve um sistema mecânico* cross linked *constituído por componentes que funcionam de modo independente e possuem fiabilidades iguais a* p_i, $i = 1, 2, 3, 4$.



(a) *Determine a função estrutura do sistema por decomposição fulcral em torno da componente 3.* (1.5)

- **Structure function by pivotal decomposition around component 3**
  - **Sub-system associated to** $(1_3, \underline{X})$



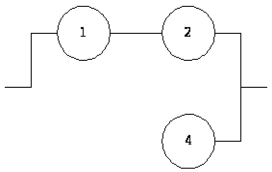
Minimal path sets

$$\mathcal{P}_1 = \{2\}$$
$$\mathcal{P}_2 = \{4\}$$
$$p^* = 2 \text{ minimal path sets}$$

Structure function

$$\phi(1_3, \underline{X}) \overset{(1.13)}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) = 1 - (1 - X_2)(1 - X_4).$$

  - **Sub-system associated to** $(0_3, \underline{X})$



Minimal path sets

$$\mathcal{P}_1 = \{1, 2\}$$
$$p^* = 1 \text{ minimal path set}$$

Structure function

$$\phi(0_3, \underline{X}) \overset{(1.13)}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) = 1 - (1 - X_1 X_2).$$

  - **Structure function of the original system**

$$\phi(\underline{X}) \overset{(1.7)}{=} X_3 \times \phi(1_3, \underline{X}) + (1 - X_3) \times \phi(0_3, \underline{X})$$
$$= X_3 \times [1 - (1 - X_2)(1 - X_4)] + (1 - X_3) \times [1 - (1 - X_1 X_2)].$$

(b) *Admita agora que as componentes do sistema funcionam de forma associada (positiva). Calcule um par de limites inferior e superior o mais estritos possível para a fiabilidade deste sistema mecânico quando* $p_i = p = 0.975$, $i = 1, 2, 3, 4$. *Comente.* (1.5)

- **Components**
  $p_i = p = 0.975$, $i = 1, \ldots, 4$
  Since the 4 components form a coherent system and operate in a positively associated fashion, we can apply Theorem 1.70, namely result (1.42).

- **Minimal path sets**

$$\mathcal{P}_1 = \{1, 2\}$$
$$\mathcal{P}_2 = \{2, 3\}$$
$$\mathcal{P}_3 = \{3, 4\}$$
$$p^* = 3 \text{ minimal path sets}$$

- **Minimal cut sets**

$$\mathcal{K}_1 = \{1, 3\}$$
$$\mathcal{K}_2 = \{2, 3\}$$
$$\mathcal{K}_3 = \{2, 4\}$$
$$q = 3 \text{ minimal cut sets}$$

- **Lower bound for the reliability** $r(\underline{p})$

$$r(\underline{p}) \overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i \in \mathcal{P}_j} p_i \overset{p_i = p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \overset{\#\mathcal{P}_j = 2, \forall j}{=} p^2 \overset{p=0.975}{=} 0.975^2 = 0.950625.$$

- **Upper bound for the reliability**

$$r(\underline{p}) \overset{(1.42)}{\leq} \min_{j=1,\ldots,q}\left[1 - \prod_{i \in \mathcal{K}_j}(1 - p_i)\right]$$

$$\begin{aligned}
&\overset{p_i=p}{=} \min_{j=1,\dots,q}\left[1-(1-p)^{\#\mathcal{K}_j}\right] \\
&\overset{\#\mathcal{K}_j=2,3}{=} \min\left\{1-(1-p)^2,\, 1-(1-p)^3\right\} \\
&= 1-(1-p)^2 \\
&\overset{p=0.975}{=} 1-(1-0.975)^2 \\
&= 0.999375.
\end{aligned}$$

- **Comment**
Since the components are positively associated we cannot apply Theorem 1.68 (for independent components!) which sometimes provides stricter bounds than the ones we just obtained by using Theorem 1.70.

4. *Uma máquina automática é usada para encher e selar latas de 0.1 l de um produto líquido. Considerando as hipóteses que entender mais razoáveis e amostras de dimensão $n = 4$ e um valor nominal do desvio-padrão de 0.02 l, responda às questões seguintes:*

(a) *Por que tipo de carta para a média optaria por forma a procurar-se respeitar o valor mínimo de 0.1 l? Justifique.*
Obtenha os respectivos limites de controlo de modo que o número esperado de amostras recolhidas até falso alarme seja igual a 500 amostras. (1.0)

- **Quality characteristic**
X = quantity of a liquid product in a can
$X \sim \text{Normal}(\mu, \sigma^2)$

- **Suitable $\bar{X}-$ chart**
A LOWER ONE-SIDED $\bar{X}-$ chart should be adopted because the quantity of the liquid product in the cans should be very close to the minimum value μ_0 and any decrease of μ should detected as soon as possible.

- **Control statistic**
$\bar{X}_N$ = mean of the N^{th} random sample of size n

- **Distribution**
$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 0.1$, $\sigma_0 = 0.02$ and $n = 4$
$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta < 0$) represents the magnitude of the shift (a decrease!) in μ and θ the magnitude of the shift in the standard deviation σ.

- **Control limits of the lower one-sided $\bar{X}-$ chart**
$LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
$UCL_\mu = +\infty$

- **Probability of triggering a signal**
Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta,\theta\right) \\
&= 1-\left[\Phi\left(\frac{UCL_\mu-\mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu-\mu}{\frac{\sigma}{\sqrt{n}}}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= 1-\left[1-\Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\right] \\
&= \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right),\ \delta\le 0,\ \theta\ge 1.
\end{aligned}$$

- **Run length**
We are dealing with a Shewhart chart, therefore the number of samples collected until the chart triggers a signal given (δ,θ), $RL_\mu(\delta,\theta)$, has the following distribution:
$$RL_\mu(\delta,\theta) \sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right)$$

- **Obtaining γ_μ**
The constant γ_μ is such that $ARL_\mu(0,1) = 500$, that is,

$$\begin{aligned}
\gamma_\mu :\ & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1) \\
& \Phi(-\gamma_\mu) = \frac{1}{ARL_\mu(0,1)} \\
& \gamma_\mu = \Phi^{-1}\left(1-\frac{1}{ARL_\mu(0,1)}\right) \\
& \gamma_\mu = \Phi^{-1}(0.998) \\
& \gamma_\mu \overset{table}{=} 2.8782
\end{aligned}$$

- **Lower control limit**

$$\begin{aligned}
LCL_\mu &= \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\
&= 0.1 - 2.8782 \times \frac{0.02}{\sqrt{4}} \\
&= 0.071218
\end{aligned}$$

(b) *Admita que o valor esperado sofreu um "shift" para 0.07 l. Obtenha o valor de $ARL_\mu(\delta, 1)$.*
Admita agora que o valor nominal do desvio-padrão também se alterou, por sinal, para 0.025 l. Compare e comente o valor de ARL obtido nesta situação com $ARL_\mu(\delta, 1)$. (1.5)

- **Magnitude of the shift in μ**

$$\begin{aligned}
\delta &= \frac{\mu-\mu_0}{\frac{\sigma_0}{\sqrt{n}}} \\
&= \frac{0.07-0.1}{\frac{0.02}{\sqrt{4}}} \\
&= -3
\end{aligned}$$

- **Out-of-control ARL**

$$\begin{aligned}
ARL_\mu(\delta,1) &= \frac{1}{\xi_\mu(\delta,1)} \\
&= \frac{1}{\Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)} \\
&= \frac{1}{\Phi\left[\frac{-2.8782-(-3)}{1}\right]} \\
&\simeq \frac{1}{\Phi(0.12)} \\
&\overset{table}{=} \frac{1}{0.5478}
\end{aligned}$$

$$\simeq \quad 1.825484$$

- **Magnitude of the shift in σ**

$$\begin{aligned} \theta &= \frac{\sigma}{\sigma_0} \\ &= \frac{0.025}{0.02} \\ &= 1.25 \end{aligned}$$

- **Another out-of-control ARL**

$$\begin{aligned} ARL_\mu(\delta,\theta) &= \frac{1}{\xi_\mu(\delta,\theta)} \\ &= \frac{1}{\Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)} \\ &= \frac{1}{\Phi\left[\frac{-2.8782-(-3)}{1.25}\right]} \\ &\simeq \frac{1}{\Phi(0.10)} \\ &\overset{table}{=} \frac{1}{0.5398} \\ &\simeq 1.852538 \end{aligned}$$

- **Comment**
  $ARL_\mu(-3,1) < ARL_\mu(-3,1.25)$, that is, the average time this lower one-sided $\bar{X}$ chart takes to detect a shift from $\mu_0 = 0.1$ to $\mu = 0.07$ increases if this shift in μ is associated to a shift in σ.
  This is a counter-intuitive and undesirable result because the more severe the shifts the smaller should be the out-of-control ARL. This result suggest that an upper one-sided S^2 for σ should be use in addition to the lower one-sided $\bar{X}$ chart.

5. *Na tabela abaixo pode encontrar o número de erros de alinhamento na fase final de produção de uma aeronave.*

Aeronave	*201*	*202*	*203*	*204*	*205*	*206*	*207*	*208*	*209*	*210*
Erros de alinhamento	*7*	*6*	*6*	*7*	*4*	*7*	*8*	*12*	*9*	*9*

Pretende conceber-se um esquema CUSUM *unilateral superior para a detecção de aumentos do valor nominal do número esperado de erros de alinhamento por aeronave, $\lambda_0 = 8$, para $\lambda_1 = 9$.*

(a) *Obtenha (e arredonde convenientemente) o valor de referência óptimo $k = \frac{\lambda_1-\lambda_0}{ln(\lambda_1/\lambda_0)}$ para o esquema acima referido, averigue se alguma das três primeiras observações foi responsável por um sinal por parte do esquema* CUSUM *unilateral, sem* head start *e com limite superior igual a $UCL = 5$.* (1.0)

- **Quality characteristic**
  Y = number of alignment errors per aircraft
- **Distribution assumption**
  Y_N = number of alignment errors in the N^{th} aircraft, $N \in I\!N$
  $Y_N \sim \text{Poisson}(\lambda = \lambda_0)$, IN CONTROL, where $\lambda_0 = 8$
  $Y_N \sim \text{Poisson}(\lambda = \lambda_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of the shift in λ

- **Upper one-sided CUSUM chart for Poisson data**
  $LCL = 0$
  $UCL = x = 5$
  $u = 0$ (no head-start)
- **Reference value**
  When we plan to minimize the out-of-control ARL of the upper one-sided CUSUM chart when there is a shift from $\lambda_0 = 8$ to $\lambda_1 = 9$, the *optimal* reference value is equal to

$$\begin{aligned} k^* &= \frac{\lambda_1-\lambda_0}{\ln(\lambda_1)-\ln(\lambda_0)} \\ &= \frac{9-8}{\ln(9)-\ln(8)} \\ &\simeq 8.490187. \end{aligned}$$

  We shall adopt the integer reference value $k = 8$.
- **Control statistic**

$$Z_N = \begin{cases} 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in I\!N, \end{cases}$$

- **Observed values of the control statistic**

$$\begin{aligned} z_0 &= 0 \\ z_1 &= \max\{0, 0 + (7-8)\} \\ &= 0 \\ z_2 &= \max\{0, 0 + (6-8)\} \\ &= 0 \\ z_3 &= \max\{0, 0 + (6-8)\} \\ &= 0. \end{aligned}$$

- **Comment**
  Since $z_1, z_2, z_3 \in [LCL, UCL] = [0, 5]$ none of the three first observations were responsible for a signal while using such an upper one-sided CUSUM chart for Poisson data.

(b) *Calcule as quatro primeiras entradas da diagonal da matriz de probabilidades de transição associada ao esquema na presença de um "shift" do número esperado de erros de alinhamento por aeronave de $\lambda_0 = 8$ para $\lambda_1 = 9$.* (1.5)

- **Transition probability matrix**
  According to Example 10.9 and result (10.8) of the lecture notes, the control statistic is associated to an absorbing Markov chain governed by the following transition probability matrix

$$\mathbf{P}(\theta) = \begin{bmatrix} F_\theta(k) & P_\theta(k+1) & P_\theta(k+2) & P_\theta(k+3) & \cdots & P_\theta(k+UCL) & 1-F_\theta(k+UCL) \\ F_\theta(k-1) & P_\theta(k) & P_\theta(k+1) & P_\theta(k+2) & \cdots & P_\theta(k+UCL-1) & 1-F_\theta(k+UCL-1) \\ F_\theta(k-2) & P_\theta(k-1) & P_\theta(k) & P_\theta(k+1) & \cdots & P_\theta(k+UCL-2) & 1-F_\theta(k+UCL-2) \\ F_\theta(k-3) & P_\theta(k-2) & P_\theta(k-1) & P_\theta(k) & \cdots & P_\theta(k+UCL-3) & 1-F_\theta(k+UCL-3) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ F_\theta(k-UCL) & P_\theta(k-UCL+1) & P_\theta(k-UCL+2) & P_\theta(k-UCL+3) & \cdots & P_\theta(k) & 1-F_\theta(k) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

  where $\theta = \lambda_1 - \lambda_0$ and F_θ and P_θ represent the c.d.f. and the p.f. of a r.v. with a Poisson($\lambda_0 + \theta = \lambda_1$) distribution.
- **First four entries of the diagonal**

Since we are dealing with an absorbing and spatially homogenous Markov chain, with a reflecting barrier, all but the first of these four entries are equal. The first one is given by

$$\begin{aligned} F_{Poisson(\lambda_1)}(k) &= F_{Poisson(9)}(8) \\ &\overset{tabela}{=} 0.4557 \end{aligned}$$

and the remaining three are equal to

$$\begin{aligned} P_{Poisson(\lambda_1)}(k) &= P_{Poisson(9)}(8) \\ &= F_{Poisson(9)}(8) - F_{Poisson(9)}(7) \\ &\overset{tabela}{=} 0.4557 - 0.3239 \\ &= 0.1318. \end{aligned}$$

6. *Com o objectivo de controlar o valor esperado (μ) e a variância (σ^2) de determinada característica de qualidade, considerou-se um esquema conjunto que faz uso de duas cartas* Shewhart *— uma carta padrão para μ e outra unilateral superior para σ^2 — cujos limites de controlo são tais que:*

- *o número esperado de amostras recolhidas até à emissão de falso alarme por parte da carta para μ (σ^2) é de 500 (1000) amostras e $n = 5$.*

(a) *Determine a probabilidade de ser emitido um sinal válido pelo esquema conjunto somente após a recolha de 50 amostras quando $(\delta, \theta) = (0.1, 1.25)$?*

Nota: Na impossibilidade de obter valor exacto obtenha um intervalo de valores para esta probabilidade e para as que se seguirem. (1.5)

- **Quality characteristic**
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistics**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  S^2_N = variance of the N^{th} random sample of size n, $N \in I\!N$
- **Distributions**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $n = 5$
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or an increase!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ
  $\frac{(n-1)S^2_N}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
  $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL
- **Control limits of the standard $\bar{X}-$ chart and the upper one-sided S^2- chart**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$

- **Probabilities of triggering signals**
  Taking into account the distributions of the control statistics, the individual charts for μ and σ trigger signals with probabilities equal to:

$$\begin{aligned} \xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta, \theta\right) \\ &= 1 - \left[\Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \ \delta \in I\!R, \ \theta \geq 1, \\ \xi_\sigma(\theta) &= P\left(S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\ &= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] \\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right), \ \theta \geq 1, \end{aligned}$$

  respectively.
- **Run length of the individual charts**
  We are dealing with Shewhart charts, thus, the number of samples collected until each chart triggers a signal given (δ, θ), $RL_\mu(\delta,\theta)$ and $RL_\sigma(\theta)$, have the following distribution:

$$\begin{aligned} RL_\mu(\delta,\theta) &\sim \text{Geometric}\left(\xi_\mu(\delta,\theta)\right) \\ RL_\sigma(\theta) &\sim \text{Geometric}\left(\xi_\sigma(\theta)\right). \end{aligned}$$

- **Obtaining γ_μ and γ_σ**
  The constants γ_μ and γ_σ are such that $ARL_\mu(0,1) = 500$ and $ARL_\sigma(1) = 1000$, i.e.

$$\begin{aligned} \gamma_\mu \ :\ & \frac{1}{\xi_\mu(0,1)} = ARL_\mu(0,1) \\ & 1 - [\Phi(\gamma_\mu) - \Phi(-\gamma_\mu)] = \frac{1}{ARL_\mu(0,1)} \\ & \gamma_\mu = \Phi^{-1}\left(1 - \frac{1}{2 \times ARL_\mu(0,1)}\right) \\ & \gamma_\mu = \Phi^{-1}(0.999) \\ & \gamma_\mu \overset{table}{=} 3.0902 \\ \gamma_\sigma \ :\ & \frac{1}{\xi_\sigma(1)} = ARL_\sigma(1) \\ & 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = \frac{1}{ARL_\sigma(1)} \\ & \gamma_\sigma = F^{-1}_{\chi^2_{(n-1)}}\left(1 - \frac{1}{ARL_\sigma(1)}\right) \\ & \gamma_\sigma = F^{-1}_{\chi^2_{(5-1)}}(0.999) \\ & \gamma_\sigma \overset{table}{=} 18.47 \end{aligned}$$

- **Probability of a signal by the joint scheme for μ and σ**
  The joint scheme triggers a signal if either of the individual charts triggers an alarm. Moreover, the control statistics of the individual charts are independent given (δ, θ). As a consequence, the joint scheme for μ and σ triggers a signal with probability equal to:

$$\begin{aligned} \xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \delta, \theta\right) \\ &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta) \end{aligned}$$

$$= \xi_\mu(\delta,\theta) + [1-\xi_\mu(\delta,\theta)] \times \xi_\sigma(\theta).$$

When $(\delta,\theta) = (0.1, 1.25)$, we get:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= 1 - \left[\Phi\left(\frac{\gamma_\mu-\delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu-\delta}{\theta}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{3.0902-0.1}{1.25}\right) - \Phi\left(\frac{-3.0902-0.1}{1.25}\right)\right] \\
&\simeq 1 - [\Phi(2.39) - \Phi(-2.55)] \\
&\stackrel{table}{=} 1 - (0.9916 - 0.0054) \\
&= 0.0138 \\
\xi_\sigma(\theta) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right) \\
&= 1 - F_{\chi^2_{(5-1)}}\left(\frac{18.47}{1.25^2}\right) \\
&\simeq 1 - F_{\chi^2_{(4)}}(11.8208) \\
&\in (0.01; 0.025)
\end{aligned}$$

because

$$\begin{aligned}
F^{-1}_{\chi^2_{(4)}}(0.975) = 11.14 &< 11.8208 < 13.28 = F^{-1}_{\chi^2_{(4)}}(0.99) \\
1 - 0.99 &< \xi_\sigma(1.25) < 1 - 0.975.
\end{aligned}$$

Then a signal is triggered by the joint scheme, when $(\delta,\theta) = (0.1, 1.25)$, with probability:

$$\begin{aligned}
\xi_{\mu,\sigma}(0.1, 1.25) &= \xi_\mu(0.1, 1.25) + [1-\xi_\mu(0.1, 1.25)] \times \xi_\sigma(1.25) \\
&\in (0.0138 + (1-0.0138) \times 0.01;\ 0.0138 + (1-0.0138) \times 0.025) \\
&= (0.023662;\ 0.038455).
\end{aligned}$$

- **Run length**

  The run length of this joint scheme for μ and σ, $RL_{\mu,\sigma}(\delta,\theta)$, has the following distribution:

  $$RL_{\mu,\sigma}(\delta,\theta) \sim \text{Geometric}(\xi_{\mu,\sigma}(\delta,\theta)).$$

- **Requested probability**

  The probability that the joint scheme triggers a signal exactly after 50 samples equals:

  $$\begin{aligned}
P\left[RL_{\mu,\sigma}(\delta,\theta) = m\right] &= [1-\xi_{\mu,\sigma}(\delta,\theta)]^{m-1} \times \xi_{\mu,\sigma}(\delta,\theta) \\
&\in \left((1-0.023662)^{50-1} \times 0.023662; (1-0.038455)^{50-1} \times 0.038455\right) \\
&= (0.0034464; 0.523806).
\end{aligned}$$

(b) *Obtenha a probabilidade de ocorrência de sinal erróneo de Tipo III (IV) quando $\theta = 1.25$ ($\delta = 0.1$).* (1.0)

- **Probability of a misleading signal of type III**

  $$\begin{aligned}
PMS_{III}(\theta) &\stackrel{Table\ 10.12}{=} \frac{1 - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}{[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)]^{-1} - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]} \\
&\stackrel{\theta=1.25}{=} \frac{1 - [\Phi(3.0902/1.25) - \Phi(-3.0902/1.25)]}{[F_{\chi^2_{(5-1)}}(18.47/1.25^2)]^{-1} - [\Phi(3.0902/1.25) - \Phi(-3.0902/1.25)]} \\
&\simeq \frac{1 - [\Phi(2.47) - \Phi(-2.47)]}{[F_{\chi^2_4}(11.08208)]^{-1} - [\Phi(2.47) - \Phi(-2.47)]} \\
&\stackrel{table,(a)}{=} \left(\frac{1-(0.9932-0.0068)}{0.975^{-1} - (0.9932-0.0068)}; \frac{1-(0.9932-0.0068)}{0.99^{-1} - (0.9932-0.0068)}\right) \\
&\simeq (0.346576; 0.573815)
\end{aligned}$$

- **Probability of a misleading signal of type IV**

  $$\begin{aligned}
PMS_{IV}(\delta) &\stackrel{Table\ 10.12}{=} \frac{1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma)}{[\Phi(\gamma_\mu-\delta) - \Phi(-\gamma_\mu-\delta)]^{-1} - F_{\chi^2_{(n-1)}}(\gamma_\sigma)} \\
&\stackrel{\delta=0.1}{=} \frac{1 - F_{\chi^2_{(5-1)}}(18.47)}{[\Phi(3.0902-0.1) - \Phi(-3.0902-0.1)]^{-1} - F_{\chi^2_{(5-1)}}(18.47)} \\
&\stackrel{a)}{\simeq} \frac{0.001}{[\Phi(2.99) - \Phi(-3.19)]^{-1} - (1-0.001)} \\
&\stackrel{table}{=} \frac{0.001}{(0.9986-0.0097)^{-1} - 0.999} \\
&\simeq 0.322121.
\end{aligned}$$

7. *Uma companhia inspecciona lotes fazendo uso de um plano de amostragem dupla com $n_1 = 100$, $c_1 = 0$, $n_2 = 100$ e $c_2 = 2$.*

   (a) *Determine um valor aproximado para a probabilidade de aceitação do lote na primeira fase deste plano de amostragem dupla quando o verdadeiro valor da fracção de defeituosas é igual a $p = 0.015$.* (1.5)

   - **Double sampling plan (for attributes)**

     $n_1 = 100$, $n_2 = 100$ (sample sizes)

     $c_1 = 0$, $c_2 = 2$ (acceptance numbers)

   - **Auxiliary r.v. and their approximate distributions**

     D_i = number of defective units in the i^{th} sample $\stackrel{a}{\sim}$ Binomial(n_i, p), $i = 1, 2$

   - **Probability of accepting the lot in the first stage of the plan**

     $$\begin{aligned}
P_a^I(p) &\stackrel{(13.16)}{=} P(D_1 \le c_1) \\
&\simeq F_{Binomial(n_1,p)}(c_1) \\
&\stackrel{p=0.015}{=} F_{Binomial(100,0.015)}(0) \\
&= \binom{100}{0} \times 0.015^0 \times (1-0.015)^{100-0} \\
&= 0.985^{100} \\
&\simeq 0.220609
\end{aligned}$$

   (b) *Confronte este plano de amostragem dupla com um plano de amostragem simples com $n = 100$, $c = 2$ no que diz respeito à dimensão média da amostra, considerando para o efeito o valor de p da alínea anterior.* (1.0)

   - **Average sample number of the double sampling plan**

     $$\begin{aligned}
ASN(p) &\stackrel{(13.21)}{=} n_1 + n_2 \times P(c_1 < D_1 \le c_2) \\
&\simeq n_1 + n_2 \times \left[F_{Binomial(n_1,p)}(c_2) - F_{Binomial(n_1,p)}(c_1)\right] \\
&\stackrel{p=0.015}{=} 100 + 100 \times \left[F_{Binomial(100,0.015)}(2) - F_{Binomial(100,0.015)}(0)\right] \\
&= 100 + 100 \\
&\quad \times \left[\binom{100}{1} \times 0.015^1 \times (1-0.015)^{100-1} + \binom{100}{2} \times 0.015^2 \times (1-0.015)^{100-2}\right] \\
&\simeq 158.919615
\end{aligned}$$

- **Single sampling plan**
  $n = 100$
  $c = 2$
- **Average sample number of the single sampling plan**

$$\begin{aligned} ASN'(p) &= n \\ &= 100,\ \forall p \in (0,1) \\ &< ASN(0.01) \simeq 155.459446 \end{aligned}$$

- **Comment**
  According to the ASN criterion, the single sampling plan is preferable to the double sampling plan because $ASN'(0.015) < ASN(0.015)$.
  Confronting the two sampling plans based solely on the average sample number is unwise — we should also compare the their primary OC curves.