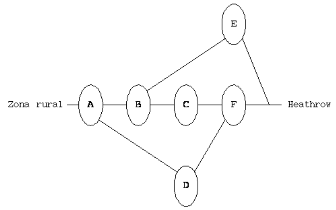
Fiabilidade e Controlo de Qualidade — LMAC

1o. Teste — 2o. Semestre – 2004/05

Duração: 1 hora e 45 minutos — 14/Maio/05 (Sab.) – 10h

1. *O esquema abaixo descreve as etapas de uma deslocação em transportes públicos de uma zona rural do sul da Inglaterra ao aeroporto de Heathrow. Há a possibilidade de ocorrência de avaria em cada uma das etapas da deslocação, contribuindo assim para que um passageiro perca o seu vôo.*

   Considere que os transportes B, D, E e F (A e C, resp.) possuem probabilidade de avaria q (2q, resp.) e funcionam de modo independente.



   (a) *Identifique as fiabilidades dos transportes, os caminhos mínimos, os cortes mínimos e a função estrutura.* (2.5)

   - **Reliabilities of the components/transports**
     $p_i = 1 - P(\text{breakdown of transport } i)$, $i = A, \ldots, F$
     $p_B = p_D = P_E = p_F = 1 - q$
     $p_A = p_C = 1 - 2q$
   - **Minimal path sets**
$$\begin{aligned}\mathcal{P}_1 &= \{A, B, E\}\\ \mathcal{P}_2 &= \{A, D, F\}\\ \mathcal{P}_3 &= \{A, B, C, F\}\\ p^* &= 3 \text{ minimal path sets}\end{aligned}$$
   - **Minimal cut sets**
$$\begin{aligned}\mathcal{K}_1 &= \{A\}\\ \mathcal{K}_2 &= \{B, D\}\\ \mathcal{K}_3 &= \{B, F\}\\ \mathcal{K}_4 &= \{C, D, E\}\\ \mathcal{K}_5 &= \{E, F\}\\ q^* &= 5 \text{ minimal cut sets}\end{aligned}$$
   - **Structure function**
     By considering $X_i \sim \text{Bernoulli}(p_i)$, $i = A, \ldots, F$ and applying result (1.13) or (1.14), we can conlude that the structure function of this system equals:
$$\begin{aligned}\phi(\underline{X}) &\overset{(1.13)}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} X_i\right)\\ &= 1 - (1 - X_A X_B X_E)(1 - X_A X_D X_F)(1 - X_A X_B X_C X_F)\\ &\overset{(1.14)}{=} \prod_{j=1}^{p^*}\left[1 - \prod_{i \in \mathcal{K}_j}(1 - X_i)\right]\\ &= [1 - (1 - X_A)] \times [1 - (1 - X_B)(1 - X_D)] \times [1 - (1 - X_B)(1 - X_F)]\\ &\quad \times [1 - (1 - X_C)(1 - X_D)(1 - X_E)] \times [1 - (1 - X_E)(1 - X_F)].\end{aligned}$$

   (b) *Calcule um par de limites inferior e superior o mais estritos possível para a fiabilidade desta deslocação a Heathrow.* (2.0)

   - **Bounds for the reliability** $r(\underline{p})$
     Since the components form a coherent system and operate in an independent fashion, we can apply Theorem 1.68, use the fact that $p_A = p_C = 1 - 2q$ and $p_B = p_D = P_E = p_F = 1 - q$ and succesively get:
     - **Lower bound**
$$\begin{aligned}r(\underline{p}) &\geq \prod_{j=1}^{q^*}\left[1 - \prod_{i \in \mathcal{K}_j}(1 - p_i)\right]\\ &= [1 - (1 - p_A)] \times [1 - (1 - p_B)(1 - p_D)] \times [1 - (1 - p_B)(1 - p_F)]\\ &\quad \times [1 - (1 - p_C)(1 - p_D)(1 - p_E)] \times [1 - (1 - p_E)(1 - p_F)]\\ &= (1 - 2q) \times (1 - q^2) \times (1 - q^2) \times (1 - 2qq^2) \times (1 - q^2)\\ &= (1 - 2q) \times (1 - q^2)^3 \times (1 - 2qq^2)\end{aligned}$$
     - **Upper bound**
$$\begin{aligned}r(\underline{p}) &\leq 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} p_i\right)\\ &= 1 - (1 - p_A p_B p_E)(1 - p_A p_D p_F)(1 - p_A p_B p_C p_F)\\ &= 1 - \left[1 - (1 - 2q)(1 - q)^2\right] \times \left[1 - (1 - 2q)(1 - q)^2\right]\\ &\quad \times \left[1 - (1 - 2q)^2(1 - q)^2\right].\end{aligned}$$

2. *Um veículo pesado possui 2 eixos, com 6 pneus cada (3 de cada lado). Este veículo opera quando os dois eixos estão a funcionar. Um eixo fica inoperacional se mais de um dos 3 pneus de cada lado estiver furado.*

   (a) *Obtenha um limite inferior para a fiabilidade do veículo, assumindo que os 12 pneus do veículo funcionam de forma associada (positiva) e que a fiabilidade de cada pneu é de* 97.5%. (3.0)

   - **System**
     The vehicle described in the text is a system with the following characteristics:
     - Each SIDE OF THE AXIS corresponds to a $2 - \text{out} - \text{of} - 3$ system because each axis is non operational if more than one of the 3 tires is non operational, that is, the system is operational if at least 2 of the 3 tires is operational;
     - each AXIS is nothing but two $2 - \text{out} - \text{of} - 3$ sub-systems put in series;
     - the VEHICLE only operates when both axes are operational, therefore the vehicle corresponds to $2 - \text{out} - \text{of} - 3$ sub-systems put in series.

- **Reliability of the components of each $2-\text{out}-\text{of}-3$ sub-system**
  $p_i = p^* = 0.975,\ i = 1,2,3$

- **Lower bound for the reliability of each $2-\text{out}-\text{of}-3$ sub-system**
  Since the 3 components form a coherent sub-system and operate in a positively associated fashion, we can apply Theorem 1.70, namely result (1.42), after having identified the minimal path sets.

  - **Minimal path sets**

$$
\begin{aligned}
\mathcal{P}_1 &= \{1,2\} \\
\mathcal{P}_2 &= \{1,3\} \\
\mathcal{P}_3 &= \{2,3\} \\
p^* &= 3 \text{ minimal path sets}
\end{aligned}
$$

  - **Lower bound for the reliability**

$$
\begin{aligned}
r_{2-out-of-3}(\underline{p}) &\overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i\in\mathcal{P}_j} p_i \\
&\overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\
&\overset{\#\mathcal{P}_j=2,\forall j}{=} p^2 \\
&\overset{p=0.975}{=} 0.975^2 \\
&= 0.950625.
\end{aligned}
$$

- **Lower bound for the reliability of the vehicle**
  Now, we can capitalize on the previous bound for the reliability of each of the four sub-systems and apply Theorem 1.65 to provide the requested lower bound or the reliability of the vehicle:

$$
\begin{aligned}
r_{vehicle}(\underline{p}) &\overset{(1.34)}{\geq} \prod_{i=1}^{4} r_{2-out-of-3,\,i}(\underline{p}) \\
&= 0.950625^4 \\
&= 0.816652.
\end{aligned}
$$

(b) *Considere agora que os "tempos" até que os pneus fiquem furados (em milhares de Km) são i.i.d. com distribuição* Weibull *com parâmetro de escala $\delta = 1$ e de forma $\alpha = 2$. Obtenha a função de fiabilidade da vida do veículo para uma deslocação de 400 Km.* (2.5)

- **Individual durations of the components of each $2-\text{out}-\text{of}-3$ sub-system**
  $T_i \overset{i.i.d.}{\sim} \text{Weibull}(\delta = 1, \alpha = 2),\ i = 1,2,3$
  $F_i(t) = F(t) = 1 - e^{-t^2},\ t \geq 0$

- **Duration of each $2-\text{out}-\text{of}-3$ sub-system**
  $S_j = \text{duration of the } j^{th}\ 2-\text{out}-\text{of}-3\text{sub-system},\ j = 1,\ldots,4$
  $S_i \overset{(2.6)}{=} T_{(n-k+1)} \overset{n=3,\,k=2}{=} T_{(2)}$

- **Reliability function of S_j**
  For $j = 1,\ldots,4$,

$$
\begin{aligned}
R_{S_j}(t) &\overset{(2.8)}{=} F_{Binomial(3,F(t))(3-2)} \\
&= \sum_{m=0}^{1} \binom{3}{m} \left(1-e^{-t^2}\right)^m \left[1-\left(1-e^{-t^2}\right)\right]^{3-m} \\
&= e^{-3t^2} + 3\left(1-e^{-t^2}\right)e^{-2t^2} \\
&\overset{t=0.4}{\simeq} 0.94088
\end{aligned}
$$

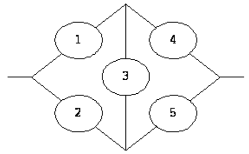
- **Duration of the vehicle**
  $S_{vehicle} = \min_{j=1,\ldots,4} S_j$

- **Reliability function of $S_{vehicle}$**

$$
\begin{aligned}
R_{S_{vehicle}}(t) &\overset{(2.3)}{=} \prod_{j=1}^{4} R_{S_j}(t) \\
&= \left[e^{-3t^2} + 3\left(1-e^{-t^2}\right)e^{-2t^2}\right]^4 \\
&\overset{t=0.4}{\simeq} 0.94088^4 \\
&\simeq 0.783678.
\end{aligned}
$$

3. *Um sistema em ponte possui componentes, com durações (em horas) i.i.d. e função taxa de falha comum*

$$
\lambda(t) = \begin{cases} t, & 0 < t \leq 1.5 \\ 3-t, & 1.5 < t \leq 2 \\ t-1, & t > 2. \end{cases}
$$



(a) *Qual o comportamento monótono da função taxa de falha das durações das componentes e da duração da estrutura? Justifique.* (3.0)

- **Individual durations and common hazard rate function**
  $T_i,\ i = 1,\ldots,5$, i.i.d. r.v. with hazard rate function $\lambda(t)$.

- **Stochastic aging character of T_i**
  $\lambda(t)$ is not a monotone (or monotonic) function. Thus, $T_i \notin IHR$ or $T_i \notin DHR$. However,

$$
\begin{aligned}
\frac{1}{t}\Lambda(t) &= \frac{1}{t}\int_0^t \lambda(u)\,du \\
&= \frac{1}{t} \times \begin{cases} 0, & t < 0 \\ \int_0^t u\,du, & 0 \leq t \leq 1.5 \\ \int_0^t \frac{0}{1.5} u\,du + \int_{1.5}^t (3-u)\,du, & 1.5 < t \leq 2 \\ \int_0^t \frac{0}{1.5} u\,du + \int_{1.5}^2 (3-u)\,du + \int_2^t (u-1)\,du, & t > 2 \end{cases} \\
&= \frac{1}{t} \times \begin{cases} 0, & t < 0 \\ \frac{t^2}{2}, & 0 \leq t \leq 1.5 \\ \frac{1.5^2}{2} + \left(3u - \frac{u^2}{2}\right)\Big|_{1.5}^t, & 1.5 < t \leq 2 \\ -2.25 + 3\times 2 - \frac{2^2}{2} + \left[(t^2 - t) - (2^2 - 2)\right], & t > 2 \end{cases} \\
&\ldots \\
&= \frac{1}{t} \times \begin{cases} 0, & t < 0 \\ \frac{t^2}{2}, & 0 \leq t \leq 1.5 \\ -2.25 + 3t - \frac{t^2}{2}, & 1.5 < t \leq 2 \\ \frac{t^2}{2} - t + 1.75, & t > 2 \end{cases} \\
\frac{d\frac{1}{t}\Lambda(t)}{dt} &= \begin{cases} \frac{1}{2}, & 0 \leq t \leq 1.5 \\ \frac{2.25}{t^2} - \frac{1}{2}, & 1.5 < t \leq 2 \\ \frac{1}{2} - \frac{1.75}{t^2}, & t > 2. \end{cases}
\end{aligned}
$$

Please note that:

- $\frac{1}{2} > 0$;
- $\frac{2.25}{t^2} - \frac{1}{2}$ is a decreasing function, thus, the minimum value is this function in $(1.5, 2]$ is $\frac{2.25}{2^2} - \frac{1}{2} = 0.0625 > 0$;
- $\frac{1}{2} - \frac{1.75}{t^2}$ is an increasing function, thus, the minimum value is this function in $(2, +\infty]$ is $\frac{1}{2} - \frac{1.75}{2^2} = 0.0625 > 0$.

As a consequence $\frac{1}{t}\Lambda(t)$ is an increasing monotone function for $t \geq 0$, thus, $T_i \in IHRA$, $i = 1, \ldots, 5$.

- **Duration of the bridge system**
  T
- **Stochastic ageing character of T**
  According to Table 3.2, the $IHRA$ property of the components of a system is preserved after the formation of a coherent system, like the bridge system. Thus, $T \in IHRA$.

(b) *Estabeleça um limite inferior para a duração esperada da vida da estrutura, sabendo que a duração esperada comum das componentes é de aproximadamente 5.62 horas.* (2.0)

- **Lower bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - the 5 components have durations $T_i \stackrel{i.i.d.}{\sim} IHRA$, $i = 1, \ldots, 5$ and, thus, according to Proposition 3.36,
    $T_i \stackrel{i.i.d.}{\sim} NBUE$, $i = 1, \ldots, 5$;
  - the expected value of the duration of each of the 5 components is equal to $\mu^* = \mu_i = E(T_i) \simeq 5.62$.
  - the minimal path sets are

$$\begin{aligned}\mathcal{P}_1 &= \{1, 24\}\\ \mathcal{P}_2 &= \{2, 5\}\\ \mathcal{P}_3 &= \{1, 3, 5\}\\ \mathcal{P}_4 &= \{2, 3, 4\}\\ p &= 5 \text{ minimal path sets.}\end{aligned}$$

  Now, we can apply Theorem 3.65, and conclude that

$$\begin{aligned}\mu &= E(T)\\ &\geq \max_{j=1,\ldots,p}\left\{\left(\sum_{i\in\mathcal{P}_j}\mu_i^{-1}\right)^{-1}\right\}\\ &\stackrel{\mu_i=\mu^*}{=} \max_{j=1,\ldots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\}\\ &= \frac{\mu^*}{\min_{j=1,\ldots,p}\{\#\mathcal{P}_j\}}\\ &= \frac{\mu^*}{2}\\ &= 2.81.\end{aligned}$$

4. *Um veículo de teste foi sujeito a uma rodagem de 25000Km, tendo-se registado os Kms percorridos entre falhas eléctricas graves sucessivas: 63, 51, 14166, 1825, 1288, 1314, 472, 3463, 486, 1017Km.*

(a) *Averigue a adequação do modelo* exponencial *a este conjunto de dados considerando um nível de significância de* 5%. (2.0)

- **Life test**
  Since the test was scheduled to end after exactly $t_0 = 25000Km$ and the exercise mentions just one vehicle submitted to this life test, we are dealing with a
  - Type I/item censored testing with replacement.
- **R.v.**
  $T_{(i)}$ = time (in Km) of the i^{th} serious electrical failure of the military vehicle
  $Z_i = T_{(i)} - T_{(i-1)}$ = run time (in Km) between the i^{th} and $(i-1)^{th}$ serious electrical failure of the military vehicle
  $Z_i \stackrel{i.i.d.}{\sim} Z$, $i \in I\!N$
- **Censored data**
  $n = 1$
  $r = 10$ electrical failures during the life test
  $(z_1, \ldots, z_r) = 63, 51, 14166, 1825, 1288, 1314, 472, 3463, 486, 1017)$
  $(t_{(1)}, \ldots, t_{(r)}) = (63, 114, 14820, 16105, 17393, 18707, 19179, 22642, 23128, 24145)$
- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned}\tilde{t} &= n \times t_0\\ &= 1 \times 25000\\ &= 25000Km\end{aligned}$$

- **Hypotheses**
  $H_0 : Z \sim \text{Exponential}(\lambda)$
  $H_1 : Z \sim \text{Weibull}(\lambda^{-1}, \alpha)$, $\alpha \neq 1$
- **Significance level**
  $\alpha_0 = 5\%$
- **Test statistic** (Bartlett's test)

$$\begin{aligned}B_r &\stackrel{(5.19)}{=} \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r Z_i}{r}\right) - \frac{1}{r}\sum_{i=1}^r \ln(Z_i)\right]\\ &\stackrel{a}{\sim}_{H_0} \chi^2_{(r-1)}\end{aligned}$$

- **Rejection region of H_0**

$$\begin{aligned}W &= \left(0, F^{-1}_{\chi^2_{(r-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(r-1)}}(1-\alpha_0/2), +\infty\right)\\ &\stackrel{r=10,\,\alpha_0=0.05}{=} (0, 2.700) \cup (19.02, +\infty)\end{aligned}$$

- **Decision**
  The observed value of the test statistic is

$$b_r = \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r z_i}{r}\right) - \frac{1}{r}\sum_{i=1}^r \ln(z_i)\right]$$

$$
\begin{aligned}
&= \frac{2\times 10}{1+\frac{10+1}{6\times 10}} \times \left[\ln\left(\frac{24145}{10}\right) - \frac{1}{10} 69.902263\right] \\
&= 18.575006 \\
&\notin (0, 2.700) \cup (19.02, +\infty),
\end{aligned}
$$

therefore we should not reject H_0 for any significance level $\alpha \leq 5\%$.

(b) *Após ter enunciado as hipóteses de trabalho que entender convenientes, obtenha uma estimativa centrada da função de fiabilidade aos 1200 Km e um intervalo de confiança a 95% para a kilometragem associada a 10% das falhas.* (3.0)

- **Distribution assumption**
  $T_i \overset{i.i.d.}{\sim}$ Exponential(λ), $i = 1, \ldots, 6$.
  This is fairly reasonable since we did not reject H_0 in (a).

- **Unknown parameter**
  $R_T(t) = e^{-\lambda t}$

- **Unbiased estimate of $R_T(t)$**
  According to Table 5.14, the UMVUE of $R_T(t)$ is, for $t = 1200 < \tilde{t} = 25000$ and $r > 0$, equal to

$$
\begin{aligned}
\tilde{R}_T(t) &= \left(1 - \tilde{t}^{-1} \times t\right)^r \\
&= \left(1 - \frac{1}{25000} \times 1200\right)^{10} \\
&\simeq 0.611462.
\end{aligned}
$$

- **Confidence interval for λ**
  According to Table 5.16 of the lecture notes,

$$
\begin{aligned}
CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L; \lambda_U] \\
&= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2\times\tilde{t}}\right] \\
CI_{95\%}(\lambda) &\overset{a)}{=} \left[\frac{F^{-1}_{\chi^2_{(20)}}(0.025)}{2\times 1\times 25000}; \frac{F^{-1}_{\chi^2_{(22)}}(0.975)}{2\times 1\times 25000}\right] \\
&= \left[\frac{9.591}{50000}; \frac{36.78}{50000}\right] \\
&\simeq [0.000192; 0.000736].
\end{aligned}
$$

- **Another unknown parameter**
  $F_T^{-1}(0.1) = -\frac{1}{\lambda}\ln(1-0.1)$, which is a decreasing function of $\lambda > 0$.

- **Confidence interval for $F_T^{-1}(0.1)$**

$$
\begin{aligned}
CI_{95\%}\left(F_T^{-1}(0.1)\right) &= \left[\frac{1}{\lambda_U}\ln(1-0.1); \frac{1}{\lambda_L}\ln(1-0.1)\right] \\
&\simeq [143.237917; 549.291784].
\end{aligned}
$$