ame de 1a. Época / 2o. Teste — 2o. Semestre – 2002/03
ração: 3 horas / 1 hora e 45 minutos — 28/06/03 – 09 horas – V1.12

1. *Foram registados e posteriormente ordenados instantes de ocorrência de falha (em dias) de 20 sistemas de controlo de tráfego aéreo:*

2.55	*5.95*	*10.86*	*19.17*	*23.85*	*25.83*	*33.46*	*34.56*	*46.24*	*48.51*
55.56	*64.82*	*77.93*	*97.39*	*132.03*	*155.68*	*241.60*	*253.91*	*260.53*	*273.05*

$\sum_{i=1}^{20} t_i = 1863.48 \quad \sum_{i=1}^{20} \ln(t_i) = 78.50$

(a) *Averigue a adequação do modelo* exponencial *a este conjunto de dados considerando para o efeito um nível de significância de* 10%. (2

- **Times to failure**
  $T_i =$ time to failure of the i^{th}air traffic control system, $i = 1, \ldots, n$
  $T_i \stackrel{i.i.d.}{\sim} T$, $i = 1, \ldots, n$
- **Complete and ordered data**
  $n = 20$
  $(t_{(1)}, \ldots, t_{(n)}) = (2.55, \ldots, 273.05)$
- **Hypotheses**
  $H_0 : T \sim \text{Exponential}(\lambda)$
  $H_1 : T \sim \text{Weibull}(\lambda^{-1}, \alpha)$, $\alpha \neq 1$
- **Significance level**
  $\alpha_0 = 10\%$
- **Test statistic** (Bartlett's test)

$$B_r \stackrel{(5.17)}{=} \frac{2n}{1+\frac{n+1}{6n}} \left\{ \ln\left(\frac{\sum_{i=1}^n T_{(i)}}{n}\right) - \frac{1}{n}\sum_{i=1}^n \ln\left[T_{(i)}\right] \right\} \stackrel{a}{\sim}_{H_0} \chi^2_{(n-1)}$$

- **Rejection region of H_0**

$$\begin{aligned} W &= \left(0, F^{-1}_{\chi^2_{(n-1)}}(\alpha_0/2)\right) \cup \left(F^{-1}_{\chi^2_{(n-1)}}(1-\alpha_0/2), +\infty\right) \\ &\stackrel{n=20,\alpha_0=0.1}{=} (0, 10.12) \cup (30.14, +\infty) \end{aligned}$$

- **Decision**
  The observed value of the test statistic is equal to

$$\begin{aligned} b_r &= \frac{2n}{1+\frac{n+1}{6n}} \left\{ \ln\left(\frac{\sum_{i=1}^n t_{(i)}}{n}\right) - \frac{1}{n}\sum_{i=1}^n \ln\left[t_{(i)}\right] \right\} \\ &= \frac{2\times 20}{1+\frac{20+1}{6\times 20}} \left\{ \ln\left(\frac{1863.48}{20}\right) - \frac{1}{20}\times 78.50 \right\} \\ &\simeq 20.75 \\ &\notin W = (0, 10.12) \cup (30.14, +\infty), \end{aligned}$$

  therefore we should not reject H_0 for any significance level $\alpha \leq 10\%$.

Constatou-se que afinal os registos diziam respeito a um único sistema de controlo de tráfego aéreo que esteve sob observação durante 1 ano.

(b) *Após ter identificado o teste de vida associado e considerado as hipóteses de trabalho que entender convenientes, obtenha uma estimativa centrada bem como um intervalo de confiança a 95% para a função de fiabilidade do tempo até falha do referido sistema para um período de 3 meses.* (1

- **Life test**
  Since the test was scheduled to end after exactly $t_0 = 365$ days (one year) and the exercise mentions just one air traffic control system has been submitted to this life test, we are dealing with a
  - Type I/item censored testing with replacement.
- **Distribution assumption**
  Consider $T_i =$ time between the i^{th} and the $(i-1)^{th}$ failures. Since we did not reject H_0 in (a) it is fairly reasonable to admit that: $T_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1, \ldots, 20$.
- **Censored data**
  $n = 1$
  $r = 20$ failures in one year
- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in this test equals:

$$\begin{aligned} \tilde{t} &= n \times t_0 \\ &= 1 \times 365 \\ &= 365 \text{ days.} \end{aligned}$$

- **Unknown parameters**
  λ
  $R_T(t) = e^{-\lambda t}$, $t = 90$
- **Unbiased estimate of $R_T(t)$**
  According to Table 5.14, the UMVUE of $R_T(t)$ is equal to

$$\begin{aligned} \tilde{R}_T(t) &= \left(1 - \tilde{t}^{-1} \times t\right)^r \\ &= \left(1 - \frac{1}{365} \times 90\right)^{20} \\ &\simeq 0.003473734 \end{aligned}$$

  because $t = 90 < \tilde{t} = 365$ and $r > 0$.
- **Confidence interval for λ**
  According to Table 5.16 of the lecture notes,

$$\begin{aligned} CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L; \lambda_U] \\ &\simeq \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2\times\tilde{t}}\right] \\ CI_{95\%}(\lambda) &\simeq \left[\frac{F^{-1}_{\chi^2_{(40)}}(0.025)}{2\times 365}; \frac{F^{-1}_{\chi^2_{(42)}}(0.975)}{2\times 365}\right] \\ &= \left[\frac{24.43}{730}; \frac{59.34}{730}\right] \end{aligned}$$

$$\simeq \quad [0.033465753; 0.081287671].$$

- **Confidence interval for $R_T(t)$**
  Since $R_T(t) = e^{-\lambda t}$ is a decreasing function of $\lambda > 0$, we conclude that

$$\begin{aligned} CI_{95\%}\left(e^{-\lambda t}\right) &= \left[e^{-\lambda_U\, t}; e^{-\lambda_L\, t}\right] \\ &\overset{t=90}{\simeq} [0.000664889; 0.049197237]. \end{aligned}$$

2. *Uma estrutura em série é composta por duas componentes com durações i.i.d. e função taxa de falha comum igual a $\lambda(t) = 0.5t^{-0.5}, t \geq 0$.*

   (a) *Determine a função de fiabilidade da estrutura. O que pode concluir acerca do comportamento monótono da função taxa de falha da estrutura? Justifique.* (1

- **System**
  Series system
- **Individual durations and common hazard rate function**
  T_i, $i = 1, 2$, i.i.d. r.v. with common hazard rate function $\lambda(t) = 0.5t^{-0.5}$, $t \geq 0$, $i = 1, 2$.
- **Duration of the system**
  $T = \min\{T_1, T_2\}$
- **Hazard rate function of T**

$$\begin{aligned} \lambda_T(t) &\overset{(3.8)}{=} 2 \times \lambda(t) \\ &= t^{-0.5},\ t \geq 0, \end{aligned}$$

  which is a decreasing function of t.
- **Reliability function of T**
  According to Proposition 3.3, the reliability function of T is equal to

$$\begin{aligned} R_T(t) &= \exp\left[-\int_0^t \lambda_T(u)\, du\right] \\ &= \exp\left(-\int_0^t t^{-0.5}\, du\right) \\ &= \exp\left(-\left.\frac{u^{-0.5+1}}{-0.5+1}\right|_0^t\right) \\ &= \exp\left(-2t^{0.5}\right),\ t \geq 0 \\ [&= \exp\left[-\left(\frac{t}{2^{-\frac{1}{0.5}}}\right)^{0.5}\right],\ t \geq 0 \\ &= R_{Weibull(\delta=2^{-\frac{1}{0.5}}, \alpha=0.5)}(t)]. \end{aligned}$$

- **Stochastic ageing of T_i**
  Note that $\lambda_T(t)$ is a decreasing function of t. Thus, $T \in DHR$.
- **Obs.**
  According to Definition 4.21, the hazard rate function of a Weibull distribution with scale parameter δ and shape parameter α, Weibull(δ, α), is equal to

$$\lambda(t) = \frac{\alpha}{\delta} \times \left(\frac{t}{\delta}\right)^{\alpha-1},\ t \geq 0.$$

Thus,

$$T_i \overset{i.i.d.}{\sim} \text{Weibull}(\delta = 1, \alpha = 0.5),\ i = 1, 2.$$

Also note that $0 < \alpha = 0.5 < 1$, thus, according to subsection 4.3.4 (see table in page 100),

$$T_i \overset{i.i.d.}{\sim} DHR,\ i = 1, 2,$$

not to mention the fact that $\lambda(t) = t^{-0.5}$, a decreasing function of t. Now, if we apply Proposition 3.23, namely result (3.12), we can conclude that

$$T = T_{(1)} \in DHR.$$

(b) *Obtenha a duração esperada da estrutura e compare-a com um seu limite superior.* (1

- **Distribution of T**
  According to Definition 4.21, the reliability function of a Weibull distribution with scale parameter δ and shape parameter α, Weibull(δ, α), is equal to

$$\exp\left[-\left(\frac{t}{\delta}\right)^{\alpha}\right],\ t \geq 0$$

  Therefore $T \sim$ Weibull$(\delta = 2^{-\frac{1}{0.5}} = \frac{1}{4}, \alpha = 0.5)$.
- **Expected value of T**
  According to Exercise 4.22 (b),

$$\begin{aligned} E(T) &= E\left[\text{Weibull}(\delta, \alpha)\right] \\ &= \delta \times \Gamma\left(\frac{1}{\alpha} + 1\right) \\ &= \frac{1}{4} \times \Gamma\left(\frac{1}{0.5} + 1\right) \\ &= \frac{1}{4} \times \Gamma(3) \\ &= \frac{1}{4} \times 2! \\ &= 0.5. \end{aligned}$$

- **Upper bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - the 2 components have durations $T_i \overset{i.i.d.}{\sim} DHR$, $i = 1, 2$, and, thus, according to Proposition 3.36,

$$T_i \overset{i.i.d.}{\sim} NWUE,\ i = 1, 2;$$

  - the expected value of the duration of each of the 2 components is equal to

$$\begin{aligned} \mu_i &= E(T_i) \\ &= E\left[\text{Weibull}(\delta = 1, \alpha = 0.5)\right] \\ &= 1 \times \Gamma\left(\frac{1}{0.5} + 1\right) \\ &= 2. \end{aligned}$$

  Now, we can apply Theorem 3.62 and conclude that

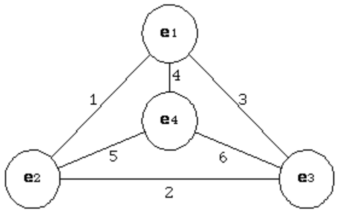
$$\mu_s = E(T)$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{n} \mu_i^{-1}\right)^{-1} \\
&= \left(\sum_{i=1}^{2} 2^{-1}\right)^{-1} \\
&= 1.
\end{aligned}
$$

- **Comment**

  The upper bound for $E(T)$ is quite different from its true value. This $\left(\frac{1-0.5}{0.5}\right) \times 100\% = 100\%$ relative discrepancy is probably due to the inaccuracy of this upper bound.

3. *Uma rede de comunicação com 4 estações de trabalho possui a configuração da figura abaixo onde* e_i *denota a estação* i $(i = 1, \ldots 4)$ *e os arcos* $1, \ldots, 6$ *representam canais de comunicação independentes.*



(a) *Determine a função de estrutura da rede de comunicação assumindo que ela fica inoperacional caso pelo menos uma das estações de trabalho fique isolada das restantes.* (1

- **Important**

  This communication network is no longer operational if at least one of the workstations $(e_1, \ldots, e_4)$ is isolated from the remaining ones because some of communications channels $(1, \ldots, 6)$ are not working.

- **Minimal cut sets**

$$
\begin{aligned}
\mathcal{K}_1 &= \{1,3,4\} \\
\mathcal{K}_2 &= \{1,2,5\} \\
\mathcal{K}_3 &= \{2,3,6\} \\
\mathcal{K}_4 &= \{4,5,6\} \\
q &= 4 \text{ minimal cut sets}
\end{aligned}
$$

- **Structure function**

  By considering $X_i \sim \text{Bernoulli}(p_i)$, $i = 1, \ldots, 6$, and applying result (1.14), we can conclude that the structure function of this system equals:

$$
\begin{aligned}
\phi(\underline{X}) &= \prod_{j=1}^{p^*} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - X_i)\right] \\
&= [1-(1-X_1)(1-X_3)(1-X_4)] \times [1-(1-X_1)(1-X_2)(1-X_5)] \\
&\quad \times [1-(1-X_2)(1-X_3)(1-X_6)] \times [1-(1-X_4)(1-X_5)(1-X_6)].
\end{aligned}
$$

(b) *Obtenha um par de limites inferior e superior o mais estritos possível para a fiabilidade da rede, caso os canais de comunicação sejam independentes e a fiabilidade de qualquer deles seja igual a* $p_i = p = 95\%$, $i = 1, \ldots, 6$. (2

- **Important**

  This communication network is operational if there is no workstation isolated from the remaining ones, either because their connected in pairs (see the first 3 minimal path sets) or because the four worstations are all connected by 3 communications channels (see the remaining minimal path sets).

- **Minimal path sets**

$$
\begin{aligned}
\mathcal{P}_1 &= \{1,6\} \\
\mathcal{P}_2 &= \{2,4\} \\
\mathcal{P}_3 &= \{3,5\} \\
\mathcal{P}_4 &= \{1,3,4\} \\
\mathcal{P}_5 &= \{1,2,5\} \\
\mathcal{P}_6 &= \{4,5,6\} \\
\mathcal{P}_4 &= \{2,3,6\} \\
p^* &= 7 \text{ minimal path sets}
\end{aligned}
$$

- **Bounds for the reliability** $r(\underline{p})$

  Since the components form a coherent system and operate in an independent fashion, we can apply Theorem 1.68, and get:

  - **Lower bound**

$$
\begin{aligned}
r(\underline{p}) &\geq \prod_{j=1}^{q^*} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - p_i)\right] \\
&\stackrel{p_i=p}{=} \prod_{j=1}^{4} \left[1 - (1-p)^{\#\mathcal{K}_j}\right] \\
&\stackrel{\#\mathcal{K}_j=3,\forall j}{=} \left[1 - (1-p)^3\right]^4 \\
&= \left[1 - (1-0.95)^3\right]^4 \\
&\simeq 0.999500
\end{aligned}
$$

  - **Upper bound**

$$
\begin{aligned}
r(\underline{p}) &\leq 1 - \prod_{j=1}^{p^*} \left(1 - \prod_{i \in \mathcal{P}_j} p_i\right) \\
&\stackrel{p_i=p}{=} 1 - \prod_{j=1}^{7} \left(1 - p^{\#\mathcal{P}_j}\right) \\
&\stackrel{\#\mathcal{P}_j=2,3}{=} 1 - (1-p^2)^3 \times (1-p^3)^4 \\
&= 1 - (1-0.95^2)^3 \times (1-0.95^3)^4 \\
&\simeq 1.000000.
\end{aligned}
$$

- **Obs.**

  Since the communication channels are independent and therefore operate in a positively associated fashion, we could have also applied Theorem 1.70 and obtain:

– **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned} r(\underline{p}) &\overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i\in\mathcal{P}_j} p_i \\ &\overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\ &\overset{\#\mathcal{P}_j=2,3}{=} p^2 \\ &= 0.95^2 \\ &= 0.9025; \end{aligned}$$

– **Upper bound for the reliability**

$$\begin{aligned} r(\underline{p}) &\overset{(1.42)}{\leq} \min_{j=1,\ldots,q} \left[1 - \prod_{i\in\mathcal{K}_j}(1-p_i)\right] \\ &\overset{p_i=p}{=} \min_{j=1,\ldots,q} \left[1-(1-p)^{\#\mathcal{K}_j}\right] \\ &\overset{\#\mathcal{K}_j=3}{=} 1-(1-p)^3 \\ &= 1-(1-0.95)^3 \\ &= 0.99875. \end{aligned}$$

Curiosuly, Theorem 1.70 provides a more (resp. less) accurate upper (resp. lower) bound for $r(\underline{p})$ than Theorem 1.68.

4. *Os dados abaixo reportam-se ao número de artigos defeituosos em 20 amostras de 150 peças cada soldadas por uma máquina recentemente adquirida.*

7	*5*	*5*	*4*	*2*	*1*	*1*	*7*	*4*	*3*
6	*5*	*6*	*15*	*7*	*8*	*0*	*12*	*10*	*9*

O fornecedor dessa mesma máquina afirmou que historicamente a verdadeira proporção de artigos defeituosos é aproximadamente igual a $p_0 = 0.04$. *Tendo em conta esta informação construiu-se uma carta de controlo unilateral superior com limite* $UCL = 14$*, tendo ocorrido um sinal de perda de controlo à 14a. observação, como se pode constatar pela tabela.*

(a) *Compare o valor de* ARL *sob controlo desta carta com o da carta padrão com limites 3-sigma. Qual das cartas lhe parece preferível usar? Justifique.* (1

- **Control statistic / quality characteristic**
  $Y_N =$ number of defective items in the N^{th} sample of size n, $N \in I\!N$
- **Distributions**
  $Y_N \sim \text{Binomial}(n, p_0)$, IN CONTROL, where $p_0 = 0.04$
  $Y_N \sim \text{Binomial}(n, p = p_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of an upward shift in p
- **Control limits of the upper one-sided** np **chart**

$$\begin{aligned} LCL &= 0 \text{ (because we are dealing with an upper one-sided chart)} \\ UCL &= 14 \end{aligned}$$

- **Run length**
  We are dealing with a Shewhart chart therefore the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:
  $$RL(\delta) \sim \text{Geometric}\,(\xi(\delta))\,.$$
- **In-control probability of triggering a signal**
  By using the Central Limit Theorem (CLT), we can obtain an approximate value of the probability that this charts triggers a signal when the process is in control:

$$\begin{aligned} \xi(\delta=0) &= P(Y_N \notin [LCL, UCL] \mid \delta = 0) \\ &\overset{Y_N\geq 0,\,LCL=0}{=} P\left(Y_N > UCL \mid \delta = 0\right) \\ &= 1 - F_{Binomial(n,p=p_0+0)}(UCL) \\ &= 1 - F_{Binomial(150,0.04)}(14) \\ &\overset{CLT}{\simeq} 1 - \Phi\left(\frac{14 - 150\times 0.04}{\sqrt{150\times 0.04\times(1-0.04)}}\right) \\ &\simeq 1 - \Phi(3.33) \\ &\overset{table}{=} 1 - 0.999566 \\ &= 0.000434. \end{aligned}$$

- **In-control ARL**

$$\begin{aligned} ARL(\delta=0) &= \frac{1}{\xi(\delta=0)} \\ &\simeq \frac{1}{0.000434} \\ &\simeq 2304.15 \end{aligned}$$

- **Control limits of the standard** np **chart with 3 sigma limits**

$$\begin{aligned} LCL' &= \max\left\{0, np_0 - 3\times\sqrt{np_0(1-p_0)}\right\} \\ &= \max\left\{0, 150\times 0.04 - 3\times\sqrt{150\times 0.04\times(1-0.04)}\right\} \\ &= \max\{0, -1.2\} \\ &= 0 \\ UCL' &= np_0 + 3\times\sqrt{np_0(1-p_0)} \\ &= 150\times 0.04 - 3\times\sqrt{150\times 0.04\times(1-0.04)} \\ &= 13.2 \end{aligned}$$

- **In-control probability of a signal by the standard chart**
  If we use the CLT once again, we get:

$$\begin{aligned} \xi'(\delta=0) &= P(Y_N \notin [LCL', UCL'] \mid \delta = 0) \\ &\overset{Y_N\geq 0,\,LCL'=0}{=} P\left(Y_N > UCL' \mid \delta = 0\right) \\ &= 1 - F_{Binomial(n,p=p_0+0)}(UCL') \\ &= 1 - F_{Binomial(150,0.04)}(13) \\ &\overset{CLT}{\simeq} 1 - \Phi\left(\frac{13 - 150\times 0.04}{\sqrt{150\times 0.04\times(1-0.04)}}\right) \\ &\simeq 1 - \Phi(2.92) \end{aligned}$$

$$\begin{aligned} &\overset{table}{=} && 1-0.9982 \\ &= && 0.0018. \end{aligned}$$

- **In-control ARL of the standard chart**

$$\begin{aligned} ARL'(\delta=0) &= \frac{1}{\xi'(\delta=0)} \\ &\simeq \frac{1}{0.0018} \\ &\simeq 555.556 \end{aligned}$$

- **Comment**
The maximization of the in-control ARL criterion suggests the use of the first chart, whereas the minimization of the out-of-control ARL criterion suggests the second one.[1] However, the standard chart (which is in fact an upper one-sided chart), with its in-control ARL over 500 samples, seems all in all a reasonable choice.

(b) *Caso tivesse ocorrido uma alteração no valor esperado do número de artigos defeituosos para $p = 0.06$, qual seria a mediana do número de amostras recolhidas até à detecção de tal alteração por parte da carta unilateral superior? Comente o seu significado.* (1

- **Shift**
The fraction of defective items has suffered a shift from $p_0 = 0.04$ to $p = p_0 + \delta = 0.06$, i.e., $\delta = 0.02$.

- **Probability of triggering a signal**

$$\begin{aligned} \xi(\delta) &= P(Y_N \notin [LCL, UCL] \mid \delta) \\ &= 1 - F_{Binomal(n,p=p_0+\delta)}(UCL) \\ &\overset{CLT}{\simeq} 1 - \Phi\left(\frac{14 - 150 \times 0.06}{\sqrt{150 \times 0.06 \times (1-0.06)}}\right) \\ &\simeq 1 - \Phi(1.72) \\ &\overset{table}{=} 1 - 0.9573 \\ &= 0.0427. \end{aligned}$$

- **Median of $RL(\delta)$**
Since $RL(\delta) \sim \text{Geometric}\,(\xi(\delta))$, we get

$$\begin{aligned} F^{-1}_{RL(\delta)}(p) &\overset{Table\ 9.2}{=} \inf\left\{m \in I\!N : F_{RL(\delta)}(m) \geq \alpha\right\} \\ &= 1 - [1-\xi(\delta)]^m \geq \alpha \\ &= [1-\xi(\delta)]^m \leq 1-\alpha \\ &= m \times \ln[1-\xi(\delta)] \leq \ln(1-\alpha) \\ &\overset{\ln[1-\xi(\delta)]<0}{=} m \geq \frac{\ln(1-p)}{\ln[1-\xi(\delta)]} \\ &\overset{\alpha=0.5}{=} m \geq \frac{\ln(1-0.5)}{\ln(1-0.0427)} \\ &= m \geq 15.883861 \\ &= 16 \end{aligned}$$

[1]In the chart selection we should have in mind what is economically more serious: the emission of a false alarm or the late ection of a shift.

- **Comment**
When the fraction defective increases from its target value $p_0 = 0.04$ to $p = 0.06$, the probability of triggering a valid signal within the first 16 samples is at least 50%, thus, suggesting a quick detection.

5. *O tempo (em minutos) que uma firma de serviços leva a instalar certo tipo de* software *é, sob controlo, uma v.a. com distribuição normal*$(25, 6^2)$.

Por forma a acompanhar o desempenho de 2 funcionários da firma foram registadas as médias dos tempos de conjuntos de 4 instalações, sendo que as observações da tabela abaixo com número ímpar (par) referem-se ao 1o. (2o.) funcionário.

1	*23.6*	*6*	*24.6*	*11*	*29.4*	*16*	*32.8*
2	*20.8*	*7*	*22.6*	*12*	*27.8*	*17*	*23.3*
3	*25.5*	*8*	*24.4*	*13*	*26.8*	*18*	*30.5*
4	*26.2*	*9*	*24.7*	*14*	*27.2*	*19*	*25.3*
5	*23.3*	*10*	*26.0*	*15*	*24.0*	*20*	*34.1*

a) *Que conclusões pode tirar acerca do desempenho dos 2 funcionários ao utilizar uma carta unilateral superior para μ com $ARL(0) = 500$?* (1

- **Quality characteristic**
X = time to install some software
$X \sim \text{Normal}(\mu, \sigma^2)$

- **Control statistic**
$\bar{X}_N$ = mean of the N^{th} random sample of size n

- **Distribution**
$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 25$, $\sigma_0 = 6$ and $n = 4$
$\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of an upward shift in μ.

- **Control limits of the upper one-sided $\bar{X}-$ chart**
$LCL = -\infty$
$UCL = \mu_0 + \gamma \frac{\sigma_0}{\sqrt{n}}$

- **Probability of triggering a signal**
Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to:

$$\begin{aligned} \xi(\delta) &= P\left(\bar{X}_N \notin [LCL, UCL] \mid \delta\right) \\ &\overset{LCL=-\infty}{=} 1 - \Phi\left(\frac{UCL - \mu}{\frac{\sigma_0}{\sqrt{n}}}\right) \\ &= 1 - \Phi(\gamma_\mu - \delta),\ \delta \geq 0. \end{aligned}$$

- **Run length**
We are dealing with a Shewhart chart, therefore the number of samples collected until the chart triggers a signal given δ, $RL(\delta)$, has the following distribution:

$$RL(\delta) \sim \text{Geometric}\,(\xi(\delta))$$

- **Obtaining γ**
  The constant γ is such that $ARL(0) = 500$, that is,

$$
\begin{aligned}
\gamma_\mu \quad : \quad & \frac{1}{\xi_\mu(0)} = ARL(0,1) \\
& 1 - \Phi(\gamma) = \frac{1}{ARL(0)} \\
& \gamma = \Phi^{-1}\left[1 - \frac{1}{ARL(0)}\right] \\
& \gamma = \Phi^{-1}(0.998) \\
& \gamma \overset{table}{=} 2.8782
\end{aligned}
$$

- **Upper control limit**

$$
\begin{aligned}
UCL &= \mu_0 + \gamma \frac{\sigma_0}{\sqrt{n}} \\
&= 25 + 2.8782 \times \frac{6}{\sqrt{4}} \\
&\simeq 33.6346
\end{aligned}
$$

- **Comment**
  The only observation above UCL is the 20^{th}, unsurprisingly associated to the 2^{nd} employee. Please note that the even mean values in the table are associated to this employee and those values suggest not only that the 2^{nd} employee tends to take more time to install the software, but also that the process is out-of-control.

b) *Qual a probabilidade de ser emitido sinal o mais tardar ao fim de 40 instalações, caso tenha ocorrido um* shift *de $\mu_0 = 25$ para $\mu = 26$?* (1

- **Shift**
  The expected value has shifted from $\mu_0 = 25$ to $\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}} = 26$, i.e.,

$$
\begin{aligned}
\delta &= \frac{\mu - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} \\
&= \frac{26 - 25}{\frac{6}{\sqrt{4}}} \\
&= \frac{1}{3}.
\end{aligned}
$$

- **Probability of triggering a signal**

$$
\begin{aligned}
\xi(\delta) &= 1 - \Phi(\gamma_\mu - \delta) \\
&= 1 - \Phi\left(2.8782 - \frac{1}{3}\right) \\
&\simeq 1 - \Phi(2.55) \\
&\overset{table}{=} 1 - 0.9946 \\
&= 0.0054
\end{aligned}
$$

- **Requested probability**
  Since $RL(\delta) \sim \text{Geometric}(\xi(\delta))$, we get:

$$
\begin{aligned}
P[RL(\delta) \le m] &= 1 - [1 - \xi(\delta)]^m \\
&\overset{m=10}{\simeq} 1 - (1 - 0.0054)^{10} \\
&\simeq 0.052707.
\end{aligned}
$$

- **Obs.**
  It is very unlikely that the detection of such a shift is detected within the first 40 installations, i.e., within the first 10 samples. This is not a surprising result — it is well known that the Shewhart charts, such as the one we are using, are slow in detecting small and moderate shifts like one with magnitude equal to $\delta = \frac{1}{3}$.

6. *Os seguintes dados representam o número de defeitos à superfície de 20 chapas rectangulares de aço: 4, 0, 2, 3, 4, 3, 1, 2, 5, 0, 2, 5, 1, 7, 8, 10, 3, 6, 4, 5.*

   (a) *O que poderá dizer acerca do estado do processo de produção de chapas rectangulares de aço. Justifique.* (0

- **Quality characteristic / control statistic**
  Y_N = number of defects in the N^{th} rectangular steel plate, $N \in I\!N$
  $Y \sim \text{Poisson}(\lambda)$

- **Estimate of the nominal value of λ**

$$
\begin{aligned}
\hat{\lambda}_0 &= \bar{y} \\
&= \frac{1}{20} \times (4 + \ldots + 5) \\
&= 3.75
\end{aligned}
$$

- **Estimates of the 3 sigma control limits of the standard u chart**

$$
\begin{aligned}
\widehat{LCL} &= \max\left\{0, \hat{\lambda}_0 - 3 \times \sqrt{\hat{\lambda}_0}\right\} \\
&= \max\left\{0, 3.75 - 3 \times \sqrt{3.75}\right\} \\
&= \max\{0, -2.059475\} \\
&= 0 \\
\widehat{UCL} &= \hat{\lambda}_0 + 3 \times \sqrt{\hat{\lambda}_0} \\
&= 3.75 + 3 \times \sqrt{3.75} \\
&\simeq 9.559475
\end{aligned}
$$

- **State of the process**
  The fact that $y_{16} = 10 > \widehat{UCL} = 9.559475$ suggests that the process is out-of-control and that the target and control limits should be re-estimated after leaving out y_{16}.

- **New estimate of λ_0**

$$
\begin{aligned}
\hat{\lambda}_0' &= \frac{1}{n-1} \sum_{i=1, i \neq 16}^{n} y_i \\
&= \frac{1}{n-1} \times (n \times \bar{y} - y_{16}) \\
&= \frac{3.75 \times 20 - 10}{20 - 1} \\
&\simeq 3.421053
\end{aligned}
$$

- **New estimates of the 3 sigma control limits of the standard u chart**

$$
\begin{aligned}
\widehat{LCL}' &= \max\left\{0, \hat{\lambda}_0' - 3 \times \sqrt{\hat{\lambda}_0'}\right\} \\
&= \max\left\{0, 3.421053 - 3 \times \sqrt{3.421053}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \max\{0, -2.127973\} \\
&= 0 \\
\widehat{UCL}' &= \hat{\lambda}_0' + 3 \times \sqrt{\hat{\lambda}_0'} \\
&= 3.421053 + 3 \times \sqrt{3.421053} \\
&\simeq 8.967973
\end{aligned}
$$

- **State of the process**
  After eliminating y_{16} and recalculating the estimates of the target and the control limits of this charts, we can add that all the remaining observations are within the estimated control limits and the process is deemed in-control.

(b) *Obtenha as três primeiras entradas da 1a. linha da matriz de probabilidades de transição associada à carta CUSUM, com limites de controlo e valor de referência inteiros e iguais a $LCL = 2$, $UCL = 10$ e $k = 4$ (respectivamente), na ausência de causas assinaláveis.* (2

- **CUSUM chart for Poisson data**
  $LCL = 2$
  $UCL = x = 10$
  $k = 4$ (reference value)
  $u = 0$ (no head-start)

- **Control statistic**

$$
Z_N = \begin{cases} 0, & N = 0 \\ Z_{N-1} + (Y_N - k), & N \in I\!N, \end{cases}
$$

  which is similar to the control statistic of an upper one-sided CUSUM chart for binomial data in Example 10.9, equation (10.4).

- **First 3 entries of the transition probability matrix**
  First note that the control statistic is associated to an absorbing Markov chain with transient states $\{2, 3, \dots, 10\}$. Thus, in the absence of assignable causes, the 3 requested entries are equal to:

$$
\begin{aligned}
p_{2\,2}(0) &= P(Z_N = 2 | Z_{N-1} = 2, \delta = 0) \\
&= P(Z_{N-1} + (Y_N - k) = 2 | Z_{N-1} = 2, \delta = 0) \\
&\overset{Y_N\ indep\ Z_{N-1}}{=} P(2 + Y_N - k = 2 | \delta = 0) \\
&= P(Y_N = k | \delta = 0) \\
&\simeq e^{-3.42} \times \frac{3.42^4}{4!} \\
&\simeq 0.186469; \\
p_{2\,3}(0) &= P(Z_N = 3 | Z_{N-1} = 2, \delta = 0) \\
&\dots \\
&= P(Y_N = k + 1 | \delta = 0) \\
&\simeq e^{-3.42} \times \frac{3.42^5}{5!} \\
&\simeq 0.127545; \\
p_{2\,4}(0) &= P(Z_N = 4 | Z_{N-1} = 2, \delta = 0) \\
&\dots \\
&= P(Y_N = k + 2 | \delta = 0) \\
&\simeq e^{-3.42} \times \frac{3.42^6}{6!} \\
&\simeq 0.072700.
\end{aligned}
$$

- **Obs.**
  We ought to add that the first entry is not equal to $F_{Y_N}(k)$, as in the upper one-sided CUSUM chart for Poisson data.

7. *Retome o Exercício 5, considerando para o efeito um esquema conjunto para μ e σ^2 que faça uso de cartas Shewhart unilaterais superiores, cujos limites de controlo são tais que a probabilidade de emissão de falso alarme é em qualquer das 2 cartas igual a 0.005.*

   a) *Qual a probabilidade de vir a ser emitido um sinal válido entre as 10 primeiras amostras recolhidas após um* shift *simultâneo em μ e σ com magnitude $(\delta, \theta) = (0.1, 1.1)$?* (1

   - **Quality characteristic**
     $X \sim \text{Normal}(\mu, \sigma^2)$
   - **Control statistics**
     $\bar{X}_N =$ mean of the N^{th} random sample of size n
     $S^2_N =$ variance of the N^{th} random sample of size n, $N \in I\!N$
   - **Distributions**
     $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $n = 5$
     $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \geq 0$) represents the magnitude of the shift (an increase!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ
     $\frac{(n-1)S^2_N}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
     $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL
   - **Control limits of the upper one-sided $\bar{X}-$ chart and the upper one-sided S^2- chart**
     $LCL_\mu = -\infty$
     $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
     $LCL_\sigma = 0$
     $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$
   - **Probabilities of triggering signals**
     Taking into account the distributions of the control statistics, the individual charts for μ and σ trigger signals with probabilities equal to

$$
\begin{aligned}
\xi_\mu(\delta, \theta) &= P\left(\bar{X}_N \not\in [LCL_\mu, UCL_\mu] \mid \delta, \theta\right) \\
&= 1 - \Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) \\
&= 1 - \Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right), \ \delta \geq 0, \ \theta \geq 1, \\
\xi_\sigma(\theta) &= P\left(S^2_N \not\in [LCL_\sigma, UCL_\sigma] \mid \theta\right)
\end{aligned}
$$

$$= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right]$$
$$= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right),\ \theta \geq 1,$$

respectively.

- **Obtaining γ_μ and γ_σ**

  The constants γ_μ and γ_σ are such that $\xi_\mu(0,1) = \xi_\sigma(1) = 0.005$, i.e.

$$\begin{aligned}
\gamma_\mu \quad &: \quad 1 - \Phi(\gamma_\mu) = 0.005 \\
&\quad\ \ \gamma_\mu = \Phi^{-1}(0.995) \\
&\quad\ \ \gamma_\mu \overset{table}{=} 2.5758 \\
\gamma_\sigma \quad &: \quad 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = 0.005 \\
&\quad\ \ \gamma_\sigma = F_{\chi^2_{(4-1)}}(0.995) \\
&\quad\ \ \gamma_\sigma \overset{table}{=} 12.84
\end{aligned}$$

- **Probability of a signal by the joint scheme for μ and σ**

  The joint scheme triggers a signal if either of the individual charts triggers an alarm. Moreover, the control statistics of the individual charts are independent given (δ, θ). As a consequence, the joint scheme for μ and σ triggers a signal with probability equal to:

$$\begin{aligned}
\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \delta, \theta\right) \\
&= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta) \\
&= \xi_\mu(\delta,\theta) + [1 - \xi_\mu(\delta,\theta)] \times \xi_\sigma(\theta).
\end{aligned}$$

  When $(\delta,\theta) = (0.1, 1.1)$, we get:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= 1 - \Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) \\
&= 1 - \Phi\left(\frac{2.5758 - 0.1}{1.1}\right) \\
&\simeq 1 - \Phi(2.25) \\
&\overset{table}{=} 1 - 0.9878 \\
&= 0.0122 \\
\xi_\sigma(\theta) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right) \\
&= 1 - F_{\chi^2_{(4-1)}}\left(\frac{12.84}{1.1^2}\right) \\
&\simeq 1 - F_{\chi^2_{(3)}}(10.61) \\
&\in (0.01; 0.025)
\end{aligned}$$

  because

$$\begin{aligned}
F^{-1}_{\chi^2_{(3)}}(0.975) = 9.348 \quad &< \quad 10.61 < 11.34 = F^{-1}_{\chi^2_{(3)}}(0.99) \\
1 - 0.99 \quad &< \quad \xi_\sigma(1.1) < 1 - 0.975.
\end{aligned}$$

  Then a signal is triggered by the joint scheme, when $(\delta,\theta) = (0.1, 1.1)$, with probability

$$\begin{aligned}
\xi_{\mu,\sigma}(0.1, 1.1) &= \xi_\mu(0.1,1.1) + [1 - \xi_\mu(0.1,1.1)] \times \xi_\sigma(1.1) \\
&\in (0.0122 + (1 - 0.0122) \times 0.01;\ 0.0122 + (1 - 0.0122) \times 0.025) \\
&= (0.022078;\ 0.036895).
\end{aligned}$$

- **Run length**

  The run length of this joint scheme for μ and σ, $RL_{\mu,\sigma}(\delta,\theta)$, has the following distribution:

$$RL_{\mu,\sigma}(\delta,\theta) \sim \text{Geometric}(\xi_{\mu,\sigma}(\delta,\theta)).$$

- **Requested probability**

  The probability that the joint scheme triggers a signal within the first 10 samples equals:

$$\begin{aligned}
P\left[RL_{\mu,\sigma}(\delta,\theta) \leq 10\right] &= 1 - [1 - \xi_{\mu,\sigma}(\delta,\theta)]^{10} \\
&\in \left(1 - (1 - 0.022078)^{10};\ 1 - (1 - 0.036895)^{10}\right) \\
&= (0.200089;\ 0.313348).
\end{aligned}$$

b) *Determine a probabilidade de ocorrência de sinal erróneo de Tipo III (IV) quando $\theta = 1.1$ ($\delta = 0.1$).* (1

Nota: *Na impossibilidade de obter valores exactos obtenha intervalo de valores para estas probabilidades.*

- **Probability of a misleading signal of type III**

$$\begin{aligned}
PMS_{III}(\theta) &\overset{Table\ 10.12}{=} \frac{1 - \Phi(\gamma_\mu/\theta)}{[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)]^{-1} - \Phi(\gamma_\mu/\theta)} \\
&\overset{\theta=1.1}{=} \frac{1 - \Phi(2.5758/1.1)}{[F_{\chi^2_{(4-1)}}(12.84/1.1^2)]^{-1} - \Phi(2.5758/1.1)} \\
&\simeq \frac{1 - \Phi(2.34)}{[F_{\chi^2_{(3)}}(10.61)]^{-1} - \Phi(2.34)} \\
&\overset{table,\,(a)}{=} \left(\frac{1 - 0.9904}{0.975^{-1} - 0.9904};\ \frac{1 - 0.9904}{0.99^{-1} - 0.9904}\right) \\
&\simeq (0.272410;\ 0.487285)
\end{aligned}$$

- **Probability of a misleading signal of type IV**

$$\begin{aligned}
PMS_{IV}(\delta) &\overset{Table\ 10.12}{=} \frac{1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma)}{[\Phi(\gamma_\mu - \delta)]^{-1} - F_{\chi^2_{(n-1)}}(\gamma_\sigma)} \\
&\overset{\delta=0.1}{=} \frac{1 - F_{\chi^2_{(4-1)}}(12.84)}{[\Phi(2.5758 - 0.1)]^{-1} - F_{\chi^2_{(4-1)}}(12.84)} \\
&\overset{a)}{\simeq} \frac{0.005}{[\Phi(2.48)]^{-1} - (1 - 0.005)} \\
&\overset{table}{=} \frac{0.005}{(0.9934)^{-1} - 0.995} \\
&\simeq 0.429411.
\end{aligned}$$

- **Obs.**

  These values are far from negligible, proving that misleading signals are a cause for concern while dealing with joint schemes for μ and σ.