Fiabilidade e Controlo de Qualidade — LMAC

Exame de Época Especial — 2o. Semestre – 2002/03

Duração: 3 horas — 13/09/03 – 09 horas – V0.01

1. *Foram registados os seguintes instantes (em anos) de falha por desgaste do tipo bushing failure de geradores de 115kV: 8.00, 8.83, 9.50, 10.75, 11.75.*

(a) *Determine algumas estimativas das funções de fiabilidade e de taxa de falha do tempo até falha.* (1.0)

- **Failure time**
  T = time to a bushing failure by a power generator of $115kV$
- **Complete data**
  The ordered failure times are $(t_{(1)}, t_{(2)}, t_{(3)}, t_{(4)}, t_{(5)}) = (8.00, 8.83, 9.50, 10.75, 11.75)$.
- **Estimates of $R(t)$ and $\lambda(t)$ (for a few points)**
  Since we are dealing with complete data, we can obtain estimates of the reliability function $R(t)$ and the hazard rate function $\lambda_T(t)$ by using the results in Table 5.1 (Blom estimates):

i	$t_{(i)}$	$t_{(i)} - t_{(i-1)}$	$\hat{R}\left(t_{(i)}\right) = \frac{n-i+0.625}{n+0.25}$	$\hat{\lambda}\left(t_{(i)}\right) = \frac{1}{(n-i+0.625)\times\left[t_{(i)}-t_{(i-1)}\right]}$
1	8.00	0.83	0.880952	0.260501
1	8.83	0.67	0.690476	0.411734

(b) *Calcule as abcissas e ordenadas dos três primeiros pontos do correspondente gráfico TTT. Que aspecto esperaria para a função taxa de falha?* (1.0)

- **Total time on test up to time $t_{(i)}$**

$$\tau\left(t_{(i)}\right) = \sum_{j=1}^{i}(n-j+1)\left[t_{(j)} - t_{(j-1)}\right] = \tau\left(t_{(i-1)}\right) + (n-i+1)\left[t_{(i)} - t_{(i-1)}\right]$$

- **Abcissae of the TTT plot**
  $\frac{i}{n}$, $i = 0, 1, \ldots, n$
- **Ordinates of the TTT plot**
  $\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$, $i = 1, \ldots, n$; equal to 0, for $i = 0$.
- **Three/Five points of the TTT plot**

i	$\tau\left(t_{(i)}\right) = \tau\left(t_{(i-1)}\right) + (n-i+1)\left[t_{(i)} - t_{(i-1)}\right]$	$\frac{i}{n}$	$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$
0	0	0	0
1	$0 + (5-1+1)\times 8.00 = 40.00$	$\frac{1}{5} = 0.2$	$\frac{40.00}{48.83} \simeq 0.8192$
2	$40.00 + (5-2+1)\times(8.83-8.00) = 43.32$	$\frac{2}{5} = 0.4$	$\frac{43.32}{48.83} \simeq 0.8872$
3	$43.32 + (5-3+1)\times(9.50-8.83) = 45.33$	$\frac{3}{5} = 0.6$	$\frac{45.33}{48.83} \simeq 0.9283$
4	$45.33 + (5-4+1)\times(10.75-9.50) = 47.83$	$\frac{4}{5} = 0.8$	$\frac{47.83}{48.83} \simeq 0.9795$
5	$47.83 + (5-5+1)\times(11.75-10.75) = 48.83$	$\frac{5}{5} = 1$	$\frac{48.83}{48.83} = 1$

- **Aspect of the TTT plot**
  According to Remark 5.5, a concave TTT plot, as the one we obtained, suggests that $T \in IHR$.

(c) *Ao adiantar-se que os instantes registados dizem respeito a um teste de vida envolvendo 20 geradores e concluído à quinta falha, calcule um intervalo de confiança a 95% para a função de fiabilidade do tempo até falha por desgaste para um período de 10 anos. Enuncie as hipóteses de trabalho que entender mais convenientes para a resolução desta questão.* (1.5)

- **Life test**
  Since the end of the test was determined by the $r = 5^{th}$ bushing failure and nothing in this exercise suggests that the $n = 20$ power generators were replaced during the life test, we are dealing with a
  - Type II/item censored testing without replacement.
- **Censored data**
  $n = 20$
  $r = 5$
  $(t_{(1)}, \ldots, t_{(r)}) = (8.00, 8.83, 9.50, 10.75, 11.75)$
- **Cumulative total time in test**
  According to Definition 5.17, the cumulative total time in test is given by:

$$\tilde{t} = \sum_{i=1}^{r} t_{(i)} + (n-r)\times t_{(r)} = 48.83 + (20-5)\times 11.75 = 225.08$$

- **Distribution assumption**
  T_i = bushing failure time of the i^{th} power generator
  $T_i \overset{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$
- **Unknown parameters**
  λ
  $R_T(t) = e^{-\lambda t}$, where $t = 10$.
- **Confidence interval for λ**
  According to Table 5.16 of the lecture notes,

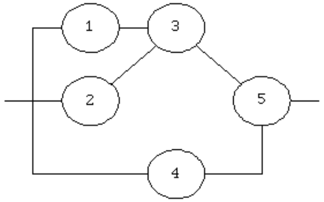
$$CI_{(1-\alpha)\times 100\%}(\lambda) = [\lambda_l; \lambda_U] = \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r)}}(1-\alpha/2)}{2\times\tilde{t}}\right]$$

$$CI_{95\%}(\lambda) = \left[\frac{F^{-1}_{\chi^2_{(10)}}(0.025)}{2\times 225.08}; \frac{F^{-1}_{\chi^2_{(10)}}(0.975)}{2\times 225.08}\right] = \left[\frac{3.247}{2\times 225.08}; \frac{20.48}{2\times 225.08}\right] \simeq [0.007213; 0.045495]$$

- **Confidence interval for $R_T(t)$**
  Since $R_T(t) = e^{-\lambda t}$ is a decreasing function of $\lambda > 0$, we can state that

$$CI_{95\%}\left(e^{-\lambda t}\right) = \left[e^{-\lambda_U t}; e^{-\lambda_L t}\right] \overset{t=10}{=} [0.634480; 0.930410].$$

2. *Um circuito integrado possui cinco componentes dispostas de acordo com a figura abaixo.*



Admita que as durações destas componentes (em milhares de horas) são v.a.s i.i.d. com função taxa de falha comum:

$$\lambda(t) = \begin{cases} t, & 0 < t \leq 2 \\ -t+4, & 2 < t \leq 2.5 \\ t-1, & t > 2.5. \end{cases}$$

(a) *Prove que, embora $\lambda(t)$ seja uma função não monótona, as durações das componentes são IHRA.* (1.0)

- **Individual durations and common hazard rate function**
  T_i, $i = 1, \ldots, 5$, i.i.d. r.v. with hazard rate function $\lambda(t)$.
- **Stochastic aging character of T_i**
  $\lambda(t)$ is not a monotone (or monotonic) function. Thus, $T_i \notin IHR$ or $T_i \notin DHR$. However,

$$\begin{aligned}
\frac{1}{t}\Lambda(t) &= \frac{1}{t}\int_0^t \lambda(u)\,du \\
&= \frac{1}{t} \times \begin{cases} 0, & t < 0 \\ \int_0^t u\,du, & 0 \leq t \leq 2 \\ \int_0^2 u\,du + \int_2^t (4-u)\,du, & 2 < t \leq 2.5 \\ \int_0^2 u\,du + \int_2^{2.5}(4-u)\,du + \int_{2.5}^t (u-1)\,du, & t > 2.5 \end{cases} \\
&\ldots \\
&= \frac{1}{t} \times \begin{cases} 0, & t < 0 \\ \frac{t^2}{2}, & 0 \leq t \leq 2 \\ -\frac{t^2}{2} + 4t - 4, & 2 < t \leq 2.5 \\ \frac{t^2}{2} - t + 2.25, & t > 2.5 \end{cases} \\
\frac{d\left[\frac{1}{t}\Lambda(t)\right]}{dt} &= \begin{cases} \frac{1}{2}, & 0 \leq t \leq 2 \\ -\frac{1}{2} + \frac{4}{t^2}, & 2 < t \leq 2.5 \\ \frac{1}{2} - \frac{2.25}{t^2}, & t > 2.5. \end{cases}
\end{aligned}$$

Please note that:

- $\frac{1}{2} > 0$;
- $-\frac{1}{2} + \frac{4}{t^2}$ is a decreasing function, thus, the minimum value is this function in $(2, 2.5]$ is $-\frac{1}{2} + \frac{4}{2.5^2} = 0.14 > 0$;
- $\frac{1}{2} - \frac{2.25}{t^2}$ is an increasing function, thus, the minimum value is this function in $(2.5, +\infty]$ is $\frac{1}{2} - \frac{2.25}{2.5^2} = 0.14 > 0$.

As a consequence $\frac{1}{t}\Lambda(t)$ is an increasing monotone function for $t \geq 0$, thus, $T_i \in IHRA$, $i = 1, \ldots, 5$.

(b) *O que poderá adiantar sobre o envelhecimento estocástico da duração do sistema?* (0.5)

- **Duration of the system**
  $T = g(T_1, \ldots, T_5)$, where $T_i \in IHRA$, $i = 1, \ldots, 5$.
- **Stochastic ageing character of T**
  According to Table 3.2, the $IHRA$ property of the components of a system is preserved after the formation of a coherent system, like the system we are dealing with. Thus, $T \in IHRA$.

(c) *Obtenha uma expressão para o valor esperado da duração de cada componente bem como outra para um limite inferior o mais estrito possível para a duração esperada do circuito integrado.* (1.5)

- **Expected value of T_i — exact expression**

$$\begin{aligned}
\mu_i &= E(T_i) \\
&\overset{T_i \geq 0, (2.10)}{=} \int_0^{+\infty} R(t)\,dt \\
&\overset{(3.3)}{=} \int_0^{+\infty} \exp\left[-\Lambda(t)\right]dt \\
&= \int_0^2 e^{-\frac{t^2}{2}}\,dt + \int_2^{2.5} e^{\frac{t^2}{2} - 4t + 4}\,dt + \int_{2.5}^{+\infty} e^{-\frac{t^2}{2} + t - 2.25}\,dt
\end{aligned}$$

- **Lower bound for $E(T)$**
  We are dealing with a coherent system characterized as follows:
  - $T_i \overset{i.i.d.}{\sim} IHRA$, $i = 1, \ldots, 5$;
  - $\mu_i = E(T_i) = \mu^*$;
  - the minimal path sets are

$$\begin{aligned}
\mathcal{P}_1 &= \{1, 3, 5\} \\
\mathcal{P}_2 &= \{2, 3, 5\} \\
\mathcal{P}_3 &= \{4, 5\} \\
p &= 3 \text{ minimal path sets.}
\end{aligned}$$

  Now, we can apply Theorem 3.69, and obtain a lower bound for $E(T)$:

$$\begin{aligned}
\mu &= E(T) \\
&\geq \max_{j=1,\ldots,p}\left\{\left(\sum_{i \in \mathcal{P}_j} \mu_i^{-1}\right)^{-1}\right\} \\
&\overset{\mu_i = \mu^*}{=} \max_{j=1,\ldots,p}\left\{\left(\frac{\#\mathcal{P}_j}{\mu^*}\right)^{-1}\right\} \\
&= \frac{\mu^*}{\min_{j=1,\ldots,p}\{\#\mathcal{P}_j\}} \\
&= \frac{\mu^*}{2} \\
[&\overset{Mathematica}{\simeq} \frac{1.269569}{2} \\
&\simeq 0.6347845].
\end{aligned}$$

(d) *Obtenha um limite para a função de fiabilidade do sistema para um período de 2500 horas, sabendo de antemão que a mediana da duração do sistema é de 3000 horas?* (1.0)

- **Lower bound for the reliability**
  Taking into account that
  - $T \in IHRA$
  - $\xi_p = F_T^{-1}(p) = 3.0$, for $p = 0.5$,
  - $t = 2.500 < \xi_p$,

  we can apply Theorem 3.40 and state that

$$\begin{aligned} R_T(t) &\geq (1-p)^{\frac{t}{\xi_p}} \\ &\geq (1-0.5)^{\frac{2.5}{3.0}} \\ &\simeq 0.561231. \end{aligned}$$

3. *Um dispositivo de detecção de níveis elevados de monóxido de carbono (CO) emite um sinal válido com probabilidade 0.60. Uma vez que tal sensor não é muito fiável decidiu-se constituir um sistema de detecção com três desses sensores.*

   (a) *Disponha sensores independentes por forma a maximizar a fiabilidade, i.e., a probabilidade de emissão de sinal válido por parte do sistema. Justifique.* (1.0)

   - **Components**
     3 detectors of high levels of CO
   - **Probability of a signal**
     $p = 0.6$, for each detector.
   - **System**
     in order to maximize the probability of a signal of a system with 3 detectors, these have to be set in parallel. The reason for this is Theorem 1.65, stated as follows: the reliability of any coherent system with positively associated components (e.g. independent components) is limited above (resp. below) by the reliability of a parallel (resp. series) system with those same components.

   (b) *Qual o número mínimo de replicações ao nível do sistema necessário para que a fiabilidade seja de pelo menos 0.995?* (1.5)

   - **Reliability of the system**
     Assuming once again that the detectors function in an independent fashion, we get:

$$\begin{aligned} r(\underline{p}) &= r(0.6, 0.6, 0.6) \\ &= 1 - \prod_{i=1}^{3}(1-p_i) \\ &= 1-(1-0.6)^3 \\ &= 0.936. \end{aligned}$$

   - **Replicating the system at its level**
     According to (1.26), if we replicate the system at its level, the resulting system has reliability equal to:

$$\begin{aligned} 1-[1-r(\underline{p})]\times[1-r(\underline{p}')] &= 1-\left\{1-[1-(1-0.6)^3]\right\}\times\left\{1-[1-(1-0.6)^3]\right\} \\ &= 1-(1-0.6)^6 \\ &= 0.995904 \\ &\geq 0.995. \end{aligned}$$

   - **Conclusion**
     One replication at this system level is enough to increase its reliability from 0.936 to a value larger than 0.995, as requested.

4. *Pretende controlar-se o número esperado de pacotes de informação transmitidos por segundo por um modem ligado a uma linha telefónica.*

   (a) *Defina uma carta que permita detectar aumentos nesse parâmetro e calcule o número esperado de amostras recolhidas até um falso alarme. Assuma que o valor nominal do parâmetro sob vigilância é igual a 10 pacotes por segundo.* (1.0)

   - **Control statistic**
     Y_N = number of packages transmitted during the N^{th} second, $N \in I\!N$
   - **Distributional assumption**
     Since we are counting events in fixed time intervals, it is reasonable to assume that:
     $Y_N \sim \text{Poisson}(\lambda_0)$, IN CONTROL, where $\lambda_0 = 10$;
     $Y_N \sim \text{Poisson}(\lambda = \lambda_0 + \delta)$, OUT OF CONTROL, where δ ($\delta > 0$) represents the magnitude of the upward shift in λ.
   - **Suitable chart**
     An UPPER ONE-SIDED u chart should be adopted because we intend to detect increases in λ.
   - **Control limits of the $3\,sigma$ upper one-sided u chart**

$$\begin{aligned} LCL &= 0 \text{ (because we are dealing with an upper one-sided chart)} \\ UCL &= \lambda_0 + \gamma\sqrt{\lambda_0} \\ &= 10 + 3\sqrt{10} \\ &= 19.486833 \end{aligned}$$

   - **Probability of triggering a false alarm**

$$\begin{aligned} \xi(0) &= P(Y_N \notin [LCL, UCL] \mid \delta = 0) \\ &\overset{Y_N \geq 0,\, LCL=0}{=} P(Y_N > UCL \mid \delta = 0) \\ &= 1 - F_{Poisson(\lambda=\lambda_0)}(UCL) \\ &= 1 - F_{Poisson(10)}(19.486833) \\ &= 1 - F_{Poisson(10)}(19) \\ &\overset{table}{=} 1 - 0.9965 \\ &= 0.0035 \end{aligned}$$

   - **In-control RL distribution and ARL**
     We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a false alarm given δ, $RL(0)$, has the following distribution and expected value:

$$\begin{aligned} RL(0) &\sim \text{Geometric}\,(\xi(0))\,; \\ ARL(0) &= \frac{1}{\xi(0)} \end{aligned}$$

$$= \frac{1}{0.0035} \simeq 285.714286.$$

(b) *Redefina o limite superior de controlo de modo a lidarmos com uma carta que assinale um aumento de 30 unidades no número esperado de pacotes de informação transmitidos por minuto com probabilidade superior a 0.10.* (1.0)

- **Shift**
  Since there was an increase of 30 packages per minute, we have now $\delta = 0.5$ (packages per second).
- **Redefining the upper control limit**
  $UCL = \lambda_0 + \gamma\sqrt{\lambda_0}$, where
  $$\begin{aligned} \gamma > 0 \quad : \quad & \xi(0.5) > 0.10 \\ & 1 - F_{Poisson(\lambda=\lambda_0+0.5)}(UCL) > 0.10 \\ = \quad & 1 - F_{Poisson(10.5)}(UCL) > 0.10 \\ = \quad & F_{Poisson(10.5)}(\lambda_0 + \gamma\sqrt{\lambda_0}) < 0.90. \end{aligned}$$
  By checking the tables, we get
  $$\begin{aligned} \gamma > 0 \quad : \quad & \lambda_0 + \gamma\sqrt{\lambda_0} < 15 \\ & \gamma < \frac{15-10}{\sqrt{10}} \\ & \gamma < 1.581139. \end{aligned}$$
  E.g. $UCL = 14$.
- **Obs.**
  To ensure that $\xi(0.5) > 0.10$, we end up dealing with:
  $$\begin{aligned} \xi(0) &= 1 - F_{Poisson(10)}(14) \\ &\overset{table}{=} 1 - 0.9165 \\ &= 0.0835; \\ ARL(0) &= \frac{1}{\xi(0)} \\ &= \frac{1}{0.0835} \\ &\simeq 11.9760479, \end{aligned}$$
  which is an unreasonably low in-control ARL.

5. *A transmissão de um fax pode ser efectuada a uma velocidade (em* b/s*) que depende da qualidade da ligação entre a máquina emissora de fax e a receptora do mesmo. Dada a natureza dos dados e alguns estudos prévios, suspeita-se que a velocidade de transmissão possua distribuição sob controlo normal*$(9600, 1000^2)$.

(a) *Os dados seguintes reportam-se à velocidade média de grupos de 9 transmissões de fax: 9635.8, 9743.9, 9361.7, 9912.6, 9250.7, 8518.9, 9421.7, 9492.2, 9306.28, 9564.44. Qual a sua opinião acerca do estado de controlo do processo de transmissão de faxes?* (1.0)

- **Quality characteristic**
  X = transmission speed
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistic**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
- **Distribution**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 9600$, $\sigma_0^2 = 1000^2$ and $n = 9$
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or an increase!) in μ.
- **Control limits of the 3 sigma standard $\bar{X}$−chart**
  $$\begin{aligned} LCL_\mu &= \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\ &= 9600 - 3 \times \frac{1000}{\sqrt{9}} \\ &= 8600 \\ UCL_\mu &= \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\ &= 9600 + 3 \times \frac{1000}{\sqrt{9}} \\ &= 10600 \end{aligned}$$
- **State of the process**
  The ten average speeds are 9635.8, 9743.9, 9361.7, 9912.6, 9250.7, 8518.9, 9421.7, 9492.2, 9306.28, 9564.44. Thus, $\bar{x}_6 = 8518.9 < LCL_\mu$ and the process is deemed out-of-control.

(b) *Calcule a probabilidade de emissão de um sinal válido quando ocorre um shift de $\mu_0 = 9600b/s$ para $\mu = 9000b/s$. Determine e comente o valor da mediana do número de amostras recolhidas até à emissão desse sinal.* (1.5)

- **Shift**
  A shift from $\mu_0 = 9600$ to $\mu = 9000$ corresponds to
  $$\begin{aligned} \delta &= \frac{\mu - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} \\ &= \frac{9000 - 9600}{\frac{1000}{\sqrt{9}}} \\ &= -1.8. \end{aligned}$$
- **Probability of triggering a signal**
  Taking into account the distribution of the control statistic, the chart for μ triggers a signal with probability equal to
  $$\begin{aligned} \xi_\mu(\delta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta\right) \\ &= 1 - \left\{\Phi\left[\frac{UCL_\mu - (\mu_0 + \delta\sigma_0/\sqrt{n})}{\frac{\sigma_0}{\sqrt{n}}}\right] - \Phi\left[\frac{LCL_\mu - (\mu_0 + \delta\sigma_0/\sqrt{n})}{\frac{\sigma_0}{\sqrt{n}}}\right]\right\} \\ &= 1 - [\Phi(\gamma_\mu - \delta) - \Phi(-\gamma_\mu - \delta)] \\ &= 1 - [\Phi(3 - \delta) - \Phi(-3 - \delta)] \\ &= 1 - \{\Phi[3 - (-1.8)] - \Phi[-3 - (-1.8)]\} \\ &= 1 - [\Phi(4.8) - \Phi(-1.2)] \\ &\overset{table}{\simeq} 1 - [0.999999 - (1 - 0.8849)] \end{aligned}$$

$$= \quad 0.115101,$$

when $\delta = -1.8$.

- **Run length**
  We are dealing with a Shewhart chart, thus, the number of samples collected until the chart triggers a signal given δ, $RL_\mu(\delta)$, has the following distribution:
  $$RL_\mu(\delta) \sim \text{Geometric}\left(\xi_\mu(\delta)\right).$$

- **Median of $RL_\mu(\delta)$**
  $$\begin{aligned}
  F^{-1}_{RL_\mu(\delta)}(p) &\stackrel{Table\,9.2}{=} \inf\left\{m \in I\!N : F_{RL_\mu(\delta)}(m) \geq p\right\} \\
  &= 1 - [1-\xi_\mu(\delta)]^m \geq p \\
  &= [1-\xi_\mu(\delta)]^m \leq 1-p \\
  &= m \times \ln\left[1-\xi_\mu(\delta)\right] \leq \ln(1-p) \\
  &\stackrel{\ln(1-p)<0}{=} m \geq \frac{\ln(1-p)}{\ln\left[1-\xi_\mu(\delta)\right]} \\
  &\stackrel{p=0.5}{=} m \geq \frac{\ln(1-0.5)}{\ln(1-0.115101)} \\
  &= m \geq 5.668443 \\
  &= 6
  \end{aligned}$$

- **Comment**
  When the expected value of the transmission speed decreases from its target value $\mu_0 = 9600$ to $\mu = 9000$, the probability of triggering a valid signal within the first 6 samples is at least 50%, meaning a rather quick detection.

6. *Pretende conceber-se um esquema CUSUM unilateral superior para a detecção de aumentos no número esperado de contentores contaminados em amostras sucessivas de 100 unidades do valor nominal 2 para 4.2, tendo-se recolhido os seguintes dados: 2, 1, 2, 6, 3, 4, 4, 4, 6, 4.*

   (a) *Após ter obtido (e arredondado convenientemente) o valor de referência óptimo para o esquema acima referido, averigue se alguma das três primeiras observações foi responsável por um sinal por parte do esquema CUSUM unilateral com limite superior igual a 5.* (1.5)

   - **Reference value, k, of the upper one-sided CUSUM chart for binomial data**
     It should be the closest integer to
     $$\begin{aligned}
     n \times \frac{\ln\left[\frac{1-p_0}{1-p_1}\right]}{\ln\left[\frac{(1-p_0)\times p_1}{(1-p_1)\times p_0}\right]} &= 100 \times \frac{\ln\left[\frac{1-0.02}{1-0.042}\right]}{\ln\left[\frac{(1-0.02)\times 0.042}{(1-0.042)\times 0.02}\right]} \\
     &= 2.96934
     \end{aligned}$$
     (see result (10.14)), that is, $k = 3$.

   - **Control statistic**
     $$Z_N = \begin{cases} 0, & N = 0 \\ \max\{0, Z_{N-1} + (Y_N - k)\}, & N \in I\!N, \end{cases}$$
     (see Example 10.9 and (10.4)).

   - **First three observed values of control statistic**
     $$\begin{aligned}
     (z_0 &= 0) \\
     z_1 &= \max\{0, z_{1-1} + (y_1 - k)\} \\
     &= \max\{0, 0 + (2-3)\} \\
     &= 0 \\
     z_2 &= \max\{0, z_{2-1} + (y_2 - k)\} \\
     &= \max\{0, 0 + (1-3)\} \\
     &= 0 \\
     z_3 &= \max\{0, z_{3-1} + (y_3 - k)\} \\
     &= \max\{0, 0 + (2-3)\} \\
     &= 0.
     \end{aligned}$$
     Therefore none of the first three observed values of control statistic were responsible for a signal.

   (b) *Calcule as duas primeiras entradas da 1a. linha da matriz de probabilidades de transição associada ao esquema na ausência de causas assinaláveis.* (1.0)

   - **Requested entries**
     The 1st. and 2nd. entries of the 1st. row of the in control transition probability matrix, $\tilde{P}(0)$, follows from Example 10.9, namely result (10.5), and are equal to
     $$\begin{aligned}
     F_{Binomial(n,p_0)}(k) &= F_{Binomial(100,0.02)}(3) \\
     &\stackrel{Mathematica}{=} 0.858962 \\
     &\simeq \Phi\left(\frac{3 + 0.5 - 100 \times 0.02}{\sqrt{100 \times 0.02 \times (1-0.02)}}\right) \\
     &\simeq \Phi(1.07) \\
     &\stackrel{table}{=} 0.8577 \\
     P_{Binomial(n,p_0)}(k+1) &= P_{Binomial(100,0.02)}(3+1) \\
     &= \binom{100}{4} 0.02^4 (1-0.02)^{100-4} \\
     &\stackrel{Mathematica}{=} 0.090208,
     \end{aligned}$$
     respectively.

7. *O número de furtos notificados anualmente nos EUA por 100000 habitantes antes de 1983 é razoavelmente descrito por uma v.a. com distribuição normal$(1420, 150^2)$. Com o objectivo de controlar o valor esperado (μ) e a variância (σ^2) de tal v.a. considere um esquema conjunto que faça uso de duas cartas* Shewhart *— uma carta padrão para μ e outra unilateral superior para σ^2 — cujos limites de controlo são tais que:*

   - *a probabilidade de emissão de sinal válido quando $(\delta, \theta) = (0, 1.8)$ pela carta para μ é igual a 0.1;*
   - *o número esperado de amostras recolhidas até à emissão de falso alarme por parte da carta para σ^2 é de 200 amostras.*

   (a) *Na tabela seguinte foram registadas as médias e as variâncias de amostras constituídas por 3 grupos de 100000 habitantes, resultados estes correspondentes ao período de 1983 a 1992:*

N	1	2	3	4	5	6	7	8	9
x_N	1653.4	1557.0	1711.8	1581.8	1423.0	1773.6	1610.3	1741.2	1716.7
s^2_N	9621.6	1651.2	7065.9	3907.2	14087.4	19136.9	63357.9	24774.6	4079.92

Após ter averiguado se as observações apontam para a alteração de algum dos parâmetros, determine a probabilidade de ser emitido um sinal válido pelo esquema conjunto somente após a recolha de 50 amostras quando $(\delta, \theta) = (0.1, 1.1)$? (2.0)

Nota: Na impossibilidade de obter valor exacto obtenha um intervalo de valores para esta probabilidade e para as que se seguirem.

- **Quality characteristic**
  X = number of robberies per 100000 inhabitants
  $X \sim \text{Normal}(\mu, \sigma^2)$
- **Control statistics**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  S^2_N = variance of the N^{th} random sample of size n, $N \in I\!N$
- **Distributions**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0, \frac{\sigma^2}{n} = \frac{\sigma_0^2}{n}\right)$, IN CONTROL, where $\mu_0 = 1420$, $\sigma_0^2 = 150^2$ and $n = 3$.
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, OUT OF CONTROL, where δ ($\delta \neq 0$) represents the magnitude of the shift (a decrease or increase!) in μ and θ ($\theta > 1$) represents a shift (an increase!) in the standard deviation σ
  $\frac{(n-1)S^2_N}{\sigma_0^2} \sim \chi^2_{(n-1)}$, IN CONTROL
  $\frac{(n-1)S^2_N}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$, OUT OF CONTROL
- **Control limits of the standard $\bar{X}-$ chart and the upper one-sided S^2- chart**
  $LCL_\mu = \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}}$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$
- **Probability of triggering signals**
  Taking into account the distribution of the control statistics, the individual chart for μ and σ trigger signals with probabilities equal to:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta, \theta\right) \\
&= 1 - \left[\Phi\left(\frac{UCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right) - \Phi\left(\frac{LCL_\mu - \mu}{\frac{\sigma}{\sqrt{n}}}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \ \delta \in I\!R,\ \theta \geq 1; \\
\xi_\sigma(\theta) &= P\left(S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \theta\right) \\
&= 1 - F_{\chi^2_{(n-1)}}\left[\frac{(n-1)\,UCL_\sigma}{\sigma^2}\right] \\
&= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right), \ \theta \geq 1.
\end{aligned}$$

Obtaining γ_μ, γ_σ and the control limits

$$\begin{aligned}
\gamma_\mu \ &: \ \xi_\mu(0, 1.8) = 0.1 \\
& 1 - \left[\Phi\left(\frac{\gamma_\mu}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu}{\theta}\right)\right] = 0.1 \\
& 2 - 2 \times \Phi\left(\frac{\gamma_\mu}{1.8}\right) = 0.1 \\
& \gamma_\mu = 1.8 \times \Phi^{-1}(0.95) \\
& \gamma_\mu \overset{table}{=} 1.8 \times 1.6449 \\
& \gamma_\mu = 2.96082 \\
\gamma_\sigma \ &: \ \frac{1}{\xi_\sigma(1)} = ARL_\sigma(1) \\
& 1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma) = \frac{1}{ARL_\sigma(1)} \\
& \gamma_\sigma = F_{\chi^2_{(n-1)}}\left[1 - \frac{1}{ARL_\sigma(1)}\right] \\
& \gamma_\sigma = F_{\chi^2_{(3-1)}}\left(1 - \frac{1}{200}\right) \\
& \gamma_\sigma = F_{\chi^2_{(2)}}(0.995) \\
& \gamma_\sigma \overset{table}{=} 10.60 \\
LCL_\mu &= \mu_0 - \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\
&= 1420 - 2.96082 \times \frac{150}{\sqrt{3}} \\
&\simeq 1163.585466 \\
UCL_\mu &= \mu_0 + \gamma_\mu \frac{\sigma_0}{\sqrt{n}} \\
&= 1420 + 2.96082 \times \frac{150}{\sqrt{3}} \\
&\simeq 1676.414534 \\
LCL_\sigma &= 0 \\
UCL_\sigma &= \frac{\sigma_0^2}{n-1} \times \gamma_\sigma \\
&= \frac{150^2}{3-1} \times 10.60 \\
&\simeq 119250.
\end{aligned}$$

- **State of the process**
  The fact that $\bar{x}_i > UCL_\mu$, for $i = 3, 6, 8, 9$, suggests that the process is out-of-control.
- **Probability of a signal by the individual charts for μ and σ**
  The individual charts for μ and σ trigger a signal with probabilities:

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{2.96082 - 0.1}{1.1}\right) - \Phi\left(\frac{-2.96082 - 0.1}{1.1}\right)\right] \\
&\simeq 1 - [\Phi(2.60) - \Phi(-2.78)] \\
&= 1 - \{\Phi(2.60) - [1 - \Phi(2.78)]\} \\
&= 1 - [0.9953 - (1 - 0.9973)] \\
&= 0.0074;
\end{aligned}$$

$$\begin{aligned}
\xi_\sigma(\theta) &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right) \\
&= 1 - F_{\chi^2_{(3-1)}}\left(\frac{10.60}{1.1^2}\right) \\
&\simeq 1 - F_{\chi^2_{(2)}}(8.760) \\
&\in (0.01; 0.025)
\end{aligned}$$

because $F^{-1}_{\chi^2_{(2)}}(0.975) = 7.378 < 8.760 < 9.210 = F^{-1}_{\chi^2_{(2)}}(0.990)$.

- **Probability of a signal by joint scheme for μ and σ**

  The joint scheme triggers a signal if either of the individual charts triggers an alarm; moreover, the control statistics of the individual charts are independent given (δ, θ). As a consequence, the joint scheme for μ and σ triggers a false alarm with probability equal to:

$$\begin{aligned}
\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \text{ or } S^2_N \notin [LCL_\sigma, UCL_\sigma] \mid \delta, \theta\right) \\
&= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta) \\
&\overset{(\delta,\theta)=(0.1,1.1)}{=} \xi_\mu(0.1, 1.1) + \xi_\sigma(1.1) \times [1 - \xi_\mu(0.1, 1.1)] \\
&\in (0.0074 + 0.01 \times (1 - 0.0074); 0.0074 + 0.025 \times (1 - 0.0074)) \\
&\simeq (0.017326; 0.032215).
\end{aligned}$$

- **Run length**

  The run length of this joint scheme for μ and σ, $RL_{\mu,\sigma}(\delta,\theta)$, has the following distribution and survival function:

$$\begin{aligned}
RL_{\mu,\sigma}(\delta,\theta) &\sim \text{Geometric}(\xi_{\mu,\sigma}(\delta,\theta)); \\
P[RL_{\mu,\sigma}(\delta,\theta) > m] &= [1 - \xi_{\mu,\sigma}(\delta,\theta)]^m, \ m \in \mathbb{N}.
\end{aligned}$$

- **Requested probability**

$$\begin{aligned}
P[RL_{\mu,\sigma}(0.1, 1.1) > 50] &= [1 - \xi_{\mu,\sigma}(0.1, 1.1)]^{50} \\
&\in \left((1 - 0.032215)^{50}; (1 - 0.017326)^{50}\right) \\
&\simeq (0.194511; 0.417324)
\end{aligned}$$

(b) *Determine a probabilidade de ocorrência de sinal erróneo de Tipo III (IV) quando $\theta = 1.1$ ($\delta =$ 0.1).* (1.0)

- **Probability of a misleading signal of type III**

$$\begin{aligned}
PMS_{III}(\theta) &\overset{Table\ 10.12}{=} \frac{1 - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]}{[F_{\chi^2_{(n-1)}}(\gamma_\sigma/\theta^2)]^{-1} - [\Phi(\gamma_\mu/\theta) - \Phi(-\gamma_\mu/\theta)]} \\
&\overset{\theta=1.2}{=} \frac{1 - [\Phi(2.96082/1.1) - \Phi(-2.96082/1.1)]}{[F_{\chi^2_{(3-1)}}(10.60/1.1^2)]^{-1} - [\Phi(2.96082/1.1) - \Phi(-2.96082/1.1)]} \\
&\simeq \frac{2 \times [1 - \Phi(2.69)]}{[F_{\chi^2_{(2)}}(8.760)]^{-1} - [2 \times \Phi(2.69) - 1]} \\
&\in \left(\frac{2 \times (1 - 0.9964)}{0.975^{-1} - (2 \times 0.9964 - 1)}; \frac{2 \times (1 - 0.9964)}{0.990^{-1} - (2 \times 0.9964 - 1)}\right) \\
&\simeq (0.219238; 0.416161)
\end{aligned}$$

- **Probability of a misleading signal of type IV**

$$PMS_{IV}(\delta) \overset{Table\ 10.12}{=} \frac{1 - F_{\chi^2_{(n-1)}}(\gamma_\sigma)}{[\Phi(\gamma_\mu - \delta) - \Phi(-\gamma_\mu - \delta)]^{-1} - F_{\chi^2_{(n-1)}}(\gamma_\sigma)}$$

$$\begin{aligned}
&\overset{\delta=0.5}{=} \frac{1 - F_{\chi^2_{(3-1)}}(10.60)}{[\Phi(2.96082 - 0.1) - \Phi(-2.96082 - 0.1)]^{-1} - F_{\chi^2_{(3-1)}}(10.60)} \\
&\simeq \frac{1 - F_{\chi^2_{(2)}}(10.60)}{[\Phi(2.86) - \Phi(-3.06)]^{-1} - F_{\chi^2_{(2)}}(10.60)} \\
&\overset{table}{=} \frac{1 - 0.995}{[0.9979 - (1 - 0.998893)]^{-1} - 0.995} \\
&\simeq 0.608471.
\end{aligned}$$

- **Obs.**

  The PMS of Type IV is unusually high!