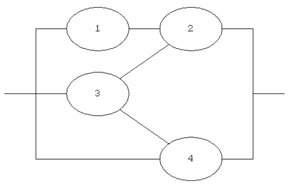
Fiabilidade e Controlo de Qualidade — LMAC

1o. Teste — 2o. Semestre – 2002/03

Duração: 1 hora e 45 minutos — 10/05/03 – 11 horas

1. *Considere que o sistema eléctrico no esquema abaixo é constituído por componentes independentes com fiabilidades iguais a p_i, $i = 1, 2, 3, 4$.*



(a) *Determine a função de estrutura e a fiabilidade do sistema.* (3.0)

- **Reliabilities of the components**
  p_i, $i = 1, \ldots, 4$
- **Minimal path sets**
  $$\begin{aligned} \mathcal{P}_1 &= \{1,2\} \\ \mathcal{P}_2 &= \{2,3\} \\ \mathcal{P}_3 &= \{4\} \\ p^* &= 3 \text{ minimal path sets} \end{aligned}$$
- **Structure function**
  By considering $X_i \sim \text{Bernoulli}(p_i)$, $i = 1, \ldots, 4$, and applying result (1.13), we can conclude that the structure function of this coherent system equals:
  $$\begin{aligned} \phi(\underline{X}) &= 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i \in \mathcal{P}_j} X_i\right) \\ &= 1 - (1 - X_1X_2)(1 - X_2X_3)(1 - X_4) \end{aligned}$$
- **Reliability**
  Taking into account that $X_i \stackrel{indep}{\sim} \text{Bernoulli}(p_i)$ and $X_i^k \sim X_i$, $k \in I\!N$, we obtain, for $\underline{p} = (p_1, \ldots, p_4)$:
  $$\begin{aligned} r(\underline{p}) &= E\left[\phi(\underline{X})\right] \\ &= E\left[1 - (1 - X_1X_2)(1 - X_2X_3)(1 - X_4)\right] \\ &= 1 - E\left[(1 - X_1X_2)(1 - X_2X_3)\right] \times E(1 - X_4) \\ &= 1 - E\left(1 - X_1X_2 - X_2X_3 + X_1X_2^2X_3\right) \times E(1 - X_4) \\ &= 1 - E\left(1 - X_1X_2 - X_2X_3 + X_1X_2X_3\right) \times E(1 - X_4) \\ &= 1 - (1 - p_1p_2 - p_2p_3 + p_1p_2p_3) \times (1 - p_4). \end{aligned}$$

(b) *Calcule a importância da fiabilidade da componente 3 e concretize-a para $p_i = p$, $i = 1, 2, 3, 4$.* (2.0)

- **Importance of the reliability of component 3**
  $$\begin{aligned} I_r(3) &\stackrel{(1.29)}{=} \frac{\partial r(\underline{p})}{\partial p_3} \\ &= \frac{\partial\left[1 - (1 - p_1p_2 - p_2p_3 + p_1p_2p_3) \times (1 - p_4)\right]}{\partial p_3} \\ &= -(-p_2 + p_1p_2) \times (1 - p_4) \\ &\stackrel{p_i=p}{=} (p - p^2) \times (1 - p) \\ &= p \times (1 - p)^2. \end{aligned}$$
- **Obs.**
  Let us remind the reader that $I_r(3) \stackrel{(1.28)}{=} r(1_3, \underline{p}) - r(0_3, \underline{p})$, where $(1_3, \underline{p}) = (p_1, p_2, 1, p_4)$, $(0_3, \underline{p}) = (p_1, p_2, 0, p_4)$ and $r(\underline{p}) = 1 - (1 - p_1p_2 - p_2p_3 + p_1p_2p_3) \times (1 - p_4)$. Thus,
  $$\begin{aligned} I_r(3) &= \left[1 - (1 - p_1p_2 - p_2 + p_1p_2) \times (1 - p_4)\right] - \left[1 - (1 - p_1p_2) \times (1 - p_4)\right] \\ &= \left[(1 - p_1p_2) - (1 - p_2)\right] \times (1 - p_4) \\ &= (p_2 - p_1p_2) \times (1 - p_4) \\ &\stackrel{p_i=p}{=} (p - p^2) \times (1 - p) \\ &= p \times (1 - p)^2. \end{aligned}$$

2. *Admita que uma aeronave com 4 motores só será capaz de voar se possuir pelo menos 3 dos motores a funcionar.*

(a) *Obtenha um limite inferior e um superior para a fiabilidade da aeronave, assumindo que os 4 motores estão associados (positivamente) e a fiabilidade de cada motor é de 97.5%.* (3.0)

- **System/components**
  $p_i = p = 0.975$, $i = 1, \ldots, 4$
  Since the $3 - \text{out} - \text{of} - 4$ system is coherent and its components operate in a positively associated fashion, we can apply Theorem 1.70, namely result (1.42), after having identified the minimal path sets and minimal cut sets of this system.
- **Minimal path sets**
  $$\begin{aligned} \mathcal{P}_1 &= \{1,2,3\} \\ \mathcal{P}_2 &= \{1,2,4\} \\ \mathcal{P}_3 &= \{1,3,4\} \\ \mathcal{P}_4 &= \{2,3,4\} \\ p^* &= 4 \text{ minimal path sets} \end{aligned}$$
- **Minimal cut sets**
  $$\begin{aligned} \mathcal{K}_1 &= \{1,2\} \\ \mathcal{K}_2 &= \{1,3\} \\ \mathcal{K}_3 &= \{1,4\} \\ \mathcal{K}_4 &= \{2,3\} \\ \mathcal{K}_5 &= \{2,4\} \\ \mathcal{K}_6 &= \{3,4\} \end{aligned}$$

$$q \;=\; 6 \text{ minimal cut sets}$$

- **Lower bound for the reliability** $r(\underline{p})$

$$\begin{aligned}
r(\underline{p}) &\stackrel{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i\in\mathcal{P}_j} p_i \\
&\stackrel{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j} \\
&\stackrel{\#\mathcal{P}_j=3,\forall j}{=} p^3 \\
&\stackrel{p=0.975}{=} 0.975^3 \\
&\simeq 0.926859.
\end{aligned}$$

- **Upper bound for the reliability**

$$\begin{aligned}
r(\underline{p}) &\stackrel{(1.42)}{\leq} \min_{j=1,\ldots,q} \left[1 - \prod_{i\in\mathcal{K}_j}(1-p_i)\right] \\
&\stackrel{p_i=p}{=} \min_{j=1,\ldots,q} \left[1-(1-p)^{\#\mathcal{K}_j}\right] \\
&\stackrel{\#\mathcal{K}_j=2,\forall j}{=} 1-(1-p)^2 \\
&\stackrel{p=0.975}{=} 1-(1-0.975)^2 \\
&= 0.999375.
\end{aligned}$$

- **Obs.**
Since the components are positively associated we cannot apply Theorem 1.68 (for independent components!) which sometimes provides stricter bounds than the ones we just obtained by using Theorem 1.70.

(b) *Admita agora que os tempos até falha dos 4 motores (em milhares de horas) são i.i.d. com distribuição Exponencial e parâmetro de escala* $\lambda^{-1} = 2$ *e obtenha a função de fiabilidade da aeronave para um período de 1000 h.* (2.0)

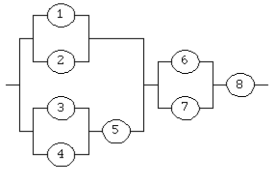
- **Individual durations of the components of the** $3-\text{out}-\text{of}-4$ **system**
$T_i \stackrel{i.i.d.}{\sim} \text{Exponential}(\lambda=0.5),\ i=1,\ldots,4$
$F_i(t) = F(t) = 1-e^{-0.5t},\ t\geq 0$

- **Duration of the** $3-\text{out}-\text{of}-4$ **system**
$T =$ duration of the $3-\text{out}-\text{of}-4$ system
$T \stackrel{(2.6)}{=} T_{(n-k+1)} \stackrel{n=4,k=3}{=} T_{(2)}$

- **Reliability function of** T

$$\begin{aligned}
R_T(t) &\stackrel{(2.8)}{=} F_{Binomial(n,F(t))}(n-k) \\
&= F_{Binomial(4,1-e^{-0.5t})}(4-3) \\
&= \sum_{m=0}^{1}\binom{4}{m}\left(1-e^{-0.5t}\right)^m\left[1-\left(1-e^{-0.5t}\right)\right]^{4-m} \\
&= e^{-2t}+4\left(1-e^{-0.5t}\right)e^{-1.5t} \\
&\stackrel{t=1}{\simeq} 0.486515
\end{aligned}$$

3. *O esquema seguinte reporta-se a uma estrutura constituída por componentes com durações i.i.d. e IHR.*



(a) *O que pode concluir acerca do comportamento monótono da sua função taxa de falha da duração de vida da estrutura? Justifique.* (2.0)

- **Individual durations**
$T_i \stackrel{i.i.d.}{\sim} IHR,\ i=1,\ldots,8$

- **Duration of the system**
Let:

$$\begin{aligned}
Z_1 &= \max\{\max\{T_1,T_2\},\min\{\max\{T_3,T_4\},T_5\}\}; \\
Z_2 &= \max\{T_6,T_7\}; \\
Z_3 &= T_8.
\end{aligned}$$

Then

$$T = \min\{Z_1,Z_2,Z_3\}.$$

- **Stochastic ageing of** T
Since $T_i \stackrel{i.i.d.}{\sim} IHR$, $i=1,\ldots,8$, we can add that $Z_3 = T_8 \in IHR$, and we can apply Proposition 3.23 and successively get:
  - $\max\{T_1,T_2\}, \max\{T_3,T_4\} \in IHR$ by result (3.14);
  - $\min\{\max\{T_3,T_4\},T_5\} \in IHR$ by result (3.11);
  - $Z_2 = \max\{T_6,T_7\} \in IHR$ by result (3.14);

  However, according to result (3.13), the maximum of two independent (but not identically distributed) IHR r.v. is not necessarily an IHR r.v., therefore we cannot infer the stochastic ageing of Z_1 or the one of $T=\min\{Z_1,Z_2,Z_3\}$.

(b) *Estabeleça um limite inferior para a função de fiabilidade da sub-estrutura constituída pelas componentes 6 e 7.* (3.0)

- **Common reliability function of components 6 and 7**
$R(t) = P(T_6>t) = P(T_7>t)$

- **Lower bound for** $R(t)$
Recall that
  - $T_i \stackrel{i.i.d.}{\sim} IHR,\ i=6,7,$

  and admit that
  - $E(T_i)=\mu,\ i=6,7.$

  Then we can apply Corollary 3.47 and state that

$$R(t) \geq \begin{cases} e^{-\frac{t}{\mu}}, & t\leq\mu \\ 0, & t>\mu. \end{cases}$$

- **Reliability function of components** $Z_2 = \max\{T_6, T_7\}$

$$\begin{aligned} R_{Z_2}(t) &= P[\max\{T_6,T_7\} > t] \\ &\stackrel{(2.5)}{=} 1-[1-R(t)]^2 \end{aligned}$$

- **Lower bound for** $R_{Z_2}(t)$
Capitalizing on the lower bound for $R(t)$, we can add that

$$\begin{aligned} [1-R(t)]^2 &\leq \begin{cases} \left(1-e^{-\frac{t}{\mu}}\right)^2, & t \leq \mu \\ (1-0)^2 = 1, & t > \mu; \end{cases} \\ 1-[1-R(t)]^2 &= R_{Z_2}(t) \\ &\geq \begin{cases} 1-\left(1-e^{-\frac{t}{\mu}}\right)^2 = e^{-\frac{t}{\mu}} \times \left(2-e^{-\frac{t}{\mu}}\right), & t \leq \mu \\ 1-1=0, & t > \mu. \end{cases} \end{aligned}$$

4. *O registo dos instantes de ocorrência de falhas (em horas) de 10 impressoras colocadas simultaneamente em teste conduziu às seguintes observações: 43, 55, 84, 172, 215, 419, 565, 586, 687, 711.*

(a) *Calcule uma estimativa da função taxa de falha da duração de uma impressora para* $t = 55\,h$. (1.5)

- **Ordered data**
$(43, 55, 84, 172, 215, 419, 565, 586, 687, 711)$

- **Nonparametric estimate of** $\lambda(t_{(i)})$
We are dealing with complete ungrouped failure data, thus, we can use the nonparametric estimate of $\lambda(t_{(i)})$ in Table 5.1,

$$\lambda(t_{(i)}) = \frac{1}{n-i+0.625} \times \frac{1}{t_{(i+1)}-t_{(i)}},$$

and get

$$\begin{aligned} \lambda(55) &= \lambda(t_{(2)}) \\ &= \frac{1}{10-2+0.625} \times \frac{1}{84-55} \\ &\simeq 0.003998. \end{aligned}$$

Admita agora que as 10 observações dizem respeito a 10 impressoras de entre um grupo de 30 colocadas simultaneamente em teste e que as durações das impressoras são v.a. i.i.d. com distribuição Exponencial.

(b) *Reavalie a estimativa que obteve em (a) e determine um intervalo de confiança a* 95% *para a fiabilidade de uma impressora para um período de* $200\,h$. (3.5)

- **Distribution assumption**
$T_i \stackrel{i.i.d.}{\sim} T \sim \text{Exponential}(\lambda)$, $i = 1, \ldots, n$.

- **Unknown parameter**
$\lambda_T(t) = \lambda$, $t > 0$.

- **Life test**
Since the end of the test was determined by the $r = 10^{th}$ failure and nothing in this exercise suggests that the $n = 20$ rubber gaskets were replaced during the life test, we are dealing with a
  - Type II/item censored testing without replacement.

- **Censored data**
$n = 30$
$r = 10$
$(t_{(1)}, \ldots, t_{(r)}) = (43, 55, 84, 172, 215, 419, 565, 586, 687, 711)$

- **Cumulative total time in test**
According to Definition 5.17, the cumulative total time in test is given by:

$$\begin{aligned} \tilde{t} &= \sum_{i=1}^{r} t_{(i)} + (n-r) \times t_{(r)} \\ &= (43 + \ldots + 711) + (30-10) \times 711 \\ &= 17757. \end{aligned}$$

- **Estimates of** λ
According to Table 5.11, the ML of λ is equal to

$$\begin{aligned} \hat{\lambda} &= \frac{r}{\tilde{t}} \\ &= \frac{10}{17757} \\ &\simeq 0.000563158. \end{aligned}$$

- **Another unknown parameter**
$R_T(t) = e^{-\lambda t}$, $t = 200$.

- **Confidence interval for** λ
According to Table 5.16 of the lecture notes,

$$\begin{aligned} CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_l; \lambda_U] \\ &= \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2 \times \tilde{t}}; \frac{F^{-1}_{\chi^2_{(2r)}}(1-\alpha/2)}{2 \times \tilde{t}}\right] \\ CI_{95\%}(\lambda) &= \left[\frac{F^{-1}_{\chi^2_{(20)}}(0.025)}{2 \times 17757}; \frac{F^{-1}_{\chi^2_{(20)}}(0.975)}{2 \times 17757}\right] \\ &= \left[\frac{9.591}{2 \times 17757}; \frac{34.17}{2 \times 17757}\right] \\ &\simeq [0.000270063; 0.000962156]. \end{aligned}$$

- **Confidence interval for** $R_T(t)$
Since $R_T(t) = e^{-\lambda t}$ is a decreasing function of $\lambda > 0$, we can state that

$$\begin{aligned} CI_{95\%}\left(e^{-\lambda t}\right) &= \left[e^{-\lambda_U t}; e^{-\lambda_L t}\right] \\ &\stackrel{t=200}{\simeq} [0.824951; 0.947420]. \end{aligned}$$