Practical Session 4

Introduction

The main goal of this brief introduction is to provide some references on contemporary applications of geometric algebra. First of all: by geometric algebra I just mean Clifford (geometric) algebras. Both Clifford algebras and quaternions are addressed herein. One should bear in mind the following: quaternions constitute a division algebra which is isomorphic to the even subalgebra $\Cl^+_{3}$ of $\Cl_{3}$. So, when speaking of quaternions, we are actually speaking of Clifford algebras.

There is a main textbook that I recommend as a (mathematical) introduction:


This textbook is elementary as far as the first chapters are concerned: Chapters 1 – 9. There are some parts in these chapters that are more demanding but these parts are just remarks or confined to the final part of each chapter. Furthermore, it has a very important characteristic: it is written with rigor, but with careful pedagogical motivation. Also, a final reading of Chapter 14 (Definitions of the Clifford algebra) should be of general interest. Of course, Chapters 10 – 23 are more demanding and more appropriate for those of you with a (more) solid mathematical background or (eventually) pursuing graduate research.

There are also two textbooks that I recommend – although they address topics with (more) direct interest for physics:


For example: the last textbook addresses special and general relativity as well as quantum mechanics. It also has a very interesting chapter (Chapter 10) where such topics as projective, conformal and non-Euclidean geometries are addressed.

However, for engineering, the following textbooks are (probably) more appropriate:


**Definition.** Basically, a real algebra $\mathcal{A}$ is a linear space over the field $\mathbb{R}$ of the reals together with a bilinear function $(\mathcal{A}, \mathcal{A}) \to \mathcal{A}$ such that $(a, b) \mapsto a \otimes b$.
Here we have denoted the bilinear function by the symbol $\otimes$. The algebraic structure is closed: $a, b, a \otimes b \in A$. Bilinear just means linear with respect to both arguments. Of course we could consider a more general case in which the algebra $A$ was defined in any field $F$. A real algebra refers to the special case where $F = \mathbb{R}$. For example: a complex algebra corresponds to $F = \mathbb{C}$.

Clifford algebras are just algebras where the bilinear function is the Clifford (or geometric) product. We are mainly interested in Clifford algebras based on $\mathbb{R}^2$, on $\mathbb{R}^3$ and on $\mathbb{R}^{1,3}$. In this first session we are just interested in $\mathbb{R}^2$ and in $\mathbb{R}^{1,1}$.

Let us begin by clarifying what is $\mathbb{R}^{p,q}$: by $\mathbb{R}^{p,q}$ we mean a quadratic space $\mathbb{R}^n$, with $p + q = n$, where a quadratic form $Q(x) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$ is defined. We abbreviate this by writing $p \langle 1 \rangle \perp q \langle -1 \rangle$. The integer $s = p - q$ is called the signature of the quadratic space $\mathbb{R}^{p,q}$.

A Euclidean quadratic space corresponds to the special case where $n = p$, i.e., where $q = 0$:

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mapsto Q(x) = x_1^2 + x_2^2 + \ldots + x_n^2.$$  

This Euclidean quadratic space can be simply denoted by $\langle 1, \ldots, 1 \rangle$.

In this particular case we will make the identification $\mathbb{R}^{n,0} \equiv \mathbb{R}^n$. Accordingly, for $\mathbb{R}^2$, we get

$$x = (x_1, x_2) \in \mathbb{R}^2 \mapsto Q(x) = x_1^2 + x_2^2.$$  

For $\mathbb{R}^3$, on the other hand, we get

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto Q(x) = x_1^2 + x_2^2 + x_3^2.$$  

However, for $\mathbb{R}^{1,1}$ (the hyperbolic plane), we make

$$x = (x_0, x_1) \in \mathbb{R}^2 \mapsto Q(x) = x_0^2 - x_1^2.$$  

And, for $\mathbb{R}^{1,3}$ (the full Minkowski spacetime), we make

$$x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mapsto Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$  

We could also define $\mathbb{R}^{3,1}$ by making

$$x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mapsto Q(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$  

The Clifford algebra based on $\mathbb{R}^{p,q}$ is denote by $C\ell_{p,q}$:

$$C\ell_{p,q}(\mathbb{R}^n) = C\ell(\mathbb{R}^{p,q}).$$  

In the Euclidean case $\mathbb{R}^{n,0}$, however, we just write $C\ell_n = C\ell_{n,0}$.

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\[
C\ell_\ast(\mathbb{R}^n) = C\ell(\mathbb{R}^{n,0})
\]

Henceforth the symbol \(\oplus\) will be used throughout. It means \textit{direct sum}. Let us denote by \(U\) and \(W\) two linear spaces that are subspaces of \(V\). Let us call \textit{multivector} a general element of \(V\). Then \(U + W\) represents the collection of multivectors in \(V\) that can be written as a sum of two multivectors, one in \(U\) and one in \(W\). It can be easily shown that \(U + W\) is a subspace of \(V\).

\[\text{Definition.}\] Let \(U\) and \(W\) be two subspaces of \(V\) such that \(V = U + W\) and the only multivector common to both \(U\) and \(W\) is the zero multivector. Then we say that \(V\) is the \textit{direct sum} of \(U\) and \(W\) and write \(V = U \oplus W\).

It can be easily proven that

\[V = U \oplus W \Rightarrow \dim(V) = \dim(U) + \dim(W)\]

Let us consider two vectors \(a, b \in \mathbb{R}^{p,q}\). We define the Clifford (or geometric product) \(u = ab \in C\ell_{p,q}\) as

\[ab = a \cdot b + a \wedge b\]

By \(a \cdot b \in \mathbb{R}\) we represent the inner (or dot) product between the two vectors. By \(a \wedge b \in \bigwedge^2 \mathbb{R}^{p,q}\) we represent the exterior (or Grassmann) product between the two vectors. One should stress that \(\alpha = a \cdot b\) is a scalar, whereas \(B = a \wedge b\) is a bivector. And, since \(a \cdot b = b \cdot a\) and \(a \wedge b = -b \wedge a\), we also get

\[
\begin{align*}
ab &= a \cdot b + a \wedge b \\
ba &= a \cdot b - a \wedge b
\end{align*}
\]

so that

\[
\begin{align*}
ab &= ba \iff a \parallel b \iff a \wedge b = 0 \iff ab = a \cdot b, \\
ab &= -ba \iff a \perp b \iff a \cdot b = 0 \iff ab = a \wedge b.
\end{align*}
\]

In Euclidean space \(\mathbb{R}^2\) we define the Clifford algebra

\[
C\ell_2 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2
\]

We also have

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where the even part is not only a subspace but also a subalgebra. Actually this subalgebra is isomorphic to \( \mathbb{C} \) and we write 
\[ C^*_2 = \mathbb{C}. \]

Let \( \mathcal{B} = \{ \mathbf{e}_1, \mathbf{e}_2 \} \) be an orthonormal basis of \( \mathbb{R}^2 \), which means
\[
\begin{align*}
\mathbf{e}_1 & \perp \mathbf{e}_2 \quad \Leftrightarrow \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \\
|\mathbf{e}_1| &= |\mathbf{e}_2| = 1 \quad \Leftrightarrow \quad \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1
\end{align*}
\]
so that
\[
\begin{align*}
\mathbf{e}_1^2 &= \mathbf{e}_1 \mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1, \\
\mathbf{e}_2^2 &= \mathbf{e}_2 \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_2 + \mathbf{e}_2 \wedge \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1.
\end{align*}
\]

For a vector \( \mathbf{r} = x \mathbf{e}_1 + y \mathbf{e}_2 \in \mathbb{R}^2 \) we define the geometric product with itself as
\[
\mathbf{r}^2 = \mathbf{r} \cdot \mathbf{r} = (x \mathbf{e}_1 + y \mathbf{e}_2)(x \mathbf{e}_1 + y \mathbf{e}_2) = x^2 \mathbf{e}_1^2 + y^2 \mathbf{e}_2^2 + xy (\mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1) = x^2 + y^2 + xy \mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1.
\]
On the other hand,
\[
|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = (x \mathbf{e}_1 + y \mathbf{e}_2)(x \mathbf{e}_1 + y \mathbf{e}_2) = x^2 |\mathbf{e}_1|^2 + y^2 |\mathbf{e}_2|^2 + 2xy (\mathbf{e}_1 \cdot \mathbf{e}_2) = x^2 + y^2.
\]
Let us impose, as an axiom, that
\[
\boxed{\mathbf{r}^2 = |\mathbf{r}|^2} \quad \Rightarrow \quad \mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_2 \mathbf{e}_1 = 0 \quad \Rightarrow \quad \mathbf{e}_1 \mathbf{e}_2 = -\mathbf{e}_2 \mathbf{e}_1.
\]
Whence,
\[
\mathbf{r}^{-1} = \frac{\mathbf{r}}{|\mathbf{r}|^2} = \frac{x}{x^2 + y^2} \mathbf{e}_1 + \frac{y}{x^2 + y^2} \mathbf{e}_2 \in \mathbb{R}^2.
\]
If \( \mathbf{m} \) is the unit vector, such that \( \mathbf{r} = |\mathbf{r}| \mathbf{m} \), then
\[
\mathbf{r}^{-1} = \frac{\mathbf{r}}{|\mathbf{r}|^2} = \frac{|\mathbf{r}|}{|\mathbf{r}|^2} \mathbf{m} = \frac{1}{|\mathbf{r}|} \mathbf{m}.
\]
The Clifford (or geometric) product between vectors is associative but not (in general) commutative. In fact
\[
\mathbf{e}_{12} = \mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_2 \cdot \mathbf{e}_1 - \mathbf{e}_1 \cdot \mathbf{e}_2 = -\mathbf{e}_2 \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_2.
\]

It is easily proven that \( \mathbf{e}_{12} \) is the unit bivector:
\[
\mathbf{e}_{12}^2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 = (\mathbf{e}_1 \mathbf{e}_2)(\mathbf{e}_1 \mathbf{e}_2) = -\mathbf{e}_1 (\mathbf{e}_2 \mathbf{e}_2) \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_1 = -\mathbf{e}_1 \mathbf{e}_1 = -\mathbf{e}_1^2 = -1.
\]

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In general, we define
\[
\begin{bmatrix}
\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \in \mathbb{R}^2 \\
\mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 \in \mathbb{R}^2
\end{bmatrix} \quad \mapsto \quad \mathbf{a} \wedge \mathbf{b} = \begin{bmatrix}
a_1 \\
b_1 \\
a_2 \\
b_2
\end{bmatrix} \mathbf{e}_{12} = (a_1 b_2 - a_2 b_1) \mathbf{e}_{12} \in \wedge \mathbb{R}^2.
\]

Hence, in general, we can also write
\[
\beta = a_1 b_2 - a_2 b_1 \in \mathbb{R} \quad \mapsto \quad \mathbf{a} \wedge \mathbf{b} = \beta \mathbf{e}_{12} \in \wedge \mathbb{R}^2.
\]

As is well-known, if \( \theta = \alpha(\mathbf{a}, \mathbf{b}) \), then
\[
\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 = ||\mathbf{a}|| ||\mathbf{b}|| \cos(\theta) \in \mathbb{R},
\]
where
\[
||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2}, \quad ||\mathbf{b}|| = \sqrt{b_1^2 + b_2^2}.
\]

Accordingly,
\[
\begin{align*}
(\mathbf{a} \wedge \mathbf{b})^2 &= (\mathbf{a} - \mathbf{b})(\mathbf{a} \cdot \mathbf{b} - \mathbf{b}) \\
&= (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} + \mathbf{b}) - \mathbf{a} \mathbf{b} \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b}) \\
&= (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} + \mathbf{b}) - \mathbf{a}^2 \mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b}) \\
&= 2(\mathbf{a} \cdot \mathbf{b})^2 - \mathbf{a}^2 \mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= (\mathbf{a} \cdot \mathbf{b})^2 - \mathbf{a}^2 \mathbf{b}^2
\end{align*}
\]

Hence,
\[
(\mathbf{a} \wedge \mathbf{b})^2 = (\mathbf{a} \cdot \mathbf{b})^2 - \mathbf{a}^2 \mathbf{b}^2 = \mathbf{a}^2 \mathbf{b}^2 \cos^2(\theta) - \mathbf{a}^2 \mathbf{b}^2 = -\mathbf{a}^2 \mathbf{b}^2 \sin^2(\theta) \leq 0.
\]

In fact,
\[
(\mathbf{a} \wedge \mathbf{b})^2 = \beta^2 \mathbf{e}_{12}^2 = -\beta^2 \quad \mapsto \quad \beta = \pm ||\mathbf{a}|| ||\mathbf{b}|| \sin(\theta)
\]
where
\[
\text{Area}(\theta) = ||\mathbf{a}|| ||\mathbf{b}|| \sin(\theta) = |a_1 b_2 - a_2 b_1|.
\]

The Pythagorean theorem, then, can be written as
\[
\mathbf{a}^2 \mathbf{b}^2 = ||\mathbf{a} \wedge \mathbf{b}||^2 + (\mathbf{a} \cdot \mathbf{b})^2.
\]

The multiplication table of \( \mathbb{C}_2 \) can be written as follows.
Accordingly, we get
\[ a e_{12} = (a_1 e_1 + a_2 e_2) e_{12} = a_1 e_2 - a_2 e_1, \]
\[ e_{12} a = e_{12} (a_1 e_1 + a_2 e_2) = -a_1 e_2 + a_2 e_1. \]
\[
\begin{bmatrix}
    a e_{12} + e_{12} a = 0.
\end{bmatrix}
\]

A basis for \( \mathbb{C} \ell_2 \) can be written as follows.

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<tbody>
<tr>
<td>1</td>
<td>( \mathbb{R} )</td>
<td>vectors ( \dim (\mathbb{C} \ell_2) = 1 + 2 + 1 = 2^2 = 4 )</td>
</tr>
<tr>
<td>1</td>
<td>( \mathbb{R}^2 )</td>
<td>bivectors ( 1 \ 2 \ 1 )</td>
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A general multivector \( u \in \mathbb{C} \ell_2 \), then, can be written as

\[
\begin{bmatrix}
    a \in \mathbb{R}^2 \\
    e_{12} \in \bigwedge \mathbb{R}^2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
    u = \alpha + a + \beta e_{12}.
\end{bmatrix}
\]

Clearly, one has

\[
\begin{bmatrix}
    u_+ = \alpha + \beta e_{12} \in \mathbb{C} \ell_2^+, \\
    u_- = a \in \mathbb{C} \ell_2^-.
\end{bmatrix}
\]

In \( \mathbb{C} \ell_2 \) the following three involutions are defined.

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<td>grade involution</td>
<td>( \hat{u} = \alpha - a + \beta e_{12} )</td>
<td>reversion</td>
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A multivector \( u \in \mathbb{C} \ell_2 \) can be decomposed into a sum of pure grade terms

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\begin{align*}
\langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 &= \alpha + \alpha e_{12} \\
\langle u \rangle_0 &= \alpha \in \mathbb{R} \\
\langle u \rangle_1 &= \alpha \in \mathbb{R}^2 \\
\langle u \rangle_2 &= \beta e_{12} \in \mathbb{R}^2.
\end{align*}

The inverse of \( u \in C \ell_2 \) is
\[ u^{-1} = \frac{\overline{u}}{\|u\|^2} \in C \ell_2. \]

As
\[
\begin{align*}
\overline{u} u &= \left( \alpha + \alpha e_{12} \right) \left( \alpha - \alpha e_{12} \right) \\
&= \alpha^2 - \alpha^2 e_{12} + \alpha^2 e_{12} + \alpha^2 - \beta a e_{12} - \beta e_{12}^2 \\
&= \alpha^2 + \beta^2 - a^2 \\
\end{align*}
\]
we conclude that
\[ u^{-1} = \frac{\alpha - \alpha e_{12}}{\alpha^2 + \beta^2 - a^2}, \quad \alpha^2 + \beta^2 \neq a^2. \]

The norm of \( u \in C \ell_2 \) is \( |u| \) such that
\[
|u|^2 = \left( \overline{u} u \right)_0 \\
= \left( \left( \alpha + \alpha e_{12} \right) \left( \alpha + \alpha e_{12} \right) \right)_0 \\
= \left( \alpha^2 + \alpha^2 e_{12} + \alpha a + a^2 - \beta a e_{12} + \alpha a e_{12} + \beta e_{12} a - \beta e_{12}^2 \right)_0 \\
= \alpha^2 + \beta^2 + a^2 \geq 0.
\]

**First Problem**

Let \( p = \frac{1}{2} \left( 1 + e_i \right) \). Compute \( p^2 \) and \( p \overline{p} \). Is it possible to calculate \( p^{-1} \)? What is the value of \( |p| \) ?

**Second Problem**

Let \( q = e_i + e_{12} \). Compute \( q^2 \) and \( r = \exp(q) \).

**Third Problem**

Let \( a = e_i - 2e_2, \ b = e_i + e_2 \) and \( r = 5e_i - e_2 \). Compute \( \alpha \) and \( \beta \) in the decomposition \( r = \alpha a + \beta b \).

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Fourth Problem

Let \( \mathbf{a} = 8\mathbf{e}_1 - \mathbf{e}_2 \) and \( \mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 \). Compute \( \mathbf{a}_\parallel = (\mathbf{a} \cdot \mathbf{b}) \mathbf{b}^{-1} \) and \( \mathbf{a}_\perp = (\mathbf{a} \land \mathbf{b}) \mathbf{b}^{-1} \).

Fifth Problem

Let \( \mathbf{a} = 3\mathbf{e}_1 - \mathbf{e}_2 \) and \( \mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2 \). Reflect first \( \mathbf{r} = 4\mathbf{e}_1 - 3\mathbf{e}_2 \) across \( \mathbf{a} \) and then the result across \( \mathbf{b} \).

Sixth Problem

Let \( \mathbf{a} = \alpha \mathbf{m} \in \mathbb{R}^2 \), where \( \alpha = |\mathbf{a}| \). Compute \( \exp(\mathbf{a}) \).

Seventh Problem

Let \( \mathbf{u} = \theta \mathbf{e}_{12} \), where \( \theta \in \mathbb{R} \). Compute \( \exp(\mathbf{u}) \).

Eight Problem

Given two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^2 \), and a third vector \( \mathbf{c} \in \mathbb{R}^2 \), determine vector \( \mathbf{x} \in \mathbb{R}^2 \) so that \( \mathbf{x} \) is to \( \mathbf{c} \) as \( \mathbf{b} \) is to \( \mathbf{a} \). By this, we mean: find \( \mathbf{x} \), such that

\[
\mathbf{x} \mathbf{c}^{-1} = \mathbf{b} \mathbf{a}^{-1}.
\]

**Answer:** We just have to make a right-multiplication of this equation (in both sides) by \( \mathbf{c} \), to obtain

\[
\mathbf{x} = \mathbf{b} \mathbf{a}^{-1} \mathbf{c}.
\]

Now, to illustrate this, let us consider

\[
\begin{align*}
\mathbf{a} &= \mathbf{e}_1 \\
\mathbf{b} &= \mathbf{e}_1 + \mathbf{e}_2 \\
\mathbf{c} &= 2\mathbf{e}_2
\end{align*}
\]

\[
\begin{align*}
\mathbf{a}^{-1} &= \frac{\mathbf{a}}{|\mathbf{a}|^2} = \mathbf{e}_1 \\
\mathbf{b} \mathbf{a}^{-1} &= (\mathbf{e}_1 + \mathbf{e}_2) \mathbf{e}_1 = 1 - \mathbf{e}_{12}
\end{align*}
\]

\[
\mathbf{x} = (\mathbf{b} \mathbf{a}^{-1}) \mathbf{c} = (1 - \mathbf{e}_{12}) (2\mathbf{e}_2) = 2(\mathbf{e}_2 - \mathbf{e}_1).
\]

The attached figure (from MATLAB) illustrates the determination of vector \( \mathbf{x} \).

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\[ xc^{-1} = ba^{-1} \quad \Rightarrow \quad x = ba^{-1}c \]