Aspects of Physics Beyond the Standard Model in the Leptonic Sector

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« Então vamos, disse Fernando Pessoa,
Vamos, disse Ricardo Reis.
O Adamastor não se voltou para ver,
Parecia-lhe que desta vez ia ser capaz de dar o grande grito.
Aqui, onde o mar se acabou e a terra espera. »

A meu Pai,
que, contrariamente a quem leia isto,
nunca chegou a saber como terminava
O Ano da Morte de Ricardo Reis.

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Resumo

Após um grande progresso teórico e experimental, é agora sabido que os neutrinos são massivos. Contudo, não há explicação teórica para a sua massa quase nula e outras questões em aberto. O mecanismo Seesaw (Balancé) responde a algumas destas, trazendo, ao mesmo tempo, nova fenomenologia que pode explicar outros problemas da Física de Partículas, como a assimetria entre matéria e anti-matéria. Nesta tese desenvolve-se uma extensão mínima do Modelo Padrão, com três neutrinos de direita. A notação é fixa e uma nova parametrização é explorada. Esta parametrização permite controlar todos os devios de unitariedade através de uma única matriz 3×3, denominada X, que também relaciona a mistura dos neutrinos leves e pesados no contexto de seesaw tipo I. Esta parametrização é adequada para um tratamento geral e exato independente da escala do termo de massa dos neutrinos de direita. Os modelos com correções às massas a um loop controladas são classificados de acordo com as hierarquias de massa que os neutrinos pesados devem ter - casos A, B e C. Os casos B e C podem ter desvios de unitariedade consideráveis. Isto quer dizer que, se um neutrino quase estéril for descoberto num futuro próximo, é expectável que as hierarquias de massas dos neutrinos pesados sejam como as do caso B - pelo menos 2 neutrinos pesados quase degenerados ou como as do caso C - pelo menos 2 neutrinos pesados com massas na escala do eV ou do KeV.

Palavras-chave: Modelo Padrão, Neutrinos, Seesaw, Desvios de Unitariedade, One-Loop

Abstract

After a great theoretical and experimental progress, it is now known that neutrinos have mass. However, there is no theoretical explanation for their almost vanishing mass and other issues. The Seesaw mechanism answers some of these and creates new phenomenology that can help answer several other open problems in Particle Physics, like the matter-antimatter asymmetry. In this thesis, a minimal extension to the Standard Model with three positive chirality neutrinos is devised, under the Seesaw Type I framework. Notation is fixed and a novel parametrization is exploited. This parametrization enables to control all deviations from unitarity through a single 3×3 matrix, which is denoted by X, that also connects the mixing of the light and heavy neutrinos in the context of type I seesaw. This parametrization is adequate for a general and exact treatment, independent of the scale of the right handed neutrino mass term. The models with controlled one-loop mass corrections are classified according to the heavy neutrino mass hierarchies they must possess - cases A, B and C. Cases B and C can have sizable deviations from unitarity. This means that if an almost sterile neutrino is discovered in the near future, heavy neutrinos mass hierarchies might be like the ones of case B - at least two almost degenerate neutrinos, or like the ones of case C - at least two eV or KeV neutrinos.

Keywords: Standard Model, Neutrinos, Seesaw, Deviations from Unitarity, One-Loop

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Nomenclature

Abbreviations

LHC Large Hadron Collider

QFT Quantum Field Theory

SM Standard Model

WB Weak Basis

Subscripts

$\alpha, \beta, \gamma, \delta$ Flavour indices.

μ, ν, σ, ρ Lorentz indices.

a, b, c Gauge group Generators indices.

i, j, k, l Mass state indices.

u, v, w Color indices.

Superscripts

$\dagger$ Conjugate Transpose.

$*$ Conjugate.

T Transpose.

Chapter 1

Introduction and Motivation

Quantum Mechanics and Relativity dramatically changed the way how we understand and explain physical phenomena. Combining both theories became, arguably, the main goal of Physics. From the effort of combining Special Relativity and Quantum Mechanics resulted Quantum Field Theory (QFT). Our best description of the behaviour of fundamental particles is a special case of a QFT - the Standard Model (SM).

1.1 Thesis Outline and Motivation

With the experimental evidence of neutrino masses and flavor oscillations, it has become very relevant to create models which explain the neutrino spectrum and its properties.

This work is organized as follows: in the rest of this section one will dwell on the history of particle physics, as it is fundamental to understand the role and importance of neutrinos in it. Then, a brief review of the SM will be made, in order to understand its flaws and limitations in the Leptonic Sector. In chapter 2, minimal extensions to the SM which include neutrino masses are presented - νSM and SIνSM (Seesaw Type I), with a review on the state of the art of Neutrino Physics: Oscillations, CP Violation and Neutrinoless Double Beta Decay. In chapter 3, the main results of this work are presented. These are models developed under the Seesaw type I framework, that can be detected experimentally very soon due to the sizable deviations from unitarity. The development of these models was possible due to the exploitation of a hidden degree of freedom in the Seesaw equations, thanks to a novel parametrization. Additionally, the fact that these models must have controlled one-loop mass corrections, constrained the heavy neutrino mass hierarchies to be of a given type. To conclude, a final chapter with the main conclusions of the work and their future prospects. Establishing the importance of these models to probe Seesaw models and favour/disfavour Majorana Neutrinos. The research work on which part of this thesis is based on can be found at [1].

1.2 Historical Introduction

Studying neutrinos plays an important role on the study of the leptonic sector. First of all, Neutrinos are special because they are the only fundamental fermion without electric charge. While other fermions are constrained to be usual Dirac particles, neutrinos are not. Moreover, their masses are so many orders of magnitude below the masses of other fundamental particles that for many years they were thought to be massless.

As of now, most physicists believe these two facts are connected. Many models have been made, which relate both aspects, and this work has been done in the framework of one of them.[1] However, 83 years have passed from the proposal of the existence of the Neutrino to the establishment of their massive nature as truth. I believe it's instructive to cover the highlights of this story [2], because it is illustrative of how ideas in Science transform from hypotheses to reality.

The History of the Neutrino began in 1930 with the proposal of their existence by Nobel prize laureate, Austrian physicist, W. Pauli. In 1930, nuclei were considered to be bound states of protons and electrons. Thus, in this model, the β decay of a nucleus - $(A; Z) \rightarrow (A; Z+1) + e^-$ - should have a discrete spectrum. This should happen because the emitted electron would have a fixed energy, which, assuming the conservation of energy, should be equal to the release energy of the reaction, different for each nuclei. However, in the early 19th century, Lise Meitner and O. Hahn measured a continuous β spectra, confirmed later by C. D. Ellis and W. A. Wooster [3]. Also, there was another problem, some nuclei had a measured spin different from the predicted one. These two issues cast some doubt on the principle of conservation of energy and the spin-statistics theorem (known at the time as the exchange theorem of statistics). In a famous letter [4] to a conference in Tübingen that he could not attend, Pauli wrote:

"Dear radioactive ladies and gentlemen, (...)

I have hit upon a desperate remedy to save the exchange theorem of statistics and the law of conservation of energy. Namely, the possibility that in the nuclei there could exist electrically neutral particles, which I will call neutrons, that have spin 1/2 and obey the exclusion principle and that further differ from light quanta in that they do not travel with the velocity of light. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton mass. - The continuous beta spectrum would then make sense with the assumption that in beta decay, in addition to the electron, a neutron is emitted such that the sum of the energies of neutron and electron is constant."

Pauli turned out to be almost right. There was indeed a neutral particle in the nuclei - the neutron (discovered by Nobel prize laureate J. Chadwick [5] some years later). This solved the spin-statistics crisis. However, its mass was of the order of the proton mass. Only a possibly massless extra neutral fermion could solve the conservation of energy. Since we needed a «neutron» with a much smaller mass, pragmatic, Nobel prize laureate, Italian physicist E. Fermi named this particle Neutrino.

After all of these advances, it was believed that the nuclei was a bound state of protons and neutrons

[1] In the seesaw mechanism, neutrinos are Majorana particles, which they can be because they're uncharged and due to this there are more neutrinos than the three currently known, with heavier masses, which explain the small masses of the light ones.

[6] and that electrons would bind electromagnetically to the nuclei, forming a neutral atom. The stability of this neutral atom was still a mystery [2]. Under these assumptions, Fermi formulated its theory of β decay [7] - where an electron/anti-neutrino pair would be produced in the transition of a neutron into a proton: $n \to p + e^- + \overline{\nu_e}$. This was one of the first examples of an explanation in terms of fundamental particles of a known nuclear physics phenomenon. This realization lead to the creation of an effective Lagrangian a la Quantum Electrodynamics (QED) - the theory that was being built by R. Feynman, S. Tomonaga and J. Schwinger to explain all interactions between electrons and photons [8, 9, 10]. The theory built around this effective Lagrangian is now known as as the Fermi Model of weak interactions and contains only vector-like [3] interactions. Comparing the predictions of this model with experimental data from β decays, lead to two very important conclusions. First, the coupling constant of such interaction would be very small comparing to QED - the name Weak interaction is born. Second, a Lagrangian with vector and axial-vector [4] interactions provided predictions consistent with experimental data. This more general model is known as Fermi-Gamow-Teller model [11], and it includes an axial vector current $\overline{\Psi}\gamma_\mu\gamma^5\Psi$, in such a way that parity was still conserved. The assumption that parity was conserved was not backed by anything, it was just something that the community felt right. Nevertheless, T.D. Lee and C.N. Yang received the 1957 Nobel Prize for predicting Parity Violation [12, 13], motivated by the $\theta - \tau$ puzzle [14] - a belief, motivated by wrongly assumed conservation of parity, that two different particles with the same quantum numbers existed just because there were two possible decays with opposite parity. Chien-Shiung Wu received eternal gratitude for conducting the experiment [15] and study of β-decay of polarized Co_{60} nuclei which declared for once and for all that Parity was not conserved in Weak Processes.

Explicit parity violation would translate into a non left-right symmetric weak interaction. As devised by Feynman and Gell-Mann [16], the correct Lagrangian for β decay would have to be of the form:

$$L_{V-A} = (\overline{\psi_p}\gamma^\mu(g_V - g_A\gamma_5)\psi_n) \cdot (\overline{\psi_e}\gamma_\mu(g_V - g_A\gamma_5)\psi_\nu) + h.c. \ , \tag{1.1}$$

where h.c. means hermitian conjugate - the hermitian conjugate of the written term should be included. Defining Chirality projectors $P_{R,L} = \frac{1\pm\gamma_5}{2}$ with the usual projector properties ($P_{R,L}^2 = P_{R,L}$, $P_{R,L}P_{L,R} = 0$, $P_{R,L} + P_{L,R} = 1$) one can define $P_{R,L}\psi = \psi_{R,L}$ and rewrite $\psi = \psi_L + \psi_R$ and L_{V-A}:

$$L_{V-A} = (\overline{\psi_p}_L\gamma^\mu(g_V - g_A\gamma_5)\psi_{n_L}) \cdot (\overline{\psi_e}_L\gamma_\mu(g_V - g_A\gamma_5)\psi_{\nu_L} + h.c. \ , \tag{1.2}$$

where the "RR" term is zero because it was experimentally observed that $g_V = g_A = 1$, which translates into that term being proportional to $P_L \cdot P_R = 0$. The crossed terms "RL" and LR" are zero, independently of g_A and g_V, for the same reason. This implies that neutrinos with positive chirality are impossible to

[2] In classical electrodynamics, a moving charge necessarily emits energy. Thus, if the atom was a nucleus with electrons orbiting around it, it would never be stable. Thankfully, Quantum Mechanics and atomic orbitals would solve this some years later. Now, it is understood that an electron on an atomic orbital doesn't emit energy because an atomic orbital is a stationary wave - an eigenmode solution to a Schrodinger's equation. Energy is quantized and can only be emitted or absorbed if it corresponds to the energy difference between two atomic orbitals. These are labeled by three quantum numbers, depicting energy (n) and angular momentum (l,m).

[3] Here, the noun vector is used in the context of tensor calculus. A vector is an object with only one Lorentz index, i.e., a (1,0) or (0,1) tensor. A (2,0) or (0,2) tensor can be represented as a $n \times n$ matrix.

[4] An axial-vector is a vector in the context of tensor calculus, which transforms usually under rotations but gains an extra sign under parity transformations.

detect via weak interaction.

A few kilometers and years away, F. Reines and C.L. Cowan, motivated by the belief that some kind of inverse β decay with a neutrino in the initial state should exist $\overline{\nu_e}+p \rightarrow n+e^+$, devised an experiment to detect this initial state (anti)-neutrino. They succeeded [17] and F. Reines won the 1995 Nobel Prize thanks to it. This was the first experimental detection of a neutrino. The idea that lead to this discovered was propelled by an apparently conserved quantity named Lepton Number (L)[5].. Experiments that would reveal a non-conservation of this quantity yeld negative results [18]. Thus it became accepted as the only distinction between neutrino and anti-neutrino. Lepton Number came upon thanks to the realization that in β^--decay ($n \rightarrow p+e^-+\overline{\nu_e}$) one could define $L = 0$ in the initial state and $L_{e^-} = 1$ and $L_{\overline{\nu_e}} = -1$ in the final state. Assuming its conservation, a β^+-decay ($p \rightarrow n + e^+ + \nu_e$) should theoretically exist with $L_{e^+} = -1$ and $L_{\nu_e} = 1$, with a similar decay with an e^- instead of a e^+ being impossible. This decay of an isolated proton is not possible due to $m_n > m_p$ but the idea turned out to be correct and the free anti-neutrino in the initially mentioned kind of β^+-decay was indeed detected. As of 2018, we know only total Lepton Number is conserved at tree level (but violated by Chiral Anomalies).

Progress until here was very experiment driven, with theory derived in an ad hoc fashion. For instance, the Lagrangian in [eq. 1.2] unsurprisingly has some issues. Cross sections of given processes calculated with it grow with energy, this violates the unitarity of the theory - cross sections need to decrease with energy. The solution to this was to postulate an Intermediate massive Vector Boson, that prediction turned out to be correct and the boson that mediates this interaction is now known as $W^\pm$ boson. Furthermore, performing calculations with this, beyond 0th order (tree level), in perturbation theory leads to infinite results. The solution for this was a technique developed for many years by H. Bethe, R. Feynman, J. Schwinger, S. Tomonaga and F. Dyson, now known as Renormalization [19, 20] .

The discovery of the muon (μ^-) in 1937 by by E.C. Stevenson and J.C. Street [21] and C.D. Anderson and S.H. Neddermeyer [22] extended the list of known particles. In 1947, a fundamental person to the development of Neutrino Physics, Italian physicist B. Pontecorvo, suggested Lepton Universality [23] - the weak interactions of μ and e would have the same cross sections, or in modern terms, the coupling of leptons to gauge bosons would be flavor independent. Taking charged leptons' different masses into account, experiments and predictions were consistent. Things changed when recent tests of lepton universality in B meson decays, performed by the LHCb, BaBar and Belle experiments, shown deviations from the Standard Model predictions [24], although yet without high enough statistical significance to claim discovery.

Moreover, these two very similar particles were naturally classified accordingly, leading to the origin of what is now known as generation or family. This triggered the question "If there are more than one charged leptons which can interact weakly, is there more than one neutrino?". To test this, B. Pontecorvo suggested the first accelerator neutrino experiment. The experiment [25] consisted of bombarding Beryllium (Be) targets with protons, this produced predominantly muon neutrinos due to helicity suppression ($\pi^+ \rightarrow \mu^+ + \nu_\mu$ vs $\pi^+ \rightarrow e^+ + \nu_e$). The produced neutrino would later interact with nucleons, if one detected the same amount of muons and electrons in the decay products, then only one neutrino would exist

[5]c.f. the end of section 2.1 for a proper explanation on this.

($\nu_e = \nu_\mu = \nu$). However, much more muons were detected, proving the existence of at least two neutrinos, one for each (known) charged lepton. L. Lederman, M. Schwartz and J. Steinberger won the 1988 Nobel Prize in Physics for this. This lead to the definition of a new apparently conserved quantity - lepton flavor number, a generalization of Total lepton number. (Total) Lepton number was just a setting of +1 to leptons and -1 to anti-leptons, lepton flavor number does that for each generation[6]. Nonetheless, it is now known that lepton flavor number is explicitly violated by Neutrino Oscillations, also, several hints point towards lepton flavor violation in rare processes [26].

Based on what was discovered until that time, at the end of the fifties it was believed that the neutrino was a massless spin 1/2 particle with only left-handed helicity, which is the same as negative chirality for massless particles. This explained parity violation in weak interactions, the $V - A$ interaction type and the experiments that only detected left-handed neutrinos. Furthermore, a Dirac particle with only one chirality is necessarily massless. In 1937, E. Majorana [27] derived a result, valid only for neutral particles, which implied that the neutrino could have mass without having another chirality state. This would also imply that the neutrino would be its own anti-particle.

Currently, we understand that neutrinos have a tiny mass so we cannot guarantee it doesn't exist a positive chirality counterpart for the neutrino - if it is a Dirac particle it must have it, if it is a Majorana it doesn't, but extra particles would be needed to make everything consistent. The fundamental question is to find out which type of particle the neutrino is.

As we entered the 60s, the effort to create a theory guided by some principles that explained all the known facts was finally converging to a solution. In 1968, S. Glashow, S. Weinberg and A. Salam [28, 29, 30] formulated the Glashow-Salam-Weinberg (GSW) model of weak-interactions, a model that would be an important part of the future Standard Model (SM). It was based on a $SU(2) \times U(1)$ gauge group and predicted the existence of weak neutral currents mediated by a new neutral boson - Z. Developments in the study of the Strong force, lead by M. Gell-Mann [31, 32], derived what is now known as Quantum Chromo Dynamics (QCD), based on a SU(3) gauge group. A full formulation of the SM was finally done, with the gauge group being $SU(3) \times SU(2) \times U(1)$. After the prediction of the Z boson an impressive number of successive successes was achieved:

1973	Neutral currents are discovered in the bubble chamber "Gargamelle" [33]
1974	J/Ψ meson discovered by groups headed by B. Richter and S. Ting, proving the existence of the charm quark (c). [34, 35]
1975	The τ charged lepton is discovered by M. Perl's group. [36] M. Perl was awarded the 1995 Nobel Prize in Physics for this.
1977	Υmeson discovered at Fermilab, proving the existence of the bottom quark (b). [37]
1979	The QCD gauge boson - the gluon (G) - is indirectly observed in three-jet events at DESY. [38]
1983	Discovery at LEP, of the mediators of eletroweak interactions - $W^\pm$ and Z. [39, 40, 41, 42]
1989	Measurement of the Z invisible width or the number of non-sterile light neutrinos: 3.27 ± 0.30. [43]
1995	Top quark (t) is discovered at Fermilab. [44, 45]
2000	First direct observation of the ν_τ at Fermilab. [46]

Table 1.1: Experimental Achievements in Particle Physics from 1973 to 2000

Due to some underlying principles, all the discovered particles needed to be theoretically massless. That contradicted experiments, and an extra particle, with very special properties was needed - the Higgs (H) particle. What is now known as simply the Higgs Mechanism was devised by Anderson, Brout and Englert, Guralnik, Hagen, Higgs, Kibble and 't Hooft [47, 48]. It provided masses to the gauge bosons,

[6] c.f. the end of section 2.1 for a proper explanation on this.

and in general to all fermions, without explicitly breaking the symmetries of the model. This turned out to be a fundamental principle, since, some years later, Nobel laureates G.'t Hooft and M. Veltman [49] proved only gauge theories with spontaneous symmetry breaking were consistent at higher orders of perturbation theory.

In 2012, the observation of a Higgs-like boson was finally announced by the ATLAS and CMS collaborations [50, 51]. All the SM particles were now experimentally detected. Everything seemed consistent with the SM besides some experiments involving neutrinos. Precision measurements of β decay spectrum end-point would be sensitive to neutrino masses, but at the time the only possible conclusion was that neutrinos were much lighter than electrons, with the bound consistent with massless neutrinos. With no irrefutable evidence for massive neutrinos, one could not conclude anything, and this remained an open problem for several years. Now, neutrinos are known to be massive and there is a dedicated experience in Karlsruhe, Germany, named KATRIN [52] (Karlsruhe Tritium Neutrino Experiment) with the goal of measuring the "mass of the electron antineutrino" [7] with high precision by examining the spectrum of electrons emitted from the beta decay of tritium.

The idea that neutrinos could be massive was proposed in 1957 by B. Pontecorvo [53]. He pinpointed that there was no symmetry preventing that (like gauge invariance prevents the photon from acquiring a mass), thus being possible that they had a very small mass. B. Pontecorvo was also the first to consider neutrino oscillations. In a seminal paper [54], he showed that if neutrinos had mass, lepton flavor number is not conserved and that neutrino states produced in weak decays are a superposition of states with definite mass. Years later, in 1962, when the existence of ν_μ was already proved, Z. Maki, M. Nakagawa and S. Sakata proposed a better model of neutrino oscillations [55]. They proposed that ν_μ and ν_e were linear combinations of two mass eigenstates, and that oscillations between one another were possible. As an answer to the lack of evidence of this oscillation phenomenon in some experiences [18], B. Pontecorvo coined the term "sterile neutrino", claiming that a massive neutrino (antineutrino) could transform into its positive chirality counterpart and become totally invisible to our experiments, since it would not interact via weak interaction [56].

The first experiment to provide an hint for oscillations, and thus, massive neutrinos was R. Davis group's experiment to detect solar neutrinos [57] trough $\nu_e + Cl_{37} \rightarrow e^- + Ar_{37}$. The measured solar neutrino flux was way below the theoretical predictions - the solar neutrino problem is born. At first, it was thought that the problem was inherent to the used solar model. However, many other measured quantities proved its consistency and only a few years later was the neutrino oscillation hypotheses accepted as the best answer [58, 59, 60]. In order for this to happen, the experiments Super Kamiokande (SK) [61] and Sudbury Neutrino Observatory (SNO) [62] played a crucial role. SK and SNO are Cherenkov effect based experiments, which detect high energy solar neutrinos from the proton proton chain due to elastic scattering of these neutrinos with electrons from the atoms in the detector. The results from these two experiments revealed, in a model independent way, an evidence of ν_e disappearance. This deficit was then understood to be due to oscillations of ν_e into ν_μ inside the sun, due to the Mikheev-Smirnov-Wolfenstein (MSW) effect [63, 64, 65], and in the path from the sun's surface to the Earth. The 2015 Nobel Prize

[7] Clarified on chapter 2, in the state of the art section.

in Physics was awarded to A. McDonald and T. Kajita, heads of the SNO and SK group, respectively, "for the discovery of neutrino oscillations, which shows that neutrinos have mass". After the success of detection of solar neutrinos, several experiments were built with the goal of detecting more energetic neutrinos - atmospheric neutrinos - like NUSEX [66] and SK in a later stage, and less energetic neutrinos - reactor neutrinos - like KamLAND [67] and MINOS [68]. These experiments helped gather data to determine, under some assumptions, the neutrino mixing angles θ_{13}, θ_{23}, θ_{12}. These mixing angles are known by reactor mixing angle, atmospheric mixing angle and solar mixing angle, respectively.

Adding to the previously mentioned deviations from the SM on lepton flavor universality measurements, the most recent measurements from MiniBooNE hint towards the existence of a sterile neutrino with a mass in the KeV scale [69], although cosmological data had already excluded a neutrino with a mass on that energy scale. However, none of these experiments has enough significance to claim discovery. Nevertheless, the community is convinced in a short period of time, conclusions regarding important properties of the neutrino and its spectrum (ordering, mass scale, nature) [8] will be taken, thanks to the data from GERDA [70], KamLAND-ZeN [71], CUORE [72] and KATRIN [52]. Adding to these, the data from a great variety of long baseline (like NOνA [73] and DUNE [74]) and short baseline experiments (like MiniBooNE [75]), with very different constraints, will be fundamental to give answers to some problems like the existence of more than 3 neutrinos, CP phases and unitarity of the mixing matrix. [9]

Theoretically, interest in neutrinos has surged when J. Pati, A. Salam and S. Glashow and H. Georgi started working on Grand Unified Theories (GUT). These models grouped leptons and quarks in the same group multiplets, and mass generation mechanism naturally lead to non-zero neutrino masses. A critical landmark of theoretical Neutrino Physics (and of this work) was the formulation of the Seesaw Mechanism by P. Minkowski, M. Gell-Mann, P. Ramond and R. Slansky, T. Yanagida and R. Mohapatra [76, 77, 78, 79] in the context of specific GUT models. The seesaw mechanism gathered interest because it provided a natural explanation to the smallness of neutrino masses comparing to the masses of charged fermions. Furthermore, it related that with the fact that the neutrino is the only known neutral fermion and thus can have special never before seen properties. With the experimental evidence of neutrino masses and flavor oscillations, the door to physics beyond the SM was opened and the seesaw mechanism looks like a promising framework to understand it. Modern reviews on the Seesaw Mechanism can be found in references [80, 81, 82, 83].

1.3 Brief Summary of the Standard Model of Particle Physics

This will be a short review, thus, some aspects of the SM will not be discussed. For a more complete treatment of the subject refer to [84, 85, 86].

The SM introduces fundamental interactions (strong, weak and electromagnetic) as a way to guarantee that the Lagrangian is locally invariant under Gauge Symmetries - where every field is in a representation of the Symmetry Group. In Quantum Field Theory [10], particles are seen as excitations (quanta) of a field.

[8] More on that on chapter 2, state of the art section

[9] More on this subject on chapter 2.

[10] In QFT it is important to state the metric one is using. From now on, everything will be written according to a (+,-,-,-)

The passage from a Classical Field Theory to a QFT does not involve a modification of the Lagrangian or of the field equations, but rather a reinterpretation of the field variables. To make this clear, the [Table 1.2] contains quantized fields in terms of classical solutions to their free massless dynamical equation:

Table 1.2: The three equations that rule the free dynamics of SM particles.

Spin	Equation	Name	Field Expansion
0	$\partial_\mu \partial^\mu \phi = 0$	Klein-Gordon eq.	$\phi = \int_p (a(p) e^{-ip_\mu x^\mu} + b^\dagger(p) e^{ip_\mu x^\mu})$
1/2	$i\gamma_\mu \partial^\mu \Psi = 0$	Dirac eq.	$\Psi = \int_p \sum_s (a_p^s u_s(p) e^{-ip_\mu x^\mu} + b_p^{s\dagger} v_s(p) e^{ip_\mu x^\mu})$
1	$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$	Proca eq.	$A^\nu = \int_p \sum_r (a_p^r \epsilon_r^\nu(p) e^{ip_\mu x^\mu} + b_p^{r\dagger} \epsilon_r^{*\nu}(p) e^{-ip_\mu x^\mu})$

The creation ($b^\dagger(p)$) and annihilation operators ($a(p)$), as the name indicates, create and annihilate an excitation (quanta) of the field , i.e, a particle [11]. One could dwell on more intrinsic aspects of QFT like partition functions, a proper Derivation of Feynman rules and propagators, how one can go from a free theory to an interacting theory (LSZ formula) or what is the meaning of the bare parameters one writes in a classical field theory Lagrangian and its relation to the measured values (Renormalization). Nevertheless, since these are not fundamental to the presented work, they will not be discussed. For more, one can refer to any Introductory QFT book [86, 87, 88].

Masses cannot be introduced in the free theory because they would explicitly spoil the required local gauge invariance. Thus, masses are introduced via Spontaneous Symmetry Breaking (SSB) of the SM Group into a smaller one. One can define the construction of the SM into two parts: Before SSB, where every field is massless and every interaction is diagonal (interaction basis) and after SSB, where the fields are massive and mixing occurs (mass basis). The Lorentz-Poincare symmetry group also plays a crucial role, the different quantum states of every fundamental particle should give rise to a irreducible representation of it, and the free dynamical equations can be deduced based on this. The SM particles currently known can be divided into Bosons and Fermions. In the Bosons category there is a scalar boson - H - the Higgs particle, responsible for the mass generation mechanism, and 4 kinds of gauge bosons: W, Z, γ and G. Fermions can be divided into Quarks and Leptons. There are three families of Quarks and other three of Leptons. Each family of Quarks has an up and down type quark, while for Leptons each family has a charged lepton and the correspondent neutrino.

Observing [Fig. 1.1] [12], the only characteristic that distinguishes particles with the same quantum numbers and from different families is their mass. However, if one considers its chirality state, positive and negative chirality states could be considered different particles, since they have different interactions. In a massless SM, this would be the case, as one wouldn't have other choice but to identify particles based on their interactions. One wouldn't have 3 families of Quarks and Leptons, but just one, with the double amount of fields - 1 Dirac field can be decomposed into two Weyl fields (chiral massless fields):

This happens because the SM is a chiral theory - it treats differently particles with different chirality. For instance, in a model with positive chirality neutrinos, the ν_R would be a sterile particle, as it doesn't interact with anything in the SM, while the ν_L would interact only via weak interaction. Since the SM is

Minkowski metric

[11] The particles created by $b^\dagger(p)$ and annihilated by $a(p)$ could be the same or the anti-particle of each other, depending on whether the field is real or not.

[12] Image not made by me. Licensed under Creative Commons Attribution 3.0 Unported license, one is free to to copy, distribute, transmit and adapt the work. Numerical values taken from Particle Data Group Booklet 2016 [89]

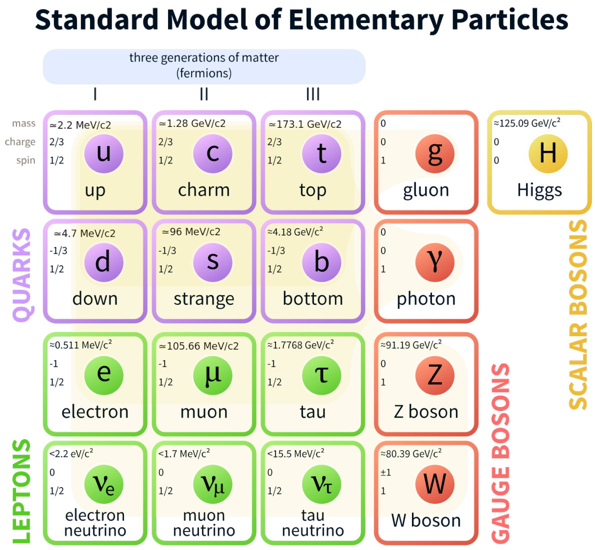


Figure 1.1: List of SM particles. Until 2012, all of them but the Higgs (H) were experimentally discovered.

	P_L	P_R
ν	ν_L	ν_R
$\overline{\nu}$	$\overline{\nu}_R$	$\overline{\nu}_L$

Table 1.3: The two particle/anti-particle states of a massless SM

a theory of massive particles, a particle is considered as a state with definite mass which is a superposition of the negative with the positive chirality state - a Dirac mass term in the SM Lagrangian can be seen as a interaction term between the positive chirality state and the negative chirality state. Thus, SSB bounds the two chirality states into, what we define as a particle state.

The SM of unification of the electroweak and strong interactions is based on the gauge group:

$$G_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y \ . \tag{1.3}$$

The massless Lagrangian for this theory is locally invariant under transformations of this group. To achieve this, the introduction of gauge fields is necessary and to every one of them should correspond a generator of the group. This group has 12 generators - 8 SU(3) bosons (G^a), 3 SU(2) bosons (W^b) and 1 U(1) boson (B).

Gauge bosons are in the adjoint representation of their corresponding gauge group (octet for $SU(3)_c$ and triplet for $SU(2)_L$) and in the vector representation of the Lorentz-Poincare group, quarks and leptons are in the fundamental or anti-fundamental representation of the Lorentz-Poincare Group, while the negative chirality ones are in in the fundamental representation of $SU(2)_L$ (doublet):

$$\Psi_L^\alpha = \begin{pmatrix} \nu_L^\alpha \\ l_L^\alpha \end{pmatrix} \ , \ \Upsilon_L^\beta = \begin{pmatrix} u_L^\beta \\ d_L^\beta \end{pmatrix} \ . \tag{1.4}$$

The positive chirality ones are in the singlet representation of $SU(2)_L$:

$$l_R^\alpha \ , \ u_R^\beta \ , \ d_R^\beta \ . \tag{1.5}$$

Regarding $SU(3)_c$, Leptons are in the singlet representation - don't take part in colored interactions - while quarks are in the triplet representation:

$$Q^\gamma = \begin{pmatrix} q_r^\gamma \\ q_g^\gamma \\ q_b^\gamma \end{pmatrix} \ , \tag{1.6}$$

where $\gamma = 1, 2, .., 6$ is the quark flavor index - one can define that to the up quark (u) corresponds the triplet Q^1 with entries q_i^1, to the down quark (d) corresponds the triplet Q^2 with entries q_i^2, to the charm quark (c) corresponds the triplet Q^3 with entries q_i^3, and so on. The order is irrelevant.

The quantum numbers of the particle spectrum of the SM in the interaction basis are given in [Table 1.4].

Table 1.4: Quantum Numbers and Representations of every SM Particle

Field	l_L^α	l_R^α	ν_L^α	u_L^α	u_R^β	d_L^β	d_R^β	ϕ^+	ϕ^0	G	B	$W^{1,2}$	W^3
T3	-1/2	0	1/2	1/2	0	-1/2	0	1/2	-1/2	0	0	± 1	0
Y	-1/2	-1	-1/2	1/6	2/3	1/6	-1/3	1/2	1/2	0	0	0	0
Q	-1	-1	0	2/3	2/3	-1/3	-1/3	1	0	0	0	± 1	0
$SU(3)_c$ Rep.	1	1	1	3	3	3	3	1	1	8	1	1	1
Lorentz-Poincare Rep.	(1/2,0)	(0,1/2)	(1/2,0)	(1/2,0)	(0,1/2)	(1/2,0)	(0,1/2)	(0,0)	(0,0)	(1/2,1/2)	(1/2,1/2)	(1/2,1/2)	(1/2,1/2)

Where $\alpha = 1, 2, 3$ is the leptonic generation index and $\beta = 1, 2, 3$ is the quark generation index- one can define that to the top quark ($t_{R,L}$) corresponds $u_{R,L}^3$ and to the muon ($\mu_{R,L}$) corresponds $l_{R,L}^2$, and so on. Y is defined by the relation $Q = Y + T_3$, which will be explained later.

The quantum numbers of the particle spectrum of the SM are chosen so that they are in the correct representation of the gauge group and that the conserved quantum number (after SSB) - electric charge - has the correct value for each one. Y is named Weak Hypercharge and is the Quantum Number corresponding to $U(1)_Y$, T_3 is named Weak Isospin and is the quantum number corresponding to $SU(2)_L$. The Quantum Number corresponding to $SU(3)_c$ is color. Quarks can have three colors (red, green and blue) and anti-quarks can have three anti-colors (anti-red, anti-green and anti-blue). They are in the triplet representation of $SU(3)_c$. Gluons can have the 8 independent combinations of these 6 (3+3) colors and anti-colors. However, every object observed in nature is colorless (or a singlet of $SU(3)_c$ like an electron), which means quarks and gluons aren't asymptotic states and don't have a spectral representation, only hadrons (combinations of these) do. This happens due to a special property of QCD: Asymptotic freedom [90, 91].

The gauge fields are necessarily bosons because to ensure the local gauge invariance of the Lagrangian it is necessary to add fields that transform like the derivative - creating what is known as the Covariant Derivative. For a field that interacts with every boson (like the quark field), the covariant derivative is:

$$D_\mu = \partial_\mu + ig_s \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} + ig \sum_{b=1}^{3} W_\mu^b \frac{\sigma^b}{2} + ig' Y B_\mu \ , \tag{1.7}$$

where the g's are the couplings of each interaction, $\frac{\lambda^a}{2}$ are the 8 generators of SU(3) (Gell-Mann matrices) and $T^b = \frac{\sigma^b}{2}$ are the 3 generators of SU(2) (Pauli matrices). The used sign notation is consistent with [92], taking all $\eta_i = 1$.

It's useful to define:

$$D_\mu^q = \partial_\mu + ig_s \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} \ , \ D_\mu^L = \partial_\mu + ig \sum_{b=1}^{3} W_\mu^b \frac{\sigma^b}{2} + ig' Y B_\mu \ , \ D_\mu^R = \partial_\mu + ig' Y B_\mu \ . \tag{1.8}$$

Since the derivative transforms like a vector under the Lorentz-Poincare group, gauge fields must have the same behaviour, and thus, be vector fields, with integer non-zero spin (vector bosons). This covariant derivative generates the interactions between the gauge bosons and the other fields:

$$\begin{aligned} L_{Fermion} = &\sum_\gamma \overline{Q^\gamma} i\gamma^\mu D_\mu^q Q^\gamma + \sum_\alpha \left(\overline{\Psi_L^\alpha} i\gamma^\mu D_\mu^L \Psi_L^\alpha + \overline{l_R^\alpha} i\gamma^\mu D_\mu^R l_R^\alpha \right) \\ &+ \sum_\beta \left(\overline{\Upsilon_L^\beta} i\gamma^\mu D_\mu^L \Upsilon_L^\beta + \overline{u_R^\beta} i\gamma^\mu D_\mu^R u_R^\beta + \overline{d_R^\beta} i\gamma^\mu D_\mu^R d_R^\beta \right) \ , \end{aligned} \tag{1.9}$$

where the indices α, β and γ have the same meaning as in [Table 1.4] and in [eq. 1.6] .

The gauge boson interactions with themselves come from their kinetic terms, which are of the form:

$$L_{kin} = -\frac{1}{4} G^{a\mu\nu} G_{a\mu\nu} - \frac{1}{4} W^{b\mu\nu} W_{b\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \ , \tag{1.10}$$

where

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_s f^{acd} G_{c\mu} G_{d\nu} \ , W_{\mu\nu}^b = \partial_\mu W_\nu^b - \partial_\nu W_\mu^b - g f^{bcd} W_{c\mu} W_{d\nu} \ , B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \ , \tag{1.11}$$

and $a = 1, ..., 8$ is summed implicitly and runs over the number of gauge bosons of SU(3) and $b = 1, 2, 3$ is summed implicitly and runs over the number of gauge bosons of SU(2). f^{abc} are the structure constants for the gauge group - for SU(2) $f^{abc} = \epsilon^{abc}$. For abelian gauge groups - like U(1) - these are zero. These terms generate self-interactions (in the gluon case) but also interactions between different gauge bosons. After electroweak unification the physical states Z, γ, W^+, W^- are revealed to be linear combinations of the gauge fields B, W^1, W^2, W^3, looking at [eq. 1.10] one can understand that this generates triple and quartic interactions between Z, γ, W^+ and W^-.

Electroweak unification is what happens when the mass of fundamental particles is generated, and here the Higgs is the leading actor. Adding to the fact that a mass term would explicitly break the gauge symmetry, without the Higgs particle the unitarity of the SM would be spoiled [93, 94]. SSB means that the vacuum of the theory at a certain point in time - spontaneously - (it's postulated that it was in the early Universe) stops having the same symmetry as the Lagrangian. The Higgs mechanism spontaneously breaks G_{SM} into $SU(3)_C \times U(1)_Q$, in order for this to happen, a scalar doublet of $SU(2)_L$ - ϕ - is added to the theory:

$$\phi = \begin{pmatrix} G^+ \\ G^0 \end{pmatrix} \ , \ V(\phi) = -\mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2 \ , \ L_{Higgs} = (D^L_\mu\phi)^\dagger \cdot (D^{\mu L}\phi) - V(\phi) \ , \tag{1.12}$$

where V(ϕ) is the most general renormalizable potential that can be added to the Lagrangian. For $\mu^2 > 0$ [13] and $\lambda > 0$, the potential has an absolute minimum [14] for $<\phi> \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. From the minimization equation one gets $<\phi^\dagger\phi> = \frac{\mu^2}{2\lambda}$. This is fine since ϕ is a scalar field, and it doesn't violate Lorentz invariance having a constant non-zero value that minimizes the potential. This is what is known as a vacuum expectation value (vev). All other SM fields with spin different from zero are compelled to have a zero vacuum expectation value. This vev can be parametrized in the following way:

$$<\phi> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \tag{1.13}$$

getting the relation:

$$<\phi^\dagger\phi> = \frac{\mu^2}{2\lambda} = \frac{v^2}{2} \rightarrow v^2 = \frac{\mu^2}{\lambda} \ . \tag{1.14}$$

To get a proper spectrum it's useful to write the ϕ field as a perturbation around its vev, taking into account all its degrees of freedom:

$$\phi = <\phi> + \begin{pmatrix} \phi^+ \\ \frac{H + i\phi_Z}{\sqrt{2}} \end{pmatrix} \ . \tag{1.15}$$

One can parametrize three degrees of freedom in the form of a global $SU(2)_L$ transformation:

$$\phi = e^{i\frac{\sigma^b}{2}\omega_b} \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix} \ , \tag{1.16}$$

and then use the freedom to apply a global $SU(2)_L$ transformation to absorb them. This is known as going to the unitary gauge:

$$\phi \rightarrow e^{-i\frac{\sigma^b}{2}\omega_b}\phi = \begin{pmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{pmatrix} = <\phi> + \begin{pmatrix} 0 \\ \frac{H}{\sqrt{2}} \end{pmatrix} \ . \tag{1.17}$$

In this gauge, the H field parametrizes the deviations from the value of ϕ that minimizes the potential. It will correspond to the Higgs field. Note that the vacuum $<\phi>$ is the kind of vacuum we need for SSB because it's not invariant under $SU(2)_L$ transformations anymore:

[13] It's postulated that in the early universe μ^2 was negative. In that case, $<\phi> = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This translates into μ^2 being a function of the temperature of the Universe.

[14] $<\phi>$ is the value of ϕ for which V(ϕ) is minimal.

$$e^{i\frac{\sigma^b}{2}\theta_b} \cdot <\phi> \neq <\phi> \Longrightarrow \frac{\sigma^b}{2} \cdot <\phi> \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{1.18}$$

However, it's invariant under the combination given by $Q = T_3 + Y$ [95] [15], where Y is a diagonal matrix in flavor space:

$$Q <\phi> = (\frac{\sigma^3}{2} + Y \cdot I_{2\times 2}) \cdot <\phi> = \begin{pmatrix} 1/2 + Y & 0 \\ 0 & -1/2 + Y \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} , \tag{1.19}$$

which is zero if $Y\phi = \frac{1}{2}\phi$.

Thus, this vacuum spontaneously breaks part of the electroweak gauge symmetry, after which only one neutral Higgs scalar, H, remains in the physical particle spectrum. It's important to note that this is not a fact. The number of scalars in the theory is not constrained and there can be more than one Higgs-like particle if one introduces more than one Higgs-like doublet. This itself is another topic of theoretical and experimental research. The Covariant derivative acting on ϕ (kinetic term on L_{Higgs}) generates the masses of the gauge bosons. This generates electroweak mixing. After a basis rotation, the physical states are identified and a relation for their couplings is given :

$$Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \ , \ A_\mu = \cos\theta_W B_\mu + \sin\theta_W W_\mu^3 \ , \ W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2) \ , \tag{1.20}$$

$$sin^2(\theta_W) = \frac{g'^2}{g^2 + g'^2} \ , \quad m_W = m_Z \cos\theta_W \ , \ e = g \sin\theta_W = g' \cos\theta_W \ , \tag{1.21}$$

$$m_W^2 = \frac{1}{4} v^2 g^2 \ , \ m_Z^2 = \frac{1}{4} v^2 (g^2 + g'^2) \ , \ m_H^2 = 2\mu^2 \ . \tag{1.22}$$

The initial vacuum was identically zero and was left invariant under the 4 generators of $SU(2)_L \times U(1)_Y$. Now, it is left invariant under one combination of two of these generators. From the Goldstone Theorem we should have 3 (4-1) Nambu-Goldstone bosons (massless scalar bosons), but we have 3 massive gauge bosons instead. In a pictoric language it is said the gauge bosons "eat" the Nambu-Goldstone bosons. If the broken symmetry was global (and not gauge/local) we would have 3 massless scalars [96, 97] [16].

Goldstone's theorem can be applied in theories without fundamental scalar fields. One can illustrate this with QCD. The pions are the pseudo-Goldstone bosons that result from the spontaneous breakdown of the chiral-flavor symmetries of QCD due to the QCD vacuum - the quark condensate $< \overline{Q}Q >$. It does not have a zero vev, induced by nonperturbative strong interactions (confirmed in Lattice QCD calculations). Considering a low energy version of QCD, with only u and d quarks, its massless Lagrangian exhibits a $SU(2)_V \times SU(2)_A = SU(2)_L \times SU(2)_R$ symmetry that is spontaneously broken by the quark condensate $< \overline{u}u + \overline{d}d >$. These symmetries are further explicitly broken by the masses of the quarks, so that the pions are not massless, but their mass is significantly smaller than typical hadron masses. After

[15] This is known as the Gell-Mann-Nishijima relation.
[16] These three degrees of freedom correspond to ϕ^+ (2, real and imaginary part) and ϕ_Z (1).

some calculations [17] one gets the Gell-Mann-Oakes-Renner formula:

$$m_\pi^2 f_\pi^2 = -\frac{(m_u + m_d)}{2} < \overline{u}u + \overline{d}d > \ , \tag{1.23}$$

where $f_\pi \approx 92MeV$ is the pion decay constant, $m_\pi = 140MeV$ is the mass of the pion. Lattice QCD calculations yield $< \overline{u}u + \overline{d}d > \approx -(300MeV)^3$. With massless quarks, the pions [18] would be massless but heavier baryons (combinations of three valence quarks) and mesons (combinations of two valence quarks) would have a mass, dynamically generated by the quark condensate. Actually, 99% of the proton's mass arises out QCD binding energy, consequence of QCD chiral symmetry breaking. Conventional wisdom that the Higgs field generated all the mass of the Universe is wrong [98].

The Covariant Derivative given in term of the physical bosons is:

$$D_\mu = \partial_\mu + i \sum_{a=1}^{8} G_\mu^a \frac{\lambda^a}{2} + i\frac{g}{\sqrt{2}}(W_\mu^+ T_+ + W_\mu^- T_-) + i\frac{g}{\cos\theta_W}(T_3 - Q\sin^2\theta_W)Z_\mu + ieQA_\mu \ , \tag{1.24}$$

where $T_\pm = T_1 \pm iT_2 = \frac{\sigma_1}{2} \pm i\frac{\sigma_2}{2} = \sigma_\pm$.

It's useful to redefine[19]:

$$D_\mu^L = \partial_\mu + i\frac{g}{\sqrt{2}}(W_\mu^+ T_+ + W_\mu^- T_-) + i\frac{g}{\cos\theta_W}(T_3 - Q\sin^2\theta_W)Z_\mu + ieQA_\mu \ , \tag{1.25}$$

$$D_\mu^R = \partial_\mu + i(Q - T_3)\frac{g}{\cos\theta_W}(\sin\theta_W \cos\theta_W A_\mu - \sin^2\theta_W Z_\mu) \ . \tag{1.26}$$

To generate Dirac fermion masses one needs to create a $SU(2)_L \times U(1)_Y$ invariant term, using ϕ and the fermion fields, that after SSB generates their mass terms. Using [Table 1.4] one can see that a term like the one in [eq. 1.27] is invariant because it has $Y = 1/2 + 1/2 - 1 = 0$ and is a $SU(2)_L$ singlet,

$$Y_{\delta\sigma}^l \begin{pmatrix} \overline{\nu_L^\delta} & \overline{l_L^\delta} \end{pmatrix} \phi l_R^\sigma = Y_{\delta\sigma}^l \begin{pmatrix} \overline{\nu_L^\delta} & \overline{l_L^\delta} \end{pmatrix} \begin{pmatrix} G^+ \\ G^0 \end{pmatrix} l_R^\sigma \xrightarrow{\text{SSB}} Y_{\delta\sigma}^l \frac{v}{\sqrt{2}} \overline{l_L^\delta} l_R^\sigma \ . \tag{1.27}$$

After SSB this generates a mass term proportional to v. The proportionality constant is $Y_{\delta\sigma}^l = \frac{\sqrt{2}M_{\delta\sigma}^l}{v}$, where $M_{\delta\sigma}^l$ is the (δ, σ)th entry of the Dirac mass matrix for the charged leptons. $Y_{\delta\sigma}^l$ are known as the Yukawa couplings, and they parametrize the couplings between the Higgs field and fermions before SSB. Their origin and exact mathematical formula is an object of study [99, 100, 101]. The same method applies for quarks. However, for up quarks (the ones with $T_3 = \frac{1}{2}$) one needs to define the adjoint doublet $\tilde{\phi} = \begin{pmatrix} G^{0*} \\ -G^- \end{pmatrix}$ which has $Y = -\frac{1}{2}$. After SSB, one has $< \tilde{\phi} > = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix}$.

[17] cf. section 7.1 on Kallen-Lehmann spectral representations and section 19.3 about Goldstone Bosons and Chiral Symmetries in QCD of [86] for details.

[18] If one considers the strange quark and applies the same reasoning, the eight light pseudoscalar mesons would be massless. For the three heavy quarks: the charm quark, bottom quark, and top quark, their masses, and hence the explicit breaking these amount to, are much larger than the QCD spontaneous chiral symmetry breaking scale $\sim (300MeV)^3$. Thus, they cannot be treated as a small perturbation around the explicit symmetry limit.

[19] D_μ^q stays the same after electroweak unification. $SU(3)_c$ is not broken.

Thus, after SSB one gets a Dirac mass matrix for up quarks, down quarks and for charged leptons:

$$L_{Yukawa} = -\overline{\Psi^{\delta}_L} Y^l_{\delta\sigma} \phi l^{\sigma}_R - \overline{\Upsilon^{\delta}_L} Y^d_{\delta\sigma} \phi d^{\sigma}_R - \overline{\Upsilon^{\delta}_L} Y^u_{\delta\sigma} \tilde{\phi} u^{\sigma}_R + h.c. \; . \tag{1.28}$$

To go from the interaction basis to the mass basis one needs to diagonalize these mass matrices. The rotated states will be the physical states. This generates mixing - particles or mass states being linear combinations of interaction states, and the interaction Lagrangian 1.9 is no longer diagonal on the fields - in the quark sector (neutrinos are massless in the SM and in that scenario it's possible to remove the mixing in the leptonic sector). In the next chapter, this procedure will be done for neutrinos in what is known as νSM, the SM with positive chirality neutrinos. The steps are the same for the quark sector in the SM, so they won't be done here.

There's only two pieces left to have the SM Lagrangian completely defined:

$$L_{SM} = L_{Fermion} + L_{Kin} + L_{Higgs} + L_{Yukawa} + L_{GF} + L_{Ghosts} \; . \tag{1.29}$$

The result in [eq. 1.17] is gauge dependent. A gauge independent formulation of the SM should use [eq. 1.15]. However, with this definition, L_{Higgs} will generate mixed quadratic terms in fields, with the three Goldstone bosons ϕ_Z and $\phi^{\pm}$, that complicate the definition of the gauge boson propagators. Using gauge independent [eq. 1.15] in $L_{Fermion}$ also introduces interactions between fermions and the unphysical Goldstone bosons. These should be taken into account when performing calculations in a general gauge. To cancel the mixed quadratic terms that arise from L_{Higgs}, it's necessary to add a new term to the SM Lagrangian:

$$L_{GF} = -\frac{1}{2\xi_G} F^a_G F_{Ga} - \frac{1}{2\xi_A} F^2_A - \frac{1}{2\xi_Z} F^2_Z - \frac{1}{\xi_W} F_- F_+, \tag{1.30}$$

where $F^a_G = \partial^{\mu} G^a_{\mu}$, $F_A = \partial^{\mu} A_{\mu}$, $F_Z = \partial^{\mu} Z_{\mu} + \xi_Z m_Z \phi_Z$, $F_{\pm} = \partial^{\mu} W^{\pm}_{\mu} \pm i\xi_W m_W \phi^{\pm}$. L_{GF} are, actually, the gauge breaking terms in L_{SM}.

The last piece is the Ghost Lagrangian. Faddeev–Popov Ghosts are unphysical particles that violate the Spin-Statistics Theorem. In theories like the SM they are bosonic (spin 0) with anti-comutation relations (fermionic). Every gauge boson correspondent to a non-Abelian Gauge Group will have a Ghost [20]. These ghost fields are necessary to achieve a linear gauge fixing condition like in L_{GF}, generating gauge field propagators with transverse and longitudinal component, thus, invertible.

$$L_{Ghost} = \sum_{i=1}^{4} \left[\overline{c}_+ \frac{\partial(\delta F_+)}{\partial \alpha_i} + \overline{c}_- \frac{\partial(\delta F_-)}{\partial \alpha_i} + \overline{c}_Z \frac{\partial(\delta F_Z)}{\partial \alpha_i} + \overline{c}_A \frac{\partial(\delta F_A)}{\partial \alpha_i} \right] c_i + \sum_{a,b=1}^{8} \overline{\omega}^a \frac{\partial(\delta F^a_G)}{\partial \beta_b} \omega^b, \tag{1.31}$$

where $\alpha = 1, 2, 3, 4$ and $\beta = 1, ..., 8$ are parameters of the correspondent gauge transformations. One can check the definitions from Appendix A of [92] taking $\eta_i = 1$ and sections 16.2 and 21.1 of [86], to understand what L_{Ghost} translates into. Since ghosts don't couple to matter fields, their contribution to one-loop corrections of physical processes involving fermions in external lines is zero, thus explaining

[20] 8 for $SU(3)$, 4 for $SU(2) \times U(1)$

ghosts further than this goes beyond the scope of this work ([88] here one can find a proper treatment of ghosts). After adding the final two terms, one can check that the Unitary gauge corresponds to $\xi_{Z,W} \to \infty$ and $\xi_{A,G} \to 1$. The Goldstone bosons correspondent to massive gauge bosons - ϕ_Z and $\phi_\pm$ - acquire gauge dependent masses. The unitary gauge - see [eq. 1.17] - only contains physical particles because having infinite masses means a decoupling from the theory. The formulation of the SM in a general gauge will be needed in further parts of this work.

To conclude, one final note regarding the SM. The electroweak sector of the SM has 17 free parameters: 9 fermion masses, 3 quark mixing angles, 1 CP violating phase in the quark sector, 1 Higgs mass, 1 Higgs vaccum expectation value (v), 1 Weinberg angle (θ_W) and 1 $SU(2)_L$ gauge coupling (g). This means that these parameters need to be fitted with experimental data. Thus, the SM doesn't predict fermion masses nor gives an explanation to the number of generations of these.[21]

Furthermore, the SM doesn't include gravity nor particles that can be dark matter candidates. Also, it doesn't explain dark energy. Another concerns are related to CP Violation. There hasn't been detected CP violation[22] in the strong sector (QCD) while there's nothing that inhibits it, and the CP Violation detected in the electroweak sector is not big enough to explain the Matter–antimatter asymmetry we observe in our universe.

Other, more theoretical, shortcomings of the SM are the hierarchy problem, related to the fine-tuning that needs to happen in higher order calculations to achieve a Higgs mass near the electroweak scale (v) and the fact that the gauge couplings don't unify at high-energy, unlike what happens in some GUT models. All these drawbacks lead to the belief that the SM is not the final theory of everything (TOE) but just a low-energy effective theory of it. The SM can be summarized in its Feynman Rules, which can be found at [92]. In this work, these were used setting all $\eta_i = 1$.

[21] However, one knows that the number of generations of leptons and quarks must be equal in order to cancel gauge anomalies that appear at one-loop corrections - c.f. Chapter 19 section 19.4 and chapter 20 section "Anomaly Cancellation" of [86]

[22] CP Violation will be discussed in the next chapter.

Chapter 2

The Leptonic Sector Beyond the Standard Model

In the literature, there are many models [102] which assume that neutrinos are Dirac Particles[1] while adding more particles and symmetries, giving reasonable explanations to the smallness of neutrino masses. If one releases the restriction of neutrinos being Dirac particles[2], many models are possible, with Majorana Neutrinos. Some contain extra particles - charged scalars and fermions - and naturally small neutrino masses due to these being radiatively generated [103, 104, 105, 106, 107].

In this chapter, one will present minimal extensions to the SM, as depicted in chapter 1, which include neutrino masses, dwell on neutrino oscillations and discuss the state of the art of neutrino physics.

2.1 νSM

On this extension, the only assumption is that neutrinos are Dirac particles. No explanation to the smallness of neutrino masses is given but unnaturally small Yukawa couplings. No extra fields are added to the SM but ν_R^α, necessary to generate Dirac mass terms, which translates into adding an extra column into [Table 1.4]: This changes the SM Lagrangian in the following way:

Table 2.1: Quantum Numbers and Representations of ν_R

Field	ν_R^α
T3	0
Y	0
Q	0
$SU(3)_c$ Rep.	1
Lorentz-Poincare Rep.	(0,1/2)

[1]Particles which obey the massive Dirac eq. and are not its own anti-particles. A Dirac particle is equivalent to two Weyl particles - massless particles which obey the Weyl eq. - c.f. [Table 1.3].

[2]Neutrinos are Dirac if one creates a symmetry such that the Majorana mass term is forbidden. Without this one is obliged to write the most general gauge invariant Lagrangian, which should include it.

$$
\begin{aligned}
L^{\nu SM}_{Fermion} =& \sum_{\gamma} \overline{Q^{\gamma}} i\gamma^{\mu} D^{q}_{\mu} Q^{\gamma} + \sum_{\alpha} \left(\overline{\Psi^{\alpha}_{L}} i\gamma^{\mu} D^{L}_{\mu} \Psi^{\alpha}_{L} + \overline{l^{\alpha}_{R}} i\gamma^{\mu} D^{R}_{\mu} l^{\alpha}_{R} + \overline{\nu^{\alpha}_{R}} i\gamma^{\mu} D^{R}_{\mu} \nu^{\alpha}_{R} \right) \\
&+ \sum_{\beta} \left(\overline{\Upsilon^{\beta}_{L}} i\gamma^{\mu} D^{L}_{\mu} \Upsilon^{\beta}_{L} + \overline{u^{\beta}_{R}} i\gamma^{\mu} D^{R}_{\mu} u^{\beta}_{R} + \overline{d^{\beta}_{R}} i\gamma^{\mu} D^{R}_{\mu} d^{\beta}_{R} \right) \\
&= L_{Fermion} + \sum_{\alpha} \overline{\nu^{\alpha}_{R}} i\gamma^{\mu} D^{R}_{\mu} \nu^{\alpha}_{R} \ ,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
L^{\nu SM}_{Yukawa} =& -\overline{\Psi^{\delta}_{L}} Y^{l}_{\delta\sigma} \phi l^{\sigma}_{R} - \overline{\Psi^{\delta}_{L}} Y^{\nu}_{\delta\sigma} \tilde{\phi} \nu^{\sigma}_{R} - \overline{\Upsilon^{\delta}_{L}} Y^{d}_{\delta\sigma} \phi d^{\sigma}_{R} - \overline{\Upsilon^{\delta}_{L}} Y^{u}_{\delta\sigma} \tilde{\phi} u^{\sigma}_{R} + h.c. \\
&= L_{Yukawa} - \overline{\Psi^{\delta}_{L}} Y^{\nu}_{\delta\sigma} \tilde{\phi} \nu^{\sigma}_{R} - \overline{\nu^{\sigma}_{R}} \tilde{\phi}^{\dagger} Y^{\nu\dagger}_{\delta\sigma} \Psi^{\delta}_{L} \ .
\end{aligned}
\tag{2.2}
$$

One important point is that the changes in $L_{Fermion}$ don't introduce any new interaction because $Y\nu^{\alpha}_{R} = 0$ as it is stated in [Table 2.1]. This implies that $D^{R}_{\mu}\nu^{\alpha}_{R} = \partial_{\mu}\nu^{\alpha}_{R}$. This addition introduces their mass terms, and new interactions in the Yukawa sector - between neutrinos and the Higgs and Goldstone Bosons. After SSB one can separate L_{Yukawa} into two terms, one with the mass matrices and another with the interactions between fermions and Higgs and Goldstone Bosons:

$$
L^{\nu SM}_{Yukawa} = L^{\nu SM}_{IntYukawa} + L^{\nu SM}_{MYukawa} \ . \tag{2.3}
$$

To simplify notation, from now on, the flavor indices will be omitted and the superscript 0 will be added to interaction states. In other words, ν^{0}_{L} is a column vector $N_f \times 1$ in flavor space. The terms from [eq. 2.3] can be written as:

$$
\begin{aligned}
L^{\nu SM}_{IntYukawa} =& -\overline{\nu^{0}_{L}} \phi^{+} Y^{l} l^{0}_{R} - \frac{1}{\sqrt{2}} \overline{l^{0}_{L}} H Y^{l} l^{0}_{R} - \frac{i}{\sqrt{2}} \overline{l^{0}_{L}} \phi_{Z} Y^{l} l^{0}_{R} \\
&- \frac{1}{\sqrt{2}} \overline{\nu^{0}_{L}} H Y^{\nu} \nu^{0}_{R} + \frac{i}{\sqrt{2}} \overline{\nu^{0}_{L}} \phi_{Z} Y^{\nu} \nu^{0}_{R} + \overline{l^{0}_{L}} \phi^{-} Y^{\nu} \nu^{0}_{R} + h.c + quark\ terms \ ,
\end{aligned}
\tag{2.4}
$$

and

$$
L^{\nu SM}_{MYukawa} = -\overline{l^{0}_{L}} M^{l} l^{0}_{R} - \overline{\nu^{0}_{L}} M^{\nu} \nu^{0}_{R} - \overline{d^{0}_{L}} M^{d} d^{0}_{R} - \overline{u^{0}_{L}} M^{u} u^{0}_{R} + h.c. \ . \tag{2.5}
$$

Thus, a general Dirac mass matrix is a $N_f \times N_f$ matrix and may have N^2_f non-zero entries. [3] As one can observe, there is a mass matrix for up and down quarks as well as for charged leptons and neutrinos. After one diagonalizes the mass matrix one finds new states - mass states that correspond to the physical particles.

The diagonalization of a mass matrix cannot be performed in the usual way - with just one diagonalizing matrix. One needs a unitary bi-diagonalization (or as mathematicians call it - Singular Value Decomposition), because one needs to diagonalize a matrix while rotating two different fields [4], thus the matrices acting on them are, in general, different. Then, one has:

$$
U^{\nu\dagger}_{L} M^{\nu} U^{\nu}_{R} = m_{\nu} \ , \ {U^{l}_{L}}^{\dagger} M^{l} U^{l}_{R} = m_{l} \ , \tag{2.6}
$$

where m_{ν} and m_l are diagonal 3×3 matrices with positive real entries, which contain the masses of the

[3] In the SM and in this extension $N_f = 3$

[4] One with positive and another with negative chirality - c.f. [eq. 2.5]

neutrinos and charged leptons, respectively, in the diagonal[5]. From [eq. 2.5] one notes that we need to transform the fields in this special way:

$$\nu_{R,L}^{\delta 0} = (U_{R,L}^{\nu})^{\delta k}\nu_{R,L}^{k}\ ,\ l_{R,L}^{\delta 0} = (U_{R,L}^{l})^{\delta k} l_{R,L}^{k}\ , \tag{2.7}$$

where the left hand side corresponds to interaction states (δ=e, μ, τ) and the right hand side corresponds to mass states ($k = 1, 2, 3$). From now on, latin alphabet indices will be used when one is referring to mass states.

Notice that ν_1, ν_2 and ν_3 are the real particles, not ν_e, ν_μ and ν_τ, which are interaction states. Referring to the mass of ν_e is nonsense[6]. Thus, an updated version of [fig. 1.1] should contain ν_1, ν_2 and ν_3 and not ν_e, ν_μ and ν_τ, but I kept it that way because listing ν_e, ν_μ and ν_τ in these tables is the most common practice. For many years the neutrino was thought to be massless, so I would say the community is still in a transitory stage. After this detour, one proceeds in writing the Lagrangian that defines the leptonic electroweak currents - the terms that are summed in α in [eq. 2.1] - in terms of the mass states:

$$L_W^l = -\frac{g}{\sqrt{2}}\left[W_\mu^+ \overline{\nu_L^j} U_{ji}^\dagger \gamma^\mu l_L^i + W_\mu^- \overline{l_L^i} U_{ij}\gamma^\mu \nu_L^j\right]\ , \tag{2.8}$$

$$L_{A,Z}^l = -\frac{g}{2\cos\theta_w}\left[Z_\mu(\overline{\nu_L^i}\delta_{ij}\gamma^\mu\nu_L^j - \overline{l_L^i}\delta_{ij}\gamma^\mu l_L^j)\right] - \left[(\frac{g\sin^2\theta_w}{\cos\theta_w}Z_\mu + eA_\mu)(\overline{l_L^i}\delta_{ij}\gamma^\mu l_L^j + \overline{l_R^i}\delta_{ij}\gamma^\mu l_R^j)\right]\ , \tag{2.9}$$

where $U = U_{PMNS} = U_L^{l\,\dagger} U_L^{\nu}$[7] is the leptonic mixing matrix and it's unitary by construction. This implies that the otherwise diagonal charged interactions become non-diagonal, giving origin to what is known as mixing. In the quark sector, the mixing matrix is known as V_{CKM}. Neutral currents remain diagonal, which means that there are no flavor Changing Neutral Currents (FCNC) at tree level in the SM (in any sector) and in this extension. Beyond tree level they are highly suppressed - c.f. $K_L^0 \to \mu^+\mu^-$ [108]. This is known as the GIM mechanism. The Yukawa Lagrangian [eq. 2.2] can also be written in terms of mass states, which, in this case, is possible to transform into functions of the fields, physical masses and the mixing matrices of the leptonic sector[8]:

$$\begin{aligned} L_{IntYukawa}^{\nu SM} =& -\frac{g}{\sqrt{2}m_W}\overline{\nu_L}U^\dagger\phi^+ m_l l_R - \frac{g}{2m_W}\overline{l_L}Hm_l l_R - \frac{ig}{2m_W}\overline{l_L}\phi_Z m_l l_R \\ & - \frac{g}{2m_W}\overline{\nu_L}Hm_\nu\nu_R + \frac{ig}{2m_W}\overline{\nu_L}\phi_Z m_\nu\nu_R + \frac{g}{\sqrt{2}m_W}\overline{l_L}\phi^- U m_\nu\nu_R + h.c + quark\ terms\ , \end{aligned} \tag{2.10}$$

and

$$L_{MYukawa}^{\nu SM} = -\overline{l_L}m_l l_R - \overline{\nu_L}m_\nu\nu_R - \overline{d_L}m_d d_R - \overline{u_L}m_u u_R + h.c.\ . \tag{2.11}$$

Usually one defines a basis - a Weak Basis (WB) - that best fits its needs. A WB is a choice of $U_{R,L}$ (c.f. [eq. 2.7]) in the quark and leptonic sector which leaves the charged current Lagrangian invariant.

[5]One obtains $U_{R,L}^\nu$ through the equation $m_\nu^2 = U_L^{\nu\dagger}M^\nu M^{\nu\dagger}U_L^\nu = U_R^{\nu\dagger}M^{\nu\dagger}M^\nu U_R^\nu$. The same applies for the charged leptons matrices.

[6]Although it has a meaning in the context of β decay, more on that on section 2.4.

[7]Named after Pontecorvo, Maki, Nakagawa and Sakata.

[8]Using $Y^i\frac{v}{\sqrt{2}} = M^i$ and $m_W = \frac{gv}{2}$ - [eq. 1.22] .

These bases need to respect some rules, in order to not spoil Gauge Invariance. This translates into the existence of invariant quantities, named Weak Basis Invariants [109, 110, 111, 112]. An example of a WB in the leptonic sector is:

$$\nu'_L = W_L \cdot \nu_L \ , \ \nu'_R = W^\nu_R \cdot \nu_R \ , \ l'_L = W_L \cdot l_L \ , \ l'_R = W^l_R \cdot l_R \ , \tag{2.12}$$

where the flavor indices are omitted. Note that one needs to rotate in the same way the negative chiral fields, with W^l_L, because they are part of the same doublet of $SU(2)_L$. Positive chiral fields are singlets of $SU(2)_L$ and, thus, can be rotated independently. For illustration purposes we will choose a basis where the charged lepton matrix is diagonal, real and positive and U_{PMNS} is the matrix that diagonalizes M^ν. For that one chooses $W_L = U'^l_L$, $W^l_R = U'^l_R$ and $W^\nu_R = I_{3\times3}$. By looking at [eq. 2.5] one notes that the mass matrices change in this way:

$$M'^l = W_L{}^\dagger M^l W^l_R = U'^{l\dagger}_L M^l U'^l_R = m'_l \ , \tag{2.13}$$

and using [eq. 2.6]:

$$M'^\nu = W_L{}^\dagger M^\nu W^\nu_R = U'^{l\dagger}_L M^\nu = U'^{l\dagger}_L U^\nu_L m_\nu U^{\nu\dagger}_R \ , \tag{2.14}$$

where m'_l is diagonal. One can make another WB transformation such that m'_l becomes m_l - diagonal with real positive entries while turning M'^ν into a hermitian matrix. This is performed by doing $W_L = K^l_L$, $W^l_R = K^l_R$, $W^\nu_R = U^\nu_R U^{\nu\dagger}_L U^l_L$ where K^l_R and K^l_L are diagonal matrices with only complex phases. Also, $U^l_R = U'^l_R \cdot K^l_R$ and $U^l_L = U'^l_L \cdot K^l_L$. The mass matrices change in this way:

$$M''^l = K^l_L{}^\dagger M'^l K^l_R = K^l_L{}^\dagger m'_l K^l_R = U^l_L{}^\dagger M^l U^l_R = m_l \ , \tag{2.15}$$

and

$$M''^\nu = K^l_L{}^\dagger M'^\nu K^\nu_R = U^l_L{}^\dagger U^\nu_L m_\nu U^{\nu\dagger}_R U^\nu_R U^{\nu\dagger}_L U^l_L = U^l_L{}^\dagger U^\nu_L m_\nu U^{\nu\dagger}_L U^l_L = U_{PMNS} m_\nu U^\dagger_{PMNS} \ , \tag{2.16}$$

achieving what was intended - a positive-definite real diagonal m_l and an hermitian M^ν. When one is in this basis, the hermitian matrix M^ν is diagonalized by the unitary matrix U_{PMNS}. In a general basis, U_{PMNS} diagonalizes $M^\nu M^{\nu\dagger}$.

The quark sector behaves in the same way, one can define a WB where the up quark matrix is diagonal, real and positive and M_d is hermitian, meaning that, in this basis, $V_{CKM} = U^{u\dagger}_L U^d_L$ is the matrix that diagonalizes M_d - the down quark mass matrix.

Thus, the following discussion is valid for the quark and leptonic sector, in this extension of the SM. For this reason, until the end of this section one will use the notation V to refer to U_{PMNS} or V_{CKM}, as it serves both purposes. The mixing matrix V is complex but some of its phases have no physical meaning, due to the fact that one has the freedom to rephase mass eigenstates, $u'_i = e^{i\phi_i} u_i$, $l'_i = e^{i\phi_i} l_i$,

$d'_j = e^{i\phi_j} d_j$ and $\nu'_j = e^{i\phi_j}\nu_j$, transforming the entries of the mixing matrix:

$$V'_{ij} = e^{i(\phi_j - \phi_i)} V_{ij} \ . \tag{2.17}$$

Thus, it's useful to look for rephasing invariants[9] such as the quartets:

$$Q_{ijkl} = VijV_{kl}V^*_{il}V^*_{kj} \ , \tag{2.18}$$

because it can be proved that invariants of higher order can always be written as combinations of quartets and the moduli.

An important thing to note is that a global rephasing of all the quark and lepton fields,

$$u'_i = e^{i\phi} u_i, l'_i = e^{i\omega} l_i, d'_j = e^{i\phi} d_j \ , \nu'_j = e^{i\omega}\nu_j \ , \tag{2.19}$$

leaves the total Lagrangian invariant, particularly, the Yukawa Lagrangian [eq. 2.2] and the charged interactions Lagrangian [eq. 2.1]. This is a global $U(1)_\phi \times U(1)_\omega$ symmetry of the Lagrangian. Thanks to Emmy Noether's Theorem [113] [10], one knows that this leads to two conserved charges, one for the leptons - Lepton Number $L = n_l - n_{\bar{l}}$ and one for the quarks - Baryon number $B = \frac{1}{3}(n_q - n_{\bar{q}})$ [11].

In the SM there is an extra symmetry. With massless neutrinos one can rephase neutrinos fields freely. Thus it is possible to perform a transformation:

$$l'_i = e^{i\omega_i} l_i, \nu'_i = e^{i\omega_i}\nu_j \ , \tag{2.20}$$

which leaves the charged interactions part of the Lagrangian of the SM [eq. 1.9] invariant and translates into an extra $SU(1) \times SU(1) \times SU(1)$ symmetry, comparing to the νSM one. Again, thanks to Emmy Noether's Theorem [113], this results in three conserved charges: $L_i = n_i - n_{\bar{i}}$, the flavor lepton numbers. The existence of neutrino masses proves that this is not an exact symmetry of nature. The $SU(3)_c \times U(1)_Q$ local gauge invariance of the SM Lagrangian is also responsible for the conservation of electrical charge Q and color.

The fact that V is complex, in general, means that CP Violation can exist. Performing a CP transformation is performing a Charge transformation - transforming a particle in its anti-particle[12] - followed by a Parity transformation - flipping the sign of the spatial coordinate[13]. One can check section 13.2 of [109] to see how SM fields transform under CP transformations. In this model, neutrinos transform like the down quarks and charged leptons transform like the up quarks[14]. Performing a CP transformation to the SM Lagrangian after SSB, one obtains the condition for CP invariance of the SM (considering only

[9] Of course the moduli of each entry, $|V_{ij}|$, is also a rephasing invariant.

[10] For me, the most important theorem in the history of physics.

[11] Defined in this way to accommodate the fact that quarks are not asymptotic states. A proton as baryon number 1.

[12] Note that this changes the sign of all the charges of the field. However, it doesn't change the chirality. A neutrino with negative chirality is transformed into a anti-neutrino with negative chirality [114].

[13] Which translates into changing the chirality of a field, since axial vectors get an extra sign under parity transformations.

[14] I mean, with the same index, like I implicitly did before [eq. 2.17], such that [eq. 2.21] is valid for both sectors

V_{CKM}) and of the νSM (considering also U_{PMNS}):

$$V_{ij}^* = e^{i(\xi_W + \xi_j - \xi_i)} V_{ij} \ , \tag{2.21}$$

where ξ_α are spurious CP phases that arise from the transformation. One can make [eq. 2.21] always true for a single entry of V, however, if one considers all of the entries, one concludes that this forces all quartets to be real and, thus, all other rephasing invariants to be real. Direct CP violation stems from the non-removable phases of V_{CKM}. Another possible approach is considering the Lagrangian before SSB. Applying the CP transformations in section 14.2 of [109] (which are the same ones as in section 13.2 with an extra weak basis transformation), one obtains the following conditions for CP invariance:

$$W_L^\dagger Y^\nu W_R^\nu = Y^{\nu *} \ , W_L^\dagger Y^l W_R^l = Y^{l*} \ , \tag{2.22}$$

and equivalent ones for the quarks. The existence of matrices W_L, W_R^l and W_R^ν (and the quark equivalents) that satisfy the above conditions is a necessary and sufficient condition for CP invariance in the νSM. Thus, one can conclude that Direct CP violation stems from the clash between the Yukawa sector and the charged currents sector.

Back to the mass basis, it's important to determine how many physical CP violating phases might exist in V. Due to the rephasing invariance, for N_f generations (V is $N_f \times N_f$ unitary matrix parametrized by N_f^2 parameters) one can remove $2N_f - 1$ phases, making the total number of parameters $(N_f - 1)^2$. $N_f(N_f - 1)/2$ of these parameters will be angles, while $(N_f - 1)(N_f - 2)/2$ will be phases. If one takes $N_f = 3$, one obtains that there is only one phase. This is a CP violating phase, Kobayashi and Maskawa [115] arrived to the above conclusion, proving that only for $N_f \geq 3$ one has CP Violating phases in the quark sector. The same statement is true for the leptonic sector, in this model. For $N_f = 3$ the imaginary part of all quartets are equal, up to a sign [116]. This is known as the Jarlskog rephasing invariant:

$$J = |Im[Q]| \tag{2.23}$$

From the unitarity constraints on the entries of V one can define what is known as unitarity triangles - c.f. section 13.5 and 13.6 of [109]. From these one gets a remarkable geometrical interpretation to J - it is twice the area of any of the six possible unitarity triangles.

For an extensive treatment of CP Violation one recommends [109, 117]. The standard parametrization [89] of the mixing matrix is the following:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} , \tag{2.24}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and δ is a Dirac-type CP violating phase. This translates into $J = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin\delta$.

However, it's important to note that, for instance, δ is not a rephasing invariant and is only meaningful under this parametrization. It is possible to parametrize the mixing matrix using a non optimal number of parameters but with everyone of them being a rephasing invariant [89]. The current best-fit values for the quark sector [89] are:

$$V_{CKM} = \begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\ 0.22438 \pm 0.00044 & 0.97359^{+0.00010}_{0.00011} & 0.04214 \pm 0.00076 \\ 0.00896^{+0.00024}_{0.00023} & 0.04133 \pm 0.00074 & 0.999105 \pm 0.000032 \end{pmatrix} , \tag{2.25}$$

and a Jarlskog invariant of $J = (3.18 \pm 0.15) \times 10^{-5}$. The current best-fit values for the leptonic sector are stated in section 2.4.

In spite of the theoretical similarities between Quarks and Dirac neutrinos, there are some major differences. The unitarity of V_{CKM} is heavily constrained [118], contrarily to the leptonic sector where sizable deviations from unitarity of the mixing matrix are not ruled out [119].

This difference stems from the fact that there are many hadron decay processes which enable the direct measurement of individual V_{CKM} entries. However, in the leptonic sector this is not possible. There is not enough precision to detect neutrino mass states (their mass scale is too small) in leptonic weak decays, so in each process one can only know the produced interaction state with certainty. Currently, one of the most reliable ways to get information regarding the mixing matrix are oscillation experiments and, even in those, one only has access to the first row and the last column of the 3×3 mixing matrix. Furthermore, what is measured in those cases are combinations of the entries of the mixing matrix, and not individual entries like in the quark case [119].

To summarize the νSM, the list of of the new Feynman Rules added to the SM is presented at Appendix A.

2.2 Seesaw Mechanism and Majorana fields

In this section one will add an extra assumption, comparing to the νSM, - neutrinos are Majorana particles.

Majorana fields are real solutions of the Dirac Equation. The Dirac Equation can be made real by going to the Majorana Basis, where all nonzero elements of the γ matrices are purely imaginary. Fields and matrices on the Majorana basis are indicated by the presence of the tilde:

$$(i\widetilde{\gamma}^{\mu}\partial_{\mu} - m)\widetilde{\nu} = 0 \ , \tag{2.26}$$

with this, one can get solutions that satisfy:

$$\widetilde{\nu} = \widetilde{\nu}^{*} \ . \tag{2.27}$$

To transform from the Majorana Basis to other Basis (Dirac or Weyl) one performs a unitary transformation:

$$\gamma^{\mu} = U\widetilde{\gamma}^{\mu}U^{\dagger} \ , \ \nu = U\widetilde{\nu} \tag{2.28}$$

The reality condition [eq. 2.27] in another basis becomes:

$$\begin{aligned} U^\dagger \nu = U^T \nu^* \Leftrightarrow \nu = UU^T \nu^* \\ \Leftrightarrow \ \nu = C\overline{\nu}^T = \nu^c \ , \ \overline{\nu} = \nu^T C = \overline{\nu^c} \ , \end{aligned} \tag{2.29}$$

where $UU^T = C\gamma_0^T$ is a unitary matrix since U is also unitary, and C is also a unitary matrix with properties:

$$\gamma_\mu C = -C\gamma_\mu^T \ , \tag{2.30}$$

derived using properties of γ_0, γ_μ and [eq. 2.28]

$$C = C^* = -C^{-1} = -C^\dagger = -C^T \ , \tag{2.31}$$

derived using the fact that UU^T is unitary, the previous property, and properties of γ_0,

$$\gamma_5 C = C\gamma_5^T \ , \tag{2.32}$$

derived using the definition of γ_5 and the first property.

ν^c is a spinor, in the same Lorentz group representation as ν, since it transforms in the same way:

$$\nu'(x')^c = C\gamma_0^T \left(exp(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}) \right)^* \nu(x)^* = \left(exp(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}) \right) \nu(x)^c \ , \tag{2.33}$$

$\overline{\nu^c}$ also transforms as $\overline{\nu}$[15], which proves that the reality condition in a general basis [eq. 2.29] is Lorentz invariant. Because of this, and to distinguish it from the C transformation, this transformation is sometimes named Lorentz-covariant conjugate [114]. This ψ^c satisfies the same Dirac equation with minimal coupling that ψ satisfies, with the term proportional to the electric charge gaining an extra sign. The C transformation only changes the sign of all additive quantum numbers and commutes with the chirality projectors. For a Weyl field, helicity and chirality is the same, and helicity involves spin and momentum - neither of these changes under a C transformation. However, the Lorentz-covariant conjugate of positive chiral field is a negative chiral field and vice-versa. CP transformations transform fields in the same way as the Lorentz-covariant conjugation - apart from some possible complex phases. All of this can be summarized in the important formula:

$$(N_{R,L})^c = P_{L,R} N^c = e^{i\xi_N} N_{L,R} \ , \tag{2.34}$$

where the last equality only applies if N is a Majorana field.

Since neutrinos don't have charge one can write in the Lagrangian new Lorentz-invariant quantities[16] that are also gauge invariant: $\overline{\nu^c}\nu$, $\overline{\nu}\nu^c$. If ν was a bosonic field quantities like $\overline{\nu}\nu^c$ would automatically vanish: $\overline{\nu}\nu^c = \overline{\nu}C\overline{\nu}^T = (\overline{\nu}C^T\overline{\nu}^T)^T = \overline{\nu}C^T\overline{\nu}^T = -\overline{\nu}C\overline{\nu}^T = -\overline{\nu}\nu^c$. The anti-commutation of the fields

[15]To prove this one needs the equation $C\gamma_0^T\sigma_{\mu\nu}^* = -\sigma_{\mu\nu}C\gamma_0^T$ that can be derived from the previously stated properties of C and the definition of $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

[16]It can be easily seen that he quantity $\overline{\nu^c}\nu^c$ is just another form of writing $\overline{\nu}\nu$

(fermion fields are Grassmann variables) is needed so that after transposing an extra minus sign is gained.[17]

Because Weyl fermions are irreducible representations of the Lorentz-Poincare Group, they can be used as building blocks of any kind of fermion field. A Dirac field in terms of Weyl fields can be written as $\nu = \nu_L + \nu_R$, where ν_L and ν_R are independent. While a Majorana field in terms of Weyl fields, can be decomposed as:

$$\nu_1 = \nu_L + (\nu_L)^c \ , \ \nu_2 = \nu_R + (\nu_R)^c \ , \tag{2.35}$$

where it's obvious that ν_1 and ν_2 obey the reality condition [eq. 2.27]. From this one concludes that if a field has a Majorana character one obtains the double amount of physical fields. However, the two components of each field are related by conjugation, meaning that the degrees of freedom are the same.

Using chirality projectors, one can define the Weyl fields $\nu_{R,L} = P_{R,L}\nu$ such that:

$$\overline{\nu}\nu = \overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L \ , \ \overline{\nu}\nu^c + \overline{\nu^c}\nu = \overline{\nu_L}\nu_L^c + \overline{\nu_L^c}\nu_L + \overline{\nu_R}\nu_R^c + \overline{\nu_R^c}\nu_R \ . \tag{2.36}$$

These new Lorentz-invariant quantities introduce the possibility of having terms like $\overline{\nu_L^c}M_L\nu_L + \overline{\nu_R^c}M_R\nu_R + h.c.$ in the Lagrangian. As long as they are gauge invariant under the chosen gauge group for our model. In a minimal extension of the SM with 3 Majorana neutrinos, $\overline{\nu_L^c}M_L\nu_L$ is not gauge invariant under the SM gauge group, thus, only the other term remains. Now that notation and definitions are settled, one is ready to define another extension of the SM - the Seesaw type I SM (SIνSM).

2.2.1 SIνSM

As it was done for νSM, one will study the Lagrangian before SSB, after SSB - in the interaction basis - and finally after SSB - in the mass basis. The assumption that neutrinos are Majorana brings nothing new to $L_{Fermion}$ in the interaction basis:

$$L_{Fermion}^{SI\nu SM} = L_{Fermion} + \sum_\alpha \overline{\nu_R^\alpha} i\gamma^\mu D_\mu^R \nu_R^\alpha = L_{Fermion}^{\nu SM} \ , \tag{2.37}$$

However, the Yukawa sector suffers a dramatic change - an extra term - a gauge invariant Majorana mass term [18]:

$$L_{Yukawa}^{SI\nu SM} = L_{Yukawa} - \overline{\Psi_L^\delta} Y_{\delta\sigma}^\nu \tilde{\phi}\nu_R^\sigma - \overline{\nu_R^\sigma}\tilde{\phi}^\dagger Y_{\delta\sigma}^{\nu\dagger}\Psi_L^\delta - \frac{1}{2}\overline{\nu_R^c}M_R\nu_R - \frac{1}{2}\overline{\nu_R}M_R^\dagger\nu_R^c = L_{Yukawa}^{\nu SM} - \frac{1}{2}\overline{\nu_R^c}M_R\nu_R - \frac{1}{2}\overline{\nu_R}M_R^\dagger\nu_R^c \ . \tag{2.38}$$

[17] Note that one writes + h.c. to remove redundant information. When writing this part explicitly one doesn't have to actually perform the operation and take into account the Grassmann nature of fermion fields. It's literally the hermitian conjugate of the part that is written.

[18] The introduction of this term in the Lagrangian imposes the Majorana nature on the fields ν as proved in pages 295-297 of [109].

Which, after SSB and using again the definitions $L^{SI\nu SM}_{Yukawa} = L^{SI\nu SM}_{IntYukawa} + L^{SI\nu SM}_{MYukawa}$:

$$\begin{aligned} L^{SI\nu SM}_{IntYukawa} = & -\overline{\nu_L}\phi^+ Y^l l_R - \frac{1}{\sqrt{2}}\overline{l_L} H Y^l l_R - \frac{i}{\sqrt{2}}\overline{l_L}\phi_Z Y^l l_R \\ -\frac{1}{\sqrt{2}}\overline{\nu_L} H Y^\nu \nu_R + & \frac{i}{\sqrt{2}}\overline{\nu_L}\phi_Z Y^\nu \nu_R + \overline{l_L}\phi^- Y^\nu \nu_R + h.c + quark\ terms = L^{\nu SM}_{IntYukawa} \ , \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} L^{SI\nu SM}_{MYukawa} = & -\overline{l_L} M^l l_R - \overline{\nu_L} M^\nu \nu_R - \overline{d_L} M^d d_R - \overline{u_L} M^u u_R - \frac{1}{2}\overline{\nu^c_R} M_R \nu_R + h.c. \\ = & L^{\nu SM}_{MYukawa} - \frac{1}{2}\overline{\nu^c_R} M_R \nu_R - \frac{1}{2}\overline{\nu_R} M^\dagger_R \nu^c_R \ , \end{aligned} \tag{2.40}$$

where M_R is a $N \times N$ matrix in flavor space (N is the number of neutrinos with positive chirality added to the SM, the most natural case would be $N = N_f$ and this choice was already made in the previous section when defining the νSM). The origin of M_R is not fixed. In the literature there are several explanations, such as the one which claims that this term results from the interaction of a new Goldstone boson related with lepton number symmetry breaking [120] with neutrinos. The most famous take on this has this term in its initial Lagrangian, and assumes it has a scale much bigger than the electroweak scale. The heavy fields ν_R are integrated out resulting in an effective theory with higher dimensional operators that result in naturally small mass terms of the type $\overline{\nu^c_L} M_L \nu_L$ [83]. This approach will not be taken here, because one intends to obtain results independently of the scale of M_R. The origin of M_R, for itself, is a very interesting topic but is out of the scope of this work. Nevertheless, the results of this work may provide hints for the origin of such term. From now on, one will assume that this term is not controlled by the Higgs Mechanism, and thus, can be of a order of magnitude bigger than the vev - v, but not as big as it is assumed in other works [121, 83] and will perform an exact approach to the study of the physical repercussions of such term. Before advancing to the identification of the mass states, let us write the above terms in a Weak Basis where M^l is diagonal - $W_L = U^l_L$, $W^l_R = U^l_R$ and $W^\nu_R = I_{3\times3}$:

$$\begin{aligned} L^{SI\nu SM}_{IntYukawa} = & -\frac{g}{\sqrt{2}m_W}\overline{\nu_L}\phi^+ m_l l_R - \frac{g}{2m_W}\overline{l_L} H m_l l_R - \frac{ig}{2m_W}\overline{l_L}\phi_Z m_l l_R \\ -\frac{g}{2m_W}\overline{\nu_L} H M'^\nu \nu_R + & \frac{ig}{2m_W}\overline{\nu_L}\phi_Z M'^\nu \nu_R + \frac{g}{\sqrt{2}m_W}\overline{l_L}\phi^- M'^\nu \nu_R + h.c + quark\ terms \ , \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} L^{SI\nu SM}_{MYukawa} = & -\overline{l_L} m_l l_R - \overline{\nu_L} M'^\nu \nu_R - \frac{1}{2}\overline{\nu^c_R} M_R \nu_R + h.c. + quark\ terms \\ & = -\overline{\nu_R} M'^{\nu\dagger} \nu_L - \frac{1}{2}\overline{\nu_R} M^\dagger_R \nu^c_R + h.c + charged-leptons\ terms + quark\ terms \\ & = L^{SI\nu SM}_{M\nu Yukawa} + charged-leptons\ terms + quark\ terms \ , \end{aligned} \tag{2.42}$$

where $M'^\nu = U^{l\dagger}_L M^\nu$ and the prime will be dropped from now on. To ease the the process of finding the mass states, one will rewrite the mass terms regarding neutrinos in a way that reveals a generalized mass matrix. Using [eq. 2.34] , one can identify:

$$\nu^c_{R,L} := \nu'_{L,R} \ , \tag{2.43}$$

which after inversion lead to:

$$\nu_{R,L} = C\overline{\nu'_{L,R}}^T \ , \ \overline{\nu_{R,L}} = -{\nu'_{L,R}}^T C^{-1} \ . \tag{2.44}$$

This means that the terms referring to neutrinos in [eq. 2.42] can be rewritten as:

$$\begin{aligned} L^{SI\nu SM}_{M\nu Yukawa} =& \nu'^T_L C^{-1} M^{\nu\dagger}\nu_L + \frac{1}{2}[\nu'^T_L C^{-1} M^\dagger_R \nu'_L] + h.c \\ =& \frac{1}{2}[\nu'^T_L C^{-1} M^{\nu\dagger}\nu_L + \nu^T_L C^{-1} M^{\nu *}\nu'_L] + \frac{1}{2}[\nu'^T_L C^{-1} M^\dagger_R \nu'_L] + h.c \ , \end{aligned} \tag{2.45}$$

where the anti-commutation of ν_L and ν'_L (Grassmann fields) was used. Thus meaning that the neutrino mass Lagrangian is:

$$L^{SI\nu SM}_{M\nu Yukawa} = \frac{1}{2}\left[\begin{pmatrix} \nu^T_L & \nu'^T_L \end{pmatrix} C^{-1} \begin{pmatrix} 0 & M^{\nu *} \\ M^{\nu\dagger} & M^\dagger_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu'_L \end{pmatrix}\right] + h.c. \ , \tag{2.46}$$

where $M^\dagger = M^* = \begin{pmatrix} 0 & M^{\nu *} \\ M^{\nu\dagger} & M^\dagger_R \end{pmatrix}$, and M is the generalized mass matrix. In a basis where the mass matrix of the charged leptons is already real, diagonal and positive, the matrix V that diagonalizes M has physical meaning - it's the generalized leptonic mixing matrix. The diagonalization can be performed via the unitary transformation[19] :

$$V^T M^* V = \begin{pmatrix} d & 0 \\ 0 & d_R \end{pmatrix} = Diag(m_1, m_2, m_3, M_1, M_2, M_3) = D \rightarrow V^\dagger M = DV^T \ . \tag{2.47}$$

Parameterizing:

$$V = \begin{pmatrix} K & R \\ S & Z \end{pmatrix} \ , \tag{2.48}$$

one identifies the new mass states n and N:

$$\begin{pmatrix} \nu_L \\ \nu'_L \end{pmatrix} = V \begin{pmatrix} n_L \\ N_L \end{pmatrix} \implies \nu^\delta_L = K^{\delta j} n_{Lj} + R^{\delta j} N_{Lj} \ , \nu'^\delta_L = S^{\delta j} n_{Lj} + Z^{\delta j} N_{Lj} \ . \tag{2.49}$$

From here one understands that K is the 3×3 mixing matrix, responsible for mixing between the light mass states. In the previous section, this matrix was unitary, U_{PMNS}, because neutrinos were Dirac fermions and this corresponded to the full mixing matrix. However, in this case, V is a unitary 6×6 matrix which means that K is not necessarily unitary. As previously stated, results from oscillation

[19]The general mathematical way would be to perform a bi-unitary diagonalization $V^\dagger M W = D$. However, using the fact that M is symmetric leads to $W = V^*$ and one obtains the diagonalization formula obtained in [eq. 2.47] that one could obtain simply from physical intuition from [eq. 2.46].

experiments are consistent with an almost unitary K [119]. By unitarity of V this implies Z is also almost unitary and that both R and S need to be suppressed and of the same order. Coming back to [eq. 2.46], after diagonalization one has:

$$L^{SI\nu SM}_{M\nu Yukawa} = -\frac{1}{2}\overline{n^c_L} d n_L - \frac{1}{2}\overline{N^c_L} d_R N_L + h.c \ , \tag{2.50}$$

using the definition of Majorana fields, one identifies 6 Majorana neutrinos:

$$n = n_L + n^c_L \ , \ N = N^c_L + N_L \ . \tag{2.51}$$

A brief treatment of the effective theory one can extract from here will be given in the next section. After that one proceeds with a general treatment of the SIνSM.

2.2.2 Effective Treatment

The previously stated effective theory [83] can be obtained from here, in a straightforward way. The key assumptions are:

$$M_R \sim d_R >> M^\nu \sim d \ , \tag{2.52}$$

where in most cases M^ν has entries of the order of the vev, at most, and M_R is assumed to take values of the order of the GUT scale - 10^{16} GeV. One can extract several relevant formulas from [eq. 2.47], such as:

$$S^\dagger M^{\nu T} = dK^T \ , K^\dagger M^\nu + S^\dagger M_R = dS^T \ , Z^\dagger M^{\nu T} = d_R R^T \ , R^\dagger M^\nu + Z^\dagger M_R = d_R Z^T \tag{2.53}$$

From the third equation one gets $R = M^\nu Z^* d_R^{-1}$, which implies that R is highly suppressed (as it should be), since $M^\nu << d_R$. From unitarity one concludes that S also needs to be highly suppressed. Under these approximations, from the second equation on [eq. 2.53] one obtains:

$$S^\dagger = -K^\dagger M^\nu M_R^{-1} \ , \tag{2.54}$$

where one set dS^T to zero because this term is small (d is suppressed while S is also suppressed) comparing to the other two. Furthermore, in approximation, [eq. 2.49] turns into:

$$\nu_L \approx K n_L \ , \ \nu'_L = \nu^c_R \approx Z N_L \rightarrow \nu_R \approx Z^* N^c_L \tag{2.55}$$

So, it can be said there is a decoupling between light and heavy neutrinos, and the heavy neutrinos are essentially sterile while the reactive neutrinos correspond to the light states. Coming back to [eq. 2.50] and using the definition of Majorana fields, one identifies 3 light Majorana neutrinos, and 3 Heavy Majorana neutrinos:

$$n = n_L + n^c_L \approx K^\dagger \nu_L + K^T \nu^c_L \ , \ N = N^c_L + N_L \approx Z^T \nu_R + Z^\dagger \nu^c_R \ , \tag{2.56}$$

where it can be seen that the degrees of freedom corresponding to negative chirality were used in the three light neutrinos and the degrees of freedom corresponding to positive chirality were used in the three heavy ones. The only missing piece of this effective approach is the promised mass term of the type $\overline{\nu_L^c} M_L \nu_L$ [83], which emerges from the deocoupling of the heavy states. Well, using [eq. 2.54] on the first equation of [eq. 2.53] one obtains:

$$d = -K^\dagger M^\nu M_R^{-1} M^{\nu T} (K^T)^{-1} \ , \tag{2.57}$$

where one can identify an effective mass matrix $M_{eff} = -M^\nu M_R^{-1} M^{\nu T}$ that is naturally small due to the suppression given by M_R^{-1}. This is a feature of the Seesaw mechanism, since it explains the small masses of the neutrinos comparing to the other fermions of the SM. Under this approximation it is safe to take K and Z as unitary since their deviations from unitarity are proportional to $RR^\dagger \sim SS^\dagger \sim (M^\nu M_R^{-1})^2$. Thus, [eq. 2.57] turns into:

$$d = K^\dagger M_{eff} K^* = K^T M_{eff}^\dagger K \ , \tag{2.58}$$

which is just a bi-diagonalization equation of a symmetric matrix like in [eq. 2.47]. The fact that d is a diagonal matrix with positive entries was used in the second equality. Using this on [eq. 2.50] while using [eq. 2.55] one obtains the expected mass term in the low energy effective part of the Lagrangian, if one recognizes $M_L = M_{eff}^\dagger$:

$$L^{SI\nu SM}_{M\nu Yukawaeff} = -\frac{1}{2}\overline{n_L^c} K^T M_{eff}^\dagger K n_L + h.c = -\frac{1}{2}\overline{\nu_L^c} M_{eff}^\dagger \nu_L + h.c \ , \tag{2.59}$$

which is naturally small according to 't Hooft naturalness criterion [122] - if one sets M_L to zero one recovers lepton number conservation and active neutrinos are massless.

It is also interesting to discuss CP Violation in this effective theory due to its similarities with the νSM case and its difference due to the fact that one is now assuming that neutrinos are Majorana particles. Since, under this approximation, K is unitary, one would naively conclude that K would correspond to U_{PMNS}. However, due to the effective mass term on [eq. 2.59], neutrino rephasings are not allowed because the effective Lagrangian is not invariant to such transformations. Thus, one can only remove N_f phases, due to charged leptons rephasing, which results in:

$$K \approx U_{PMNS} \cdot F \ , \tag{2.60}$$

where U_{PMNS} is defined on [eq. 2.24] and $F = Diag(1, e^{i\alpha_1}, e^{i\alpha_2})$. Of course that this U_{PMNS} is not the same as in [eq. 2.16. From now on, one uses a more general definition for U_{PMNS}, as the unitary part of the light neutrino mixing matrix, with only one phase. This way both mathematical definitions [eq. 2.16] and [eq. 2.58] are consistent with this definition and can be compared with oscillations data. The total number of phases is now 3 because $N_{ph} = N_f^2 - N_f - \frac{N_f(N_f-1)}{2} = \frac{N_f(N_f-1)}{2}$. These two extra phases are known as Majorana Phases.

The novelty of this model, concerning rephasing invariants, are the new ones, comparing to the νSM:

$$D_k^{ij} = K_{ki} K_{kj}^* \ , \tag{2.61}$$

which stem from the fact that neutrino rephasings are not allowed. One can also construct new unitarity triangles from these rephasing invariants - Majorana Unitarity triangles, which share the same property as Dirac-type ones - their area is equal to half of the Jarlskog invariant [eq. 2.23]. Note that Majorana phases always cancel in the quartets, and thus CP violation can be divided in two types: Dirac type and Majorana type. The equivalent of the Jarlskog rephasing invariant for Majorana type CP Violation is:

$$S_k^{ij} = Im|K_{ki} K_{kj}^*| \ , \tag{2.62}$$

where it can be proved that only two are independent and are proportional to α_1 and α_2. Majorana type CP Violation would occur if $J = 0$ and one of the S_k was different from zero. The conditions for CP invariance translate into:

$$K_{ij}^* = K_{ij} \rho_j \ , \tag{2.63}$$

where $\rho_j = -i\eta_{CP}(\nu_j) = \pm 1$ and $\eta_{CP}(\nu_j) = \pm i$ is the CP parity of the neutrino ν_j [20]. Differently from the case of Dirac fermions, here CP is conserved not only if K is real but also if the entries of K are either real or purely imaginary. For a general review on CP Violation in the Leptonic sector one recommends [123, 124].

2.2.3 Exact Treatment

Returning to the Exact treatment, one proceeds where one left off, after the identification of the mass states on [eq. 2.50]. From [eq. 2.49]] one obtains a useful equation:

$$\nu_R = S^* n_L^c + Z^* N_L^c \ . \tag{2.64}$$

First, it's important to confirm the consistency in terms of degrees of freedom. The equivalent of [eq. 2.56] in the exact treatment is:

$$n = n_L + n_L^c = K^\dagger \nu_L + R^\dagger \nu_R^c + K^T \nu_L^c + R^T \nu_R \ , \ N = N_L^c + N_L = S^T \nu_L^c + Z^T \nu_R + S^\dagger \nu_L + Z^\dagger \nu_R^c \ , \tag{2.65}$$

where [eq. 2.49] was used. Differently from the effective approach, both neutrino mass states have contributions from the degrees of freedom corresponding to negative chirality and positive chirality. However, since one knows from experiment and unitarity constraints that K and Z are almost unitary, one concludes that light neutrino states are essentially composed of the negative chiral fields in the

[20] CP parities are quantities defined in the case of CP invariance. In that case M_{eff} is real and is diagonalized by a real orthogonal matrix O ($\delta = 0$ by this). However, all the eingenvalues m_i might not be real, but one can choose a basis where m_1 is always real and positive and only m_2 and m_3 can be negative. In that case, the diagonalizing matrix is $K = OU$, where U is a diagonal matrix with $i's$ on the lines that correspond to a negative eigenvalues m_i ($i \neq 1$) and 1's on the rest. U contains the Majorana phases. After this all mass eigenvalues are real and positive and CP is conserved since the Majorana phases can only be 0 or π.

interaction basis while the heavy states are essentially composed of positive chiral fields in interaction basis. Rewriting the whole Lagrangian in terms of the mass states [eq. 2.50], [eq. 2.41] transforms into:

$$
\begin{aligned}
L^{SI\nu SM}_{IntYukawa} = &- \frac{g}{\sqrt{2}m_W}(\overline{n_L}K^\dagger + \overline{N_L}R^\dagger)\phi^+ m_l l_R - \frac{g}{2m_W}\overline{l_L}Hm_l l_R - \frac{ig}{2m_W}\overline{l_L}\phi_Z m_l l_R \\
&- \frac{g}{2m_W}(\overline{n_L}K^\dagger + \overline{N_L}R^\dagger)HM^\nu(S^*(n_L)^c + Z^*(N_L)^c) \\
&+ \frac{ig}{2m_W}(\overline{n_L}K^\dagger + \overline{N_L}R^\dagger)\phi_Z M^\nu(S^*(n_L)^c + Z^*(N_L)^c) \\
&+ \frac{g}{\sqrt{2}m_W}\overline{l_L}\phi^- M^\nu(S^*(n_L)^c + Z^*(N_L)^c) + h.c + quark\ terms\ ,
\end{aligned}
\tag{2.66}
$$

where [Eqs. 2.50], 2.64] were used and M^ν is the neutrino Dirac mass matrix in the basis where the charged lepton Dirac mass matrix is diagonal (same meaning as in [eq. 2.41]). The neutrino diagonal mass terms are given in [eq. 2.50] and the leptonic part of [eq. 2.37] transforms into:

$$
L^l_W = -\frac{g}{\sqrt{2}}\left[W^+_\mu(\overline{n^i_L}K^\dagger_{ik} + \overline{N^i_L}R^\dagger_{ik})\gamma^\mu l^k_L + W^-_\mu \overline{l^k_L}\gamma^\mu(K_{kj}n^j_L + R_{kj}N^j_L)\right]\ , \tag{2.67}
$$

and

$$
\begin{aligned}
L^l_{A,Z} = &-\frac{g}{2\cos\theta_w}\left[Z_\mu\left((\overline{n^i_L}K^\dagger_{ik} + \overline{N^i_L}R^\dagger_{ik})\gamma^\mu(K_{kj}n^j_L + R_{kj}N^j_L) - \overline{l^i_L}\delta_{ij}\gamma^\mu l^j_L\right)\right] - \\
&\left[\left(\frac{g\sin^2\theta_w}{\cos\theta_w}Z_\mu + eA_\mu\right)(\overline{l^i_L}\delta_{ij}\gamma^\mu l^j_L + \overline{l^i_R}\delta_{ij}\gamma^\mu l^j_R)\right]\ ,
\end{aligned}
\tag{2.68}
$$

which should be compared with [Es. 2.8, 2.9]. It's important to note that the terms from L^l_W can be rewritten as:

$$
\begin{aligned}
&W^-_\mu \overline{l^k_L}\gamma^\mu K_{kj}n^j_L = W^-_\mu \overline{l^k}\gamma^\mu P_L K_{kj}n^j = W^-_\mu \overline{l^k}K_{kj}\gamma^\mu P_L C\overline{n^j}^T = -W^-_\mu K_{kj}(l^k)^{c^T}C^{-1}\gamma^\mu P_L C\overline{n^j}^T \\
&= W^-_\mu \overline{n^j}K_{kj}C(\gamma^\mu P_L)^T C^{-1}(l^k)^c = -W^-_\mu K_{kj}\overline{n^j}(P_L\gamma^\mu)(l^k)^c\ ,
\end{aligned}
\tag{2.69}
$$

where the commutation relations between γ_μ and γ_5, the properties of the C matrix, [eq. 2.29] for n , [eq. B.1] and the anti-commutation of fermion fields were used - a transposition in spinor space was performed. A term like this explicitly violates lepton number and is useful when dealing with Dirac fermion number violating processes like $W^+ + W^+ \to e^+ + e^+$. When dealing with processes that conserve Dirac fermion number one does not need to care about such term. Note that for Dirac neutrinos this term is impossible - the second equality is false in that case. When discussing the Feynman rules of this model a clarification about this will be given. Note that the same transformation can be made to the Yukawa terms that concern charged leptons, neutrinos and charged Goldstone bosons and to neutrino-neutrino terms. Naturally, neutrino-neutrino terms yield a trivial equality do to their Majorana Character[21]. However, for charged Yukawa terms, the Dirac space part of the vertex is proportional to $C(P_{L,R})^T C^{-1} = P_{L,R}$, thus, no minus sign.

Along with this, a large number of new interactions appears in this model, and one of the most noticeable are the flavor Changing Neutral Currents (FCNC) of the light states, that are naturally suppressed,

[21]This proves that processes only involving Majorana particles are not a test of lepton number conservation.

as they should be, because they're proportional to the deviations of unitarity of K. Furthermore, there are new interactions between the light and heavy states, and also between heavy states and charged leptons. This generates new exotic decays (like $N \to Wl$), possible at high energy. It's possible that these decays are not CP invariant [22], this would be a source of CP Violation known as indirect CP Violation. In the leptonic sector, CP Violation in the decay is usually understood to be CP Violation at High Energies because only heavy states can decay[23].

If this kind of CP violation happened in the early universe, an interesting thing could have happened. To achieve this, one should consider the symmetric phase, before SSB, where all particles are massless but the neutrino states with positive chirality, with Yukawa Lagrangian given in [eq. 2.38]. In this epoch one identifies the heavy neutrino mass states $N_j = W^T \nu_R$, where $d_R = W^T M_R W = Diag(M_1, M_2, M_3)$ and W is a unitary matrix, since M_R is complex symmetric[24]. The relevant interactions are given in [eq. 2.39] making the changes $\phi^\pm \to G^\pm$, $\frac{H}{\sqrt{2}} \to G^0$ and $Y^\nu \to U_L^{l\dagger} \cdot Y^\nu \cdot W = Y'^\nu$. One works on a weak basis where besides a diagonal M_R one would have a diagonal M^l (if it existed), such that there is a connection between this Yukawa matrix and the one in the generalized mass matrix of the SIνSM. In this phase, when the temperature of the universe reaches the order the mass of these positive chirality neutrinos (assumed to be very heavy), these can decay and produce a CP asymmetry ϵ_j. The asymmetry is not washed out because their decay happens out-of-equilibrium[25] [125], yielding a CP asymmetry that generates Leptogenesis that can be converted in Baryogenesis with the B + L violation contained in the SM, and generate the observed asymmetry between matter and anti-matter [126]. This is known as thermal Leptogenesis [124]. The CP asymmetry is defined as [126]:

$$\epsilon_j = \frac{\Gamma(N_j \to \tilde{\phi}\overline{\Psi}) - \Gamma(N_j \to \tilde{\phi}^\dagger \Psi)}{\Gamma(N_j \to \tilde{\phi}\overline{\Psi}) + \Gamma(N_j \to \tilde{\phi}^\dagger \Psi)} \ , \tag{2.70}$$

where $\Gamma(N_j \to \tilde{\phi}\overline{\Psi}) = \sum_i \left[\Gamma(N_j \to G^0 \overline{\nu_i} + \Gamma(N_j \to G^- \overline{l_i})\right]$
and $\Gamma(N_j \to \tilde{\phi}^\dagger \Psi) = \sum_i \left[\Gamma(N_j \to G^{0*} \nu_i) + \Gamma(N_j \to G^+ l_i)\right]$.
At tree level, $\Gamma(N_j \to \tilde{\phi}\overline{\Psi})$ and $\Gamma(N_j \to \tilde{\phi}^\dagger \Psi)$ have the same value:

$$\Gamma(N_j \to \tilde{\phi}\overline{\Psi}) = \sum_i \frac{Y'^\nu_{ji} Y'^{\nu *}_{ij}}{16\pi} M_j = \frac{(Y'^\nu Y'^{\nu\dagger})_{jj}}{16\pi} M_j = \frac{(Y^\nu Y^{\nu\dagger})_{jj}}{16\pi} M_j \ , \tag{2.71}$$

because there is only one possible diagram for each case. Note that $Y'^\nu Y'^{\nu\dagger} = U_L^{l\dagger} Y^\nu W W^\dagger Y^{\nu\dagger} U_L^l = U_L^{l\dagger} Y^\nu Y^{\nu\dagger} U_L^l \equiv Y^\nu Y^{\nu\dagger}$[26], since W is unitary. Thus, $\epsilon_j^{tree} = 0$. To achieve a non-zero value one has to go to at least one-loop order, where the interference of the tree level contribution and the one-loop

[22] One decay is more likely than its CP conjugate. This is only possible when there is more than one diagram for the decay, since the CP violation stems from the interference between them.

[23] Light states, due to their mass scale, are kinematically forbidden to decay to other SM particles.

[24] C.f. Takagi Factorization.

[25] For a reaction to be considered out-of equilibrium, its rate in one direction ($N \to lG$) must be bigger than the other $lG \to N$). When the temperature of the universe drops below a certain value, N decays much more than it is produced via the inverse reaction, and the produced asymmetry is not washed out.

[26] In the last equality we made the redefinition $U_L^{l\dagger} Y^\nu \to Y^\nu$, since this quantity coincides with the Y'^ν in [eq. 2.41], and the prime was dropped from that point on: no prime in [eq. 2.66 although it is the same quantity.] but with the prime dropped.

corrections in the decay width can generate it [127]. At one-loop level:

$$\epsilon_j = \frac{1}{8\pi(Y^\nu Y^{\nu\dagger})_{jj}} \sum_{i\neq j} \left[\frac{M_i}{M_j} \left(\left(1 + \frac{M_i^2}{M_j^2} \right) \cdot Log \left[\frac{\frac{M_i^2}{M_j^2}}{1+\frac{M_i^2}{M_j^2}} \right] + \frac{2-\frac{M_i^2}{M_j^2}}{1-\frac{M_i^2}{M_j^2}} \right) Im[(Y^\nu Y^{\nu\dagger})_{ji}^2] \right] \quad , \tag{2.72}$$

as stated in [127]. Considering highly hierarchical positive chirality neutrinos - $\frac{M_i^2}{M_j^2} >> 1$ - yields the famous result of the produced asymmetry in the decay of the lightest heavy neutrino:

$$\epsilon_1 \approx -\frac{3}{16\pi} \sum_{i\neq 1} \frac{M_1}{M_i} \frac{Im[(Y'^\nu Y'^{\nu\dagger})_{1i}^2]}{(Y'^\nu Y'^{\nu\dagger})_{11}} \ , \tag{2.73}$$

as in refs [126, 124]. To achieve the observed baryon asymmetry, $Y_B \sim 10^{-11}$, the CP asymmetry must be around $\epsilon_1 \sim 10^{-6}$, for thermal Leptogenesis. However, thermal Leptogenesis is based on the assumption that heavy neutrinos are efficiently generated by thermal scatterings during the reheating stage after inflation. In the scenario in which the heavy neutrinos are hierarchical in mass, successful Leptogenesis requires a specific range of mass for the lightest heavy neutrino [124, 128]. In the resonant Leptogenesis scenario [129] this tension may be avoided: if the heavy neutrinos are nearly degenerate in mass, self-energy contributions to the CP asymmetries may be resonantly enhanced, thus making thermal Leptogenesis viable at temperatures as low as the TeV.

Furthermore, one can observe that this CP asymmetry is not necessarily related to the CP violating phases of the mixing matrix, since one can obtain a non-zero ϵ_j if only $Y^\nu Y^{\nu\dagger}$ has non-real entries, i.e, non-removable phases, in a weak basis where M_R is diagonal. For more on Leptogenesis, one recommends [121, 124, 130, 131, 132].

The other kind of CP violation, related with the phases of the mixing matrix, is known as direct CP violation or low energy CP violation, since the CP violating phases reveal themselves in low energy processes. In the exact treatment of Seesaw type I models, K assumes a different form and has new CP violating sources:

$$K = U_{PMNS} \cdot F \cdot H_R \ , \tag{2.74}$$

using the polar decomposition theorem[27], $U_{PMNS} \cdot F$ parametrizes the unitary part while H is hermitian and parametrizes the deviations of unitary of K. In this scenario, K has a total of 18 parameters (9 from the unitary part and 9 from the hermitian part), however due to charged lepton rephasings one can remove 3 phases from K, that can be chosen to be removed from the unitary part, getting K as in [eq. 2.74] with a total of 15 parameters. The rephasing invariants are the same as in the effective treatment, but their actual values are not the same and they might be a function of the phases of H, which are also CP violating phases [133]. For instance, in the effective treatment S_1^{23} is given by:

$$S_1^{23eff} = Im|c_{13}s_{12}s_{13}e^{i(\delta+\phi_1-\phi_2)}| = |c_{13}s_{12}s_{13}\sin(\delta+\phi_1-\phi_2)| \ , \tag{2.75}$$

[27]Any general invertible matrix M can be decomposed in $M = U \cdot H$, where U is unitary and H is hermitian

but if one parametrizes H as:

$$H_R = \begin{pmatrix} h_1 & ae^{i\theta_1} & be^{i\theta_2} \\ ae^{-i\theta_1} & h_2 & ce^{i\theta_3} \\ be^{-i\theta_2} & ce^{-i\theta_3} & h_3 \end{pmatrix} , \tag{2.76}$$

where $a, b, c, h_1, h_2, h_3, \theta_1, \theta_2, \theta_3$ are reals, one gets:

$$\begin{aligned} S_1^{23} =& Im|(a \cdot c_{12}c_{13}e^{i\theta_1} + c \cdot s_{13}e^{i(\alpha_2-\alpha_3-\delta)} + c_{13} \cdot h_2 e^{i\alpha_1} s_{12})(b \cdot c_{12}e^{-i\theta_2}c_{13} + c \cdot e^{-i(\alpha_1+\theta_3)}s_{12}c_{13} + h_3 \cdot e^{i(\delta-\alpha_2)}s_{13})| \\ &= |a \cdot b \cdot c_{12}^2 c_{13}^2 \sin(\theta_1 - \theta_2) - a \cdot c \cdot c_{12}c_{13}^2 s_{12} \sin(\alpha_1 - \theta_1 + \theta_3) \\ &+ a \cdot c_{12}c_{13}h_3 s_{13} \sin(\delta - \alpha_2 + \theta_1) - b \cdot c \cdot c_{12}c_{13}s_{13} \sin(\delta - \alpha_2 + \theta_2 + \theta_3) + b \cdot c_{12}c_{13}^2 \cdot h_2 \cdot s_{12} \sin(\alpha_1 - \theta_2) \\ &- c^2 \cdot c_{13}s_{12}s_{13} \sin(\delta + \alpha_1 - \alpha_2 + 2\theta_3) - c \cdot c_{13}^2 \cdot h_2 \cdot s_{12}^2 \sin(\theta_3) - c \cdot h_3 s_{13}^2 \sin(\theta_3) \\ &+ c_{13} \cdot h_2 \cdot h_3 \cdot s_{12}s_{13} \sin(\delta + \alpha_1 - \alpha_2)| , \end{aligned} \tag{2.77}$$

which is pretty different from [eq. 2.75] and reveals $\theta_1, \theta_2, \theta_3$ as possible CP violating phases. These phases may alter the value of the Dirac phase δ one obtains when fitting oscillations experimental data to the effective model [134].

To summarize this model, the extra Feynman Rules one needs to add to the SM are given at Appendix B.

2.3 Neutrino Oscillations in Vacuum and Matter

In this section one will cover neutrino oscillations in vacuum and in matter. The framework used will be the effective treatment of Seesaw type I, which gives the same results as using the νSM, since the Majorana phases cancel in all observables. In the end, the consequences of using the exact treatment instead of the aforementioned one will be discussed.

Neutrino Oscillations are a quantum mechanical phenomenon that happens due to non-zero neutrino masses and mixing and the small mass difference between mass states. Our current experiments don't have the sensitivity to distinguish between mass states. We can only know with certainty, which interaction state (linear combination of mass states) they were in when its charged lepton counterpart is detected.

If one defines the "neutrino state of flavor α" as the neutrino that is created or detected together with a charged lepton $\overline{l_\alpha}$ in a leptonic W^+ [28] decay, one can use [eq. 2.8] with the change $U \to K$ to express it as a coherent sum of mass states:

$$|\nu_\alpha> = \sum_i K_{i\alpha}^\dagger |\nu_i> = \sum_i K_{\alpha i}^* |\nu_i> , \tag{2.78}$$

where K is defined in [eq. 2.60]. If immediately after this, the ν_α interacts, the probability of producing

[28] $W^+ \to \overline{l_\alpha} + \nu_\alpha$

a l_β is proportional to:

$$< \nu_\beta|\nu_\alpha >= \sum_j \sum_i < \nu_j|K_{\beta j}K^*_{\alpha i}|\nu_i >= \sum_j \sum_i < \nu_j|\delta_{ij}K_{\beta i}K^*_{\alpha i}|\nu_i >= \sum_i K_{\beta i}K^*_{\alpha i} = \delta_{\beta\alpha} \ , \tag{2.79}$$

where the orthonormality of mass states and the unitarity of K were used. However, if before interacting, they propagate during a time t and a distance L, this probability changes dramatically. One can invert [eq. 2.78], obtaining:

$$|\nu_i >= \sum_\alpha K_{\alpha i}|\nu_\alpha > \ , \tag{2.80}$$

Since the massive neutrino states [eq. 2.80] have definite mass and energy, they evolve in time as plane waves, solutions of the time-independent Schrodinger equation.

$$i\frac{\partial}{\partial_t}|\nu_i(t) >= H|\nu_i(t) >= E_i|\nu_i(t) > \rightarrow |\nu_i(t) >= e^{-iE_it}|\nu_i > \ , \tag{2.81}$$

where H is the Hamiltonian operator and $|\nu_i >= |\nu_i(t = 0) >$. Using this and [Eqs. 2.78, 2.80] one obtains the time evolution of the flavor state α:

$$|\nu_\alpha(t) >= \sum_i K^*_{\alpha i}e^{-iE_it}|\nu_i >= \sum_\beta \left(\sum_i K^*_{\alpha i}e^{-iE_it}K_{\beta i} \right) |\nu_\beta > \tag{2.82}$$

which shows that if the mixing matrix K is not the identity matrix, for $t > 0$ ν_α is a superposition of different flavors. The quantity in parentheses in [eq. 2.82) is the amplitude of the transition $\nu_\alpha \rightarrow \nu_\beta$ at time t after the production of ν_α, whose squared absolute value gives the probability of the transition $\nu_\alpha \rightarrow \nu_\beta$:

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = | < \nu_\beta|\nu_\alpha(t) > |^2 = | \sum_i K^*_{\alpha i}e^{-iE_it}K_{\beta i}|^2 = \sum_i \sum_j K^*_{\alpha i}K_{\beta i}K_{\alpha j}K^*_{\beta j}e^{-i(E_i-E_j)t} \ , \tag{2.83}$$

where one can note that this is independent of the Majorana phases, as previously stated, since this is proportional to the quartet $Q_{\alpha j\beta i}$. One can see that $P_{\nu_\alpha \rightarrow \nu_\beta}(t)$ depends on the energy differences $E_i - E_j$. In the standard approach to neutrino oscillations it is assumed that all massive neutrinos have the same momentum p, i.e. detectable neutrinos are ultra-relativistic:

$$E_i = \sqrt{p^2 + m_i^2} \approx E + \frac{m_i^2}{2E} \rightarrow E_i - E_j = \frac{\Delta m^2_{ij}}{2E} \ , \tag{2.84}$$

where $E \equiv |p|$ is the energy of a massless neutrino, and $\Delta m^2_{ij} = m_i^2 - m_j^2$. Under the ultra-relativistic neutrino approximation, it is safe to assume that the time of propagation T is given by the distance propagated:

$$T \approx L \ , \tag{2.85}$$

in natural units. This way one can rewrite $P_{\nu_\alpha \to \nu_\beta}(t)$ only in terms of known or measurable quantities:

$$P_{\nu_\alpha \to \nu_\beta}(L) = \sum_i \sum_j K^*_{\alpha i} K_{\beta i} K_{\alpha j} K^*_{\beta j} e^{-i\left(\frac{\Delta m^2_{ij}}{2E}\right)L} = \delta_{\alpha\beta} - 4 \sum_{i>j} Re\left(K^*_{\alpha i} K_{\beta i} K_{\alpha j} K^*_{\beta j}\right) \sin^2\left(\frac{\Delta m^2_{ij} \cdot L}{4E}\right)$$
$$+ 2 \sum_{i>j} Im\left(K^*_{\alpha i} K_{\beta i} K_{\alpha j} K^*_{\beta j}\right) \sin\left(\frac{\Delta m^2_{ij} \cdot L}{2E}\right). \tag{2.86}$$

[eq. 2.86] is valid for any unitary $n \times n$ mixing matrix, with n neutrino species. For $n = 3$ using the parametrization given in [eq. 2.24] for K, and ignoring the Majorana phases, one can rewrite it only in terms of the Euler angles θ_{ij} and the Dirac CP Violating phase δ. Note that the term $\propto Im\left(K^*_{\alpha i} K_{\beta i} K_{\alpha j} K^*_{\beta j}\right)$ is proportional to the Jarlskog rephasing invariant J which is a function of Dirac CP Violating phase δ. This term explicitly violates CP, and this statement becomes obvious when one analyzes the probability $P_{\overline{\nu_\alpha} \to \overline{\nu_\beta}}$:

$$P_{\overline{\nu_\alpha} \to \overline{\nu_\beta}}(L) = \sum_i \sum_j K_{\alpha i} K^*_{\beta i} K^*_{\alpha j} K_{\beta j} e^{-i\left(\frac{\Delta m^2_{ij}}{2E}\right)L} = \delta_{\alpha\beta} - 4 \sum_{i>j} Re\left(K^*_{\alpha i} K_{\beta i} K_{\alpha j} K^*_{\beta j}\right) \sin^2\left(\frac{\Delta m^2_{ij} \cdot L}{4E}\right)$$
$$- 2 \sum_{i>j} Im\left(K^*_{\alpha i} K_{\beta i} K_{\alpha j} K^*_{\beta j}\right) \sin\left(\frac{\Delta m^2_{ij} \cdot L}{2E}\right). \tag{2.87}$$

On the determination of these formulas, three main assumptions were used:

1. Neutrinos produced or detected in charged-current weak interaction processes are described by the flavor states in [eq. 2.78].
2. Massive neutrino states $|\nu_i>$ have the same momentum $E_i = E + \frac{m_i^2}{2E}$
3. The propagation time is equal to the distance L traveled by the neutrino between production and detection (in natural units).

The validity of these assumptions and how they yield the correct result independently of that is discussed in refs [135, 136].

These formulas were derived assuming the medium of propagation was the vacuum. However, we know that the neutrinos we detect could have crossed several km of the Earth, of the universe, and in the case of solar neutrinos, several km inside the sun. The probability that a neutrino of energy $E \sim MeV$ gets scattered while crossing the earth is very low. Still, the presence of matter can significantly affect neutrino propagation [65, 64, 63]. Effects due to propagation in a medium are not unheard of in Physics. The most popular one must be the propagation of light in a medium, which reduces its phase speed significantly $v_p = \frac{c}{n}$, where n is the index of refraction of the medium. These effects can be very important in the designing of experiments - the refraction of light in a medium introduced a new way of detecting highly energetic particles, using the Cherenkov effect, used, for instance, in the Super-Kamiokande neutrino experiment.

To derive the matter-effects one needs to understand the effective matter Hamiltonian. First, one should derive the vacuum one, H_0. From [eq. 2.81] and using [eq. 2.80], one obtains:

$$H^{\nu}_{0\,\alpha\beta} = K^*_{\alpha i} E_i \delta_{ij} K_{\beta j} = \frac{1}{2E} K^*_{\alpha i} Diag(m_i^2) \delta_{ij} K_{\beta j}, \tag{2.88}$$

where [eq. 2.84] was used, dropping the linear term on E since it gives rise to an overall phase factor, common to all the flavor states and thus irrelevant for oscillations. For anti-neutrinos one has:

$$H^{\overline{\nu}}_{0\,\alpha\beta} = K_{\alpha i} E_i \delta_{ij} K^*_{\beta j} = \frac{1}{2E} K_{\alpha i} Diag(m_i^2) \delta_{ij} K^*_{\beta j}. \tag{2.89}$$

In the case there are $3+p$ light neutrino states, K, and consequently, H, is a $(3+p) \times (3+p)$ flavor matrix, with K having deviations from unitary. The interaction that generates the matter effects is the electron - electron neutrino scattering, since normal matter is essentially composed of nuclei and non-relativistic and non-polarized electrons and no positrons. For low energies - below the W boson mass - this interaction is described, in an effective way, as a Fermi Interaction, by the Hamiltonian density:

$$\mathcal{H}^{\nu}_{m} = \frac{4G_F}{\sqrt{2}} \overline{\nu_e} \gamma_\alpha P_L e \bar{e} \gamma^\alpha P_L \nu_e = -\mathcal{H}^{\overline{\nu}}_{m} \ , \tag{2.90}$$

where $G_F = \frac{\sqrt{2}}{8} \frac{g^2}{m_W^2}$. From this, one understands that the the total effective Hamiltonian has a matrix representation in flavor space with previously known and new CP violating terms due to the differences between $H^{\overline{\nu}}_{0\,\alpha\beta}$ and $H^{\overline{\overline{\nu}}}_{0\,\alpha\beta}$, and $\mathcal{H}^{\nu}_{eff}$ and $\mathcal{H}^{\overline{\nu}}_{eff}$, respectively. Scattering of ν on electrons and quarks mediated by the Z boson is flavor blind, and therefore does not affect flavor transitions between active neutrinos.

The total effective Hamiltonian is given by:

$$H^{\nu}_{eff} = H^{\nu}_{0} + H^{\nu}_{m} \ , \tag{2.91}$$

where [29]

$$H^{\nu}_{m\alpha\beta} =< \mathcal{H}^{\nu}_{m} >= \sqrt{2} G_F N_e(x) \delta_{\alpha 1} \delta_{1\beta} = -H^{\overline{\nu}}_{m\alpha\beta} \ , \tag{2.92}$$

where to achieve this result one has to integrate over the space coordinates, to go from Hamiltonian density to Hamiltonian, and to calculate the matrix element of that for an initial and final state described by a single electron neutrino, localized around $x_0 = x$, and a large number of electrons almost at rest, localized in the neighbouring positions - this is a realistic description of a neutrino propagating in the matter of the Sun or of the Earth. These two operations are what is meant by $< \mathcal{H}^{\nu}_{m} >$. The result is naturally dependent on the number density $N_e(x,t) = e^\dagger e$, since for a non-relativistic scenario only the time component of $\bar{e}\gamma^\alpha P_L e$ is not negligible, as the other terms are proportional to the current, or polarization, and those are suppressed $\beta = \frac{v}{c} << 1$. Note that the existence of more than 3 light neutrino states does not alter the matter effects.

[29] It's necessary to use a Fierz identity to transform [eq. 2.90] into a more useful representation: $\mathcal{H}^{\nu}_{m} = \frac{4G_F}{\sqrt{2}} \overline{\nu_e} \gamma_\alpha P_L \nu_e \bar{e} \gamma^\alpha P_L e$

To obtain the propagation states one has to diagonalize [eq. 2.91]. For the vacuum case, the diagonalizing matrix is the leptonic mixing matrix K, as it is clear on [eq. 2.88]. For the matter dependent case, the diagonalizing matrix will contain effective oscillation parameters that depend on the neutrino energy, and of course on the matter density. The effect is thus controlled by the parameter $N_e(x)$. It can be considered constant (Earth's mantle) and it can vary with position (Sun's interior). Both scenarios have an analytical solution, under certain assumptions.

An interesting matter effect is resonance and can happen in the first case - constant $N_e(x)$. For solar neutrinos (neutrinos that are produced as electron neutrinos) considering only two flavors is enough [30], since $\nu_e = \cos\theta_{12}\cos\theta_{13}\nu_1 + \sin\theta_{12}\cos\theta_{13}\nu_2 + \sin\theta_{13}e^{-i\delta}\nu_3$ but $\cos\theta_{13} \approx 1$.

Assuming a unitary mixing matrix, one has:

$$H^{\nu}_{eff} = \frac{1}{2E}\left[\frac{m_1^2+m_2^2}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\Delta m^2}{2}\begin{pmatrix}-\cos 2\theta & \sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}\right] + \sqrt{2}G_F N_e\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad , \tag{2.93}$$

where $\Delta m^2 = m_2^2 - m_1^2$ and θ is the mixing angle since $K = \begin{pmatrix}\cos\theta & \sin\theta - \sin\theta & \cos\theta\end{pmatrix}$. This Hamiltonian can be transformed into a much simpler version, using the fact that adding a constant will add a global factor that will be the same for all flavors. Thus, not contributing to oscillations. Subtracting $\frac{Tr(H^{\nu}_{eff})}{2}$ on [eq. 2.93], one gets:

$$H'^{\nu}_{eff} = H^{\nu}_{eff} - \frac{1}{2}Tr(H^{\nu}_{eff}) = \begin{pmatrix}\frac{\sqrt{2}G_F N_e}{2} - \cos 2\theta\frac{\Delta m^2}{4E} & \sin 2\theta\frac{\Delta m^2}{4E}\\ \sin 2\theta\frac{\Delta m^2}{4E} & \cos 2\theta\frac{\Delta m^2}{4E} - \frac{\sqrt{2}G_F N_e}{2}\end{pmatrix} \quad . \tag{2.94}$$

From here one obtains the equivalence equations for the new effective parameters:

$$\cos 2\theta\frac{\Delta m^2}{4E} - \frac{\sqrt{2}G_F N_e}{2} = \cos 2\theta_m\frac{\Delta m_m^2}{4E} \ , \ \sin 2\theta\frac{\Delta m^2}{4E} = \sin 2\theta_m\frac{\Delta m_m^2}{4E} \ , \tag{2.95}$$

with solutions:

$$\tan 2\theta_m = \frac{\sin 2\theta}{\cos 2\theta - \frac{2\sqrt{2}G_F N_e E}{\Delta m^2}} \ , \ \Delta m_m^2 = \Delta m^2 \cdot \left(\sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2\sqrt{2}G_F N_e E}{\Delta m^2}\right)^2}\right) \tag{2.96}$$

The resonance happens when $\theta_m = \frac{\pi}{4}$ or, in other terms, when:

$$\cos 2\theta = \frac{2\sqrt{2}G_F N_e E}{\Delta m^2} \ , \tag{2.97}$$

which can always be satisfied for any N_e and Δm^2, provided that $\cos 2\theta$ has the same sign as Δm^2 , since E is a continuous parameter. The 2 flavor equivalent of [eq. 2.86] is:

$$P_{\nu_\alpha\to\nu_\beta}(L) = \sin^2 2\theta \sin^2\frac{\Delta m^2 L}{4E} \ , \alpha \neq \beta \tag{2.98}$$

[30] It's much simpler than considering 3 and it's illustrative of the effect.

Making the transformation $\theta \rightarrow \theta_m$ and $\Delta m^2 \rightarrow \Delta m_m^2$ one obtains the probability for the matter case. It's maximum occurs for $\theta_m = \frac{\pi}{4}$, which is the meaning of resonance. The resonance occurs for a certain value of energy:

$$E_\nu^{SI} = \frac{\Delta m^2}{2\sqrt{2} G_F N_e} \cos 2\theta \frac{c^4}{(\hbar c)^3} \ , \tag{2.99}$$

where to obtain a numerical value one is advised to use the useful relations $(\hbar c) = 197\ MeV \cdot fm$ and [31] $N_e = \frac{0.5 \times \rho}{m_p}$, with $m_p \approx 938\ MeV/c^2$, the mass of the proton. As for Δm^2 and θ, for solar neutrinos one should use the "12" subscript data given in [Table 2.2]. Under the approximation that the whole sun has a constant density and that it has its core value ($\rho \approx 100\ g\ cm^{-3}$), the resonance energy is $E_\nu \sim 1 MeV$. This is a crude approximation for the density, however it allows to take some conclusions about the general behaviour of matter effects for solar neutrinos. For low energy neutrinos $\theta_m \approx \theta$, there is no matter effect. For neutrinos with energy given by [eq. 2.99], the probability of transition between flavors is maximal. For energies above that, θ_m comes closer and closer to $\pi/2$, which means that highly energetic electron neutrinos are essentially in the ν_{2m} mass state since:

$$\nu_e = \cos\theta_m \nu_{1m} + \sin\theta_m \nu_{2m} \ , \tag{2.100}$$

and that transitions from ν_e to ν_μ are now impossible - c.f. [eq. 2.98] with $\theta_m = \pi/2$. However, if the neutrinos are detected at night this means that they crossed the Earth, and a new study of propagation would have to be made, taking this ν_e state essentially composed of ν_{2m} as the initial state.

This approximation of constant matter density is decent only for the Earth's mantle, and thus useful for atmospheric neutrinos and solar neutrinos detected at night. Of course that for atmospheric neutrinos (mostly produced as muon neutrinos) the two flavor approximation would be wrong, unless one defines one of the states as a combination of the other two.

To consider solar neutrinos in a realistic way, one would have to consider the spatial variation of N_e inside the sun. This would also translate into a time dependence, since neutrinos are propagating, and that Schrodinger's equation would have to be solved instantaneously - allowing transitions between the mass states. However one can remove this time dependence if one claims that the density changes slow enough such that the system has time to adjust to the change. This is the case since the energy splitting between mass states is very small. Thus, the system is adiabatic. Thus, highly energetic electron neutrinos, produced above the energy given by [eq. 2.99] with $\theta_m \approx \pi/2$, will remains in that eigenstate ($|\nu_{2m}>$) during all the adiabatic evolution. As the density drops to nearly zero when the neutrino leaves the sun, the state $|\nu_{2m}>$ becomes the vacuum state $|\nu_2>$ and the probability of detecting a highly energetic ν_e on Earth is:

$$P_{ee} = |<\nu_e|\nu(t)>|^2 \approx |<\nu_e|\nu_{2m}(t)>|^2 = |<\nu_e|\nu_2>|^2 \approx \sin^2\theta \ , \tag{2.101}$$

[31] Since the neutron has approximately the same mass as the proton and the mass of the electron is negligible comparing to it, $\frac{\rho}{m_p}$ gives one the number of nucleons per unit volume. It's necessary to divide by 2 to obtain the number of electrons per unit volume, as most stable matter has the same number of protons, neutrons and electrons.

which should be compared with the vacuum value, after averaging over the oscillation factor $\propto \Delta m^2$:

$$P_{ee}^{vac} = 1 - \frac{1}{2}\sin^2 2\theta \ . \tag{2.102}$$

This is known as the MSW (Mikheyev–Smirnov–Wolfenstein) effect and solved the solar neutrino problem. For a more detailed treatment one should read, besides the seminal papers [65, 64, 63], modern reviews such as [137, 138, 139].

Now that the importance of matter effects on oscillations is clarified, it is also interesting to describe the effects of a non-unitary mixing matrix on the probability of flavor oscillations. For a non-unitary K the generalization of equation [eq. 2.86] is [140, 133, 141]:

$$\begin{aligned}
P_{\nu_\alpha\to\nu_\beta}(L) = & \frac{1}{(KK^\dagger)_{\alpha\alpha}(KK^\dagger)_{\beta\beta}}\sum_i\sum_j K^*_{\alpha i}K_{\beta i}K_{\alpha j}K^*_{\beta j}e^{-i\left(\frac{\Delta m^2_{ij}}{2E}\right)L} \\
=&\frac{1}{(KK^\dagger)_{\alpha\alpha}(KK^\dagger)_{\beta\beta}}\left[|(KK^\dagger)_{\alpha\beta}|^2 - 4\sum_{i>j} Re\left(K^*_{\alpha i}K_{\beta i}K_{\alpha j}K^*_{\beta j}\right)\sin^2\left(\frac{\Delta m^2_{ij}\cdot L}{4E}\right)\right] \\
&+ \frac{1}{(KK^\dagger)_{\alpha\alpha}(KK^\dagger)_{\beta\beta}}\left[2\sum_{i>j} Im\left(K^*_{\alpha i}K_{\beta i}K_{\alpha j}K^*_{\beta j}\right)\sin\left(\frac{\Delta m^2_{ij}\cdot L}{2E}\right)\right] \ .
\end{aligned} \tag{2.103}$$

Confronting this with the unitary case, the two differences are noticeable. $\delta_{\alpha\beta}$ turns into $|(KK^\dagger)_{\alpha\beta}|^2$ and there's a overall factor of $\frac{1}{(KK^\dagger)_{\alpha\alpha}(KK^\dagger)_{\beta\beta}}$. The first one would obviously exist, but the reasoning for the second difference is not so trivial. It is a normalization factor and stems from the following relation:

$$\begin{aligned}
P(W\to\overline{l_\alpha}+\nu_\alpha) = \sum_i P(W\to\overline{l_\alpha}+\nu_i) = 1 = N^2\sum_i|K^*_{\alpha i}|^2 = N^2\sum_i K_{\alpha i}K^\dagger_{i\alpha} = N^2(KK^\dagger)_{\alpha\alpha} \ , \\
\to N = \frac{1}{\sqrt{(KK^\dagger)_{\alpha\alpha}}} \ .
\end{aligned} \tag{2.104}$$

This means that, when the mixing matrix is not unitary, [eq. 2.83] should take into account this normalization factor:

$$P_{\nu_\alpha\to\nu_\beta}(t) = |<\nu_\beta|\nu_\alpha(t)>|^2 = \left|\sum_i \frac{K^*_{\alpha i}e^{-iE_it}K_{\beta i}}{\sqrt{(KK^\dagger)_{\alpha\alpha}\sqrt{(KK^\dagger)_{\beta\beta}}}}\right|^2 = \sum_i\sum_j\frac{K^*_{\alpha i}K_{\beta i}K_{\alpha j}K^*_{\beta j}e^{-i(E_i-E_j)t}}{(KK^\dagger)_{\alpha\alpha}(KK^\dagger)_{\beta\beta}} \ , \tag{2.105}$$

and this leads to [eq. 2.103]. For oscillations in matter with deviations from unitarity the same procedure should be followed - taking into account the normalization factor. Doing this one obtains a similar set of equations for the probability of oscillation between flavors, with the difference that K is now the matrix that diagonalizes the effective matter Hamiltonian [140].

Although the effects of deviations from unitarity are very small, they are detectable. A fundamental quantity for these tests is the zero-distance term:

$$P_{\nu_\alpha\to\nu_\beta}(L=0) = \frac{1}{(KK^\dagger)_{\alpha\alpha}(KK^\dagger)_{\beta\beta}}\left[|(KK^\dagger)_{\alpha\beta}|^2\right] \ , \alpha\neq\beta \ . \tag{2.106}$$

Note that if there are deviations from unitary the probability is non-zero, contrarily to what happens when the mixing matrix is unitary. This translates into a physical effect: a flavor transition already at the source before oscillations can take place. This can be tested in nuclear neutrinos near detectors.

2.4 State of the Art of Neutrino Physics

Although deviations from unitarity of the leptonic mixing matrix might be a reality, the current paradigm on the interpretation of neutrino oscillation data is to assume that neutrinos are Majorana, the heavy states have a much bigger mass and, thus, the deviations from unitarity are negligible. Theoretical work has been performed in order to constrain the possible deviations from unitarity, using data from weak decays and from the search of Lepton flavor Violating (LFV) decays [142, 143, 144]. The most recent bounds are [142]:

$$I - |KK^\dagger| \leq \begin{pmatrix} 2.5 \times 10^{-3} & 2.4 \times 10^{-5} & 2.7 \times 10^{-3} \\ 2.4 \times 10^{-5} & 4.0 \times 10^{-4} & 1.2 \times 10^{-3} \\ 2.7 \times 10^{-3} & 1.2 \times 10^{-3} & 5.6 \times 10^{-3} \end{pmatrix} , \tag{2.107}$$

which reveals that the claim that deviations from unitarity are negligible is arguable. More on this on the next chapters. Thus, oscillation data is fitted to the effective model, with the mixing matrix, K, defined as in [eq. 2.60]. Several groups are currently performing global phenomenological fits on θ_{12}, θ_{23}, θ_{13} and δ, as well as on neutrino mass differences [145, 146]. The specific bounds vary slightly from group to group. For definiteness, in [Table 2.2], is present the current bounds on neutrino masses and parameters of the mixing matrix K defined as in [eq. 2.60] from [146]. The quantities Δm^2_{ij} are defined by $(m_i^2 - m_j^2)$.

Parameter	Best fit	1σ range	3σ range
Δm^2_{21} $[10^{-5} eV^2]$	7.55	7.39 – 7.75	7.05 – 8.14
$\|\Delta m^2_{31}\|$ $[10^{-3} eV^2](NO)$	2.50	2.47 – 2.53	2.41 – 2.60
$\|\Delta m^2_{31}\|$ $[10^{-3} eV^2](IO)$	2.42	2.38 – 2.45	2.31 – 2.51
$\sin^2\theta_{12}$	0.320	0.304 – 0.340	0.273 – 0.379
$\sin^2\theta_{23}(NO)$	0.547	0.517 – 0.567	0.445 – 0.599
$\sin^2\theta_{23}(IO)$	0.551	0.521 – 0.569	0.453 – 0.598
$\sin^2\theta_{13}(NO)$	0.02160	0.02091 –0.02243	0.0196 –0.0241
$\sin^2\theta_{13}(IO)$	0.02220	0.02144 –0.02294	0.0199 –0.0244
$\delta/\pi(NO)$	1.32	1.17 –1.53	0.87 –1.94
$\delta/\pi(IO)$	1.56	1.41 –1.69	1.12 –1.94

Table 2.2: Neutrino oscillation parameter summary from [146]. For Δm^2_{31}, $\sin^2\theta_{23}$, $\sin^2\theta_{13}$, and δ the upper (lower) row corresponds to normal (inverted) neutrino mass hierarchy.

As one can observe, two distinct cases are considered - Normal ordering (NO) and Inverted Ordering (IO). This happens because the existing data does not allow one to determine the sign of Δm^2_{31}. Thus, both possibilities are considered:

- Normal Ordering: $m_1 < m_2 < m_3$
- Inverted Ordering: $m_3 < m_1 < m_2$

This is known as the neutrino mass ordering problem. Below there is a summary of the biggest open problems, concerning the three known neutrinos:

1. Is the mass ordering Inverted or Normal?
2. Is there CP Violation in the Leptonic Sector?
3. Are the masses hierarchical or quasi-degenerate?
4. What is the mass scale of neutrinos? (From oscillation experiments we can only get their mass differences squared.)
5. Are Neutrinos Majorana or Dirac?

One could say the first problem is solved, since the most recent global fits favour the normal mass ordering over the inverted one at more than 3σ [145, 146]. Concerning the second question, one cannot declare that CP exists in the leptonic sector because the CP conserving $\delta = \pi$ is not yet ruled out, however, it is disfavoured and every data points towards $\delta \sim \frac{3\pi}{2}$. However, such claims must be taken with a grain of salt, since the performed global fits uses a unitary matrix by construction ([eq. 2.60]) [119][32].

Questions number 3 and 4 are related, since answering 4 might answer 3. Neutrinos can be quasi-degenerate if their mass is much bigger than their mass differences : $m_i \geq 0.1\ eV$. Nevertheless, cosmological studies have several bounds for their mass scale:

$$m.s. : \sum_i m_i \leq 0.12\ eV\ , m.m. : \sum_i m_i \leq 0.72\ eV\ , \tag{2.108}$$

at the 95% CL [89], where m.s. stands for most strict and m.m. stands for most moderate. The difference between the bounds stems from the distinct data sets and assumptions used [89]. To understand how these discrepancies are possible, it's instructive to understand how these bounds are determined. General relativity gives one a relation between the scale factor of the Friedmann-Robertson-Walker metric[33], $a(t)$, and the matter and energy in the Universe, through the time-time component of Einstein's equations for a flat universe:

$$\left(\frac{\dot{a(t)}}{a(t)}\right)^2 = H(t)^2 = \frac{8\pi G\rho(t)}{3} = H_0^2\frac{\rho(t)}{\rho_c^0} = H_0^2\Omega(t) = H_0^2(\Omega_\gamma(t) + \Omega_{dm}(t) + \Omega_b(t) + \Omega_\nu(t) + \Omega_\Lambda)\ , \tag{2.109}$$

[32]Interpretations could change if, for instance, sizable (close to the bound) deviations from unitarity or a KeV "heavy" neutrino were discovered.

[33]A metric that can describe an expanding flat universe like ours.

where $\rho_c^0 = \frac{3H_0^2}{8\pi G} \approx 1.774\times10^{-29}h^2\ g\ cm^{-3}$[34] is the critical density and H_0 is the Hubble parameter, both at an epoch t_0 that can be set to the current time. Ω_i are the energy densities normalized to the critical density at t_0. They correspond to: $i = \gamma \rightarrow$ photons, $i = dm \rightarrow$ "dark matter", $i = b \rightarrow$ barionic matter, $i = \nu \rightarrow$ neutrinos and $i = \Lambda \rightarrow$ "dark energy" (the energy density that comes from the introduction of the cosmological constant Λ in Einstein's equations). Note that for a flat universe, if ρ is measured at time t_0, Ω should be equal to one[35]. Using thermal models of the evolution of the universe, Ω_i values can be inferred from the observations of astronomical objects or from the Cosmic Microwave Background (CMB) . Nevertheless, different methods and data sets yield different results[36]. Of these parameters, only $\Omega_\gamma = 2.47 \times 10^{-5}h^2$ is accurately measured directly, from the CMB observations. Fortunately, Ω_ν is inferred from that. Through thermal models of the evolution of the universe it's possible to predict $\frac{\rho_\nu}{\rho_\gamma} = \frac{\Omega_\nu}{\Omega_\gamma}$ and $\frac{n_\nu}{n_\gamma}$, inferring Ω_ν and n_ν from that, since one obtains Ω_γ and n_γ from the CMB. Assuming that the neutrino number densities have all the same value $n_i \sim n_\nu$ and $\rho_\nu \approx \sum_i n_i m_i \approx n_\nu \sum_i m_i$, one obtains:

$$\Omega_\nu = \frac{\rho_\nu}{\rho_c^0} = \frac{\sum_i m_i}{n_\nu^{-1}\rho_c^0} = \frac{\sum_i m_i}{93.14h^2 eV}\ , \tag{2.110}$$

where the value $n_\nu = 339.5cm^{-3}$ [89] was used. As one can conclude several approximations and model-dependent assumptions where taken. The bound is highly dependent on the parameters of the thermal model used to determine Ω_ν and n_ν and on the measured values H_0, Ω_γ and n_γ. The thermal model also predicts an effective number of neutrinos. Again there are several model/assumption-dependent bounds. The highest and the lowest are presented:

$$N_{eff} = 3.08 \pm 0.31\ , N_{eff} = 3.41 \pm 0.22\ , \tag{2.111}$$

at the 68% CL [89]. N_{eff} is the number of neutrinos that are non-relativistic[37] at the present time and that decoupled from the thermal plasma at temperatures $T_d \sim 2\ MeV$. This means that neutrinos with masses below $T_\nu^0 K \approx 1.7\times10^{-4}\ eV$ [89] and above $T_d \sim 2\ MeV$ [89] don't contribute to this parameter[38].

Thus, a cosmological bound is not the most trustworthy to convince oneself about the possible quasi-degeneracy of neutrino masses, although it provides an hint on their mass scale. Both questions can be answered if KATRIN yields a positive result. As discussed in the first chapter, KATRIN is an experiment that has the goal of measuring the quantity m_β, from β decay : $n \rightarrow p + e^- + \overline{\nu_e}$. The experiment is projected to be sensitive to $m_\beta > 0.2\ eV$, and started acquiring data on June 2018. But what is m_β? To study β decay is convenient to consider the Kurie function:

$$\begin{aligned} K^2(E_e) &= (Q - E_e)\sum_i |K_{ei}|^2\sqrt{(Q-E_e)^2 - m_i^2} \times \Theta(Q - T_e - m_i) \\ &\approx (Q - E_e)\sqrt{(Q-E_e)^2 - m_\beta^2} \times \Theta(Q - E_e - m_\beta)\ , \end{aligned} \tag{2.112}$$

[34] $h = H_0/(100\ kms^{-1}Mpc^{-1})$

[35] Using this one can estimate the "dark energy" energy density Ω_Λ and the "dark matter" energy density Ω_{dm}. Surprisingly, the values consistent with observations are ~ 0.70 and ~ 0.25, respectively.

[36] For instance, The Hubble Space Telescope Project measured $H_0 = 73.2 \pm 1.7km\ s^{-1}Mpc^{-1}$, while the Planck Collaboration found a lower value, $H_0 = 67.8 \pm 0.9km\ s^{-1}Mpc^{-1}$.

[37] With masses much bigger than $T_\nu^0 \approx 1.9K \approx 1.7\times10^{-4}\ eV$

[38] From oscillation data one concludes that the lightest neutrino can have a mass below $T_\nu^0 \approx 1.7\times10^{-4}\ eV$.

where E_e is the electron energy, Q is the amount of energy released by the reaction and Θ is the Heaviside step function. The approximation is valid for an experiment in which the energy resolution is such that $m_k << Q - E_e$. m_β is the "electron neutrino mass" and the Kurie function depends on it thanks to the phase space factor. It is determined as a real average over all mass eigenstates contributing to the electron neutrino:

$$m_\beta^2 = \sum_i |K_{ei}|^2 m_i^2 \ , \tag{2.113}$$

where K is defined in [eq. 2.60]. Note that this is not sensitive to Majorana phases nor the Dirac phase δ, as they cancel. Ideally, with enough resolution, one could determine exactly the mass of i^{th} neutrino m_i. However, one only detects the final electron and does not know which neutrino mass state was produced, since it is not possible to resolve between them, due to their very small mass splittings. If KATRIN has a positive result, i.e, $m_\beta > 0.2\ eV$, due to the mass differences one knows from oscillation experiments this would mean that neutrinos are quasi-degenerate.

In the exact treatment one should use [eq. 2.113] with K given by [eq. 2.74]. In this scenario, not only m_β^2 is sensitive to the CP Violating phases θ_1,θ_2, θ_3 phases that come from the hermitian part of K [Eqs. 2.74, 2.76] but it's also sensitive to the Majorana phases α_1, α_2 and to the Dirac phase δ.

Another scenario is possible, if k of the three "heavy" states have a mass of the order of the light neutrinos (or higher, given that it is kinematically allowed). In that case, the electron energy spectrum would be a superposition of the light neutrino spectrum and the "heavy" neutrino spectrum, implying that [eq. 2.112] would transform into:

$$\begin{aligned} K^2(E_e) \approx (Q-E_e)\sqrt{(Q-E_e)^2 - m_\beta^2} \times \Theta(Q - E_e - m_\beta) \\ + (Q-E_e)\sum_k |K_{ek}|^2 \sqrt{(Q-E_e)^2 - m_k^2} \times \Theta(Q-E_e-m_k) \ , \end{aligned} \tag{2.114}$$

The above expression shows that a "heavy" neutrino mass, m_k can be measured by observing a kink of the kinetic energy spectrum at $E_e = Qm_k$, the point where the "heavy" neutrino spectrum ends. [147]

As for question 5, its answer can not only discern if neutrinos are Majorana particles but also, combined with the answer to question 4 (mass scale), answer question 1 (ordering)! This could be achieved by measuring Neutrinoless double β decay ($0\nu\beta\beta$) : $(A, Z) \to (A, Z+2) + e^- + e^-$, a Dirac fermion number violating process like the one given in [eq. B.7]. The decay rate for this process is given by:

$$\Gamma_{0\nu\beta\beta} = G(Q, Z)|M_{nuc}^0|^2| = G(Q, Z)|M_{nuc}|^2 \cdot |m_{\beta\beta}|^2 \ , \tag{2.115}$$

where $G(Q, Z)$ is a phase-space factor that depends on the nucleus and M_{nuc}^0 is the total nuclear matrix element. M_{nuc}^0 contains the leptonic amplitude $(W_\mu^-)^*(p_3) + (W_\nu^-)^*(p_4) \to e^-(p_1) + e^-(p_2)$ [39], which as two diagrams (channel t and u) because of the anti-symmetrization of the final state:

$$M_{nuc}^0 \propto (M_t - M_u) \ . \tag{2.116}$$

[39] $W_\mu^-)^*$ means an off-shell W^- boson. It's an intermediate state that is taken into account in the hadronic part of the amplitude.

Below is the part that is sandwiched between the spinors, for the t-channel amplitude. A lepton number violating vertex was used, thus, making the process only possible if neutrinos are Majorana.

$$iM_t = \sum_j \left[\overline{u(p_1)} \left(\frac{-ig}{\sqrt{2}} \gamma_\mu P_L (K)_{1j} \right) \frac{i(\gamma^\delta p_\delta + m_j)}{p^2 - m_j^2} \left(-\frac{-ig}{\sqrt{2}} \gamma_\nu P_L (K)_{1j} \right) v(p_2) \right] , \tag{2.117}$$

where $p = p_1 - p_3 = p_4 - p_2$, and $e^-(p_2) = (e^+(p_2))^c$ has the role of Conjugated Dirac fermion, which means that the vertex connecting the momenta p_2 and p_4 is the lepton number violating one. When calculating $|M^0_{nuc}|^2$, taking the traces of the leptonic part one notes that the terms $\propto \gamma^\delta p_\delta$ vanish [40]. Thus, the only surviving terms are the ones $\propto m_j$. Then, neglecting the neutrino mass in the denominator, it's possible to factor out an interesting quantity:

$$|M^0_{nuc}|^2 = |M_{nuc}|^2 \cdot |\sum_j (K_{1j})^2 m_j|^2 , \tag{2.118}$$

which looking at [eq. 2.115] defines:

$$m_{\beta\beta} = \sum_j (K_{1j})^2 m_j , \tag{2.119}$$

where K is defined in [eq. 2.60]. Note that this is sensitive to Majorana phases and to the Dirac phase δ [41]. Again, there are two possible extra scenarios. In the exact treatment one should use [eq. 2.119] with K given by [eq. 2.74]. In this scenario, $m^2_{\beta\beta}$ is also sensitive to the CP Violating phases θ_1,θ_2, θ_3 phases that come from the hermitian part of K [Eqs. 2.74, 2.76].

The other scenario is if p of the three "heavy" states have a mass of the order of the light neutrinos (such that it is kinematically allowed). In that case:

$$m_{\beta\beta} = \sum_i^3 (K_{ei})^2 m_i + \sum_j^p (R_{ej})^2 M_j \tag{2.120}$$

The current experimental bound on $m_{\beta\beta}$, considering only 3 light neutrinos, is depicted graphically on [fig. 2.1]:

Thus, concluding, if $0\nu\beta\beta$ is detected, neutrinos are Majorana. To clarify the connections between the various results, a quick analysis of the figure will be done. The picture shows that if $0\nu\beta\beta$ is detected in the next round of experiments and KATRIN gives a positive sign working in the reported sensitivity ($m_\beta > 0.2\ eV$), one cannot conclude if the ordering is normal or inverted. Nevertheless, if only KATRIN gives a positive signal in this region, one can conclude that neutrinos are Dirac. However, if $m_{\beta\beta}$ is detected below $10^{-2}\ eV$, one can conclude that the ordering is normal. If $m_{\beta\beta}$ is detected between 0.05 eV and 0.01 eV and KATRIN yields a negative result, then for sure the ordering is inverted. If the m_β signal is detected below $10^{-2}\ eV$, the conclusion about ordering depends on the value of the detected $m_{\beta\beta}$. If there's no positive signal from β decay and $0\nu\beta\beta$ in the next round of experiments, then inverted ordering would eventually become ruled out and the only possibility would be Majorana neutrinos with

[40] Trace of odd number of gamma matrices.

[41] $(K_{13})^2 = s^2_{13} e^{2i(\phi_2 - \delta)}$. Although when studying $0\nu\beta\beta$ it's useful to redefine $\phi_2 = \phi'_2 + \delta$ such that $m_{\beta\beta}$ only depends on ϕ_1 and ϕ'_2.

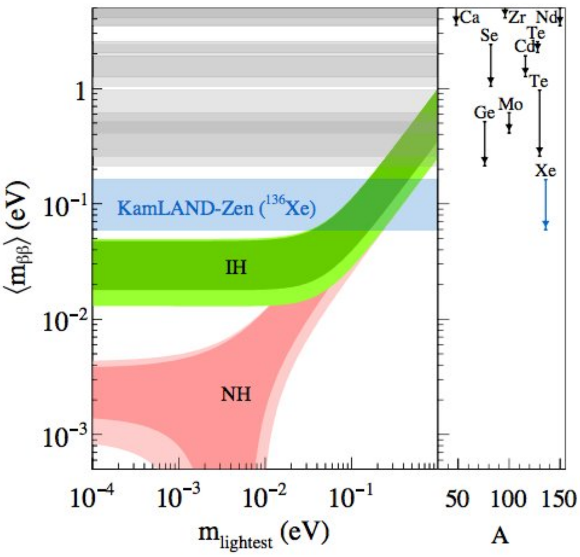


Figure 2.1: $m_{\beta\beta}$ as given in [eq. 2.119] as a function of the lightest neutrino mass and its experimental bounds from [148].

normal ordering or Dirac neutrinos with inverted ordering if the sign of Δm^3_{13} was determined, in an independent experiment, to be negative.

The interplay between the possible results of β decay and $0\nu\beta\beta$ decay experiments and the sign of Δm^2_{13} will give us answers about neutrino mass scale, hierarchy, ordering and character. Better measurements of oscillation parameters will yield a conclusion about CP Violation in the Leptonic sector. The possibility of the existence of Heavy neutrino states and sizable deviations from unitarity is still open.

An extra question that one might dwell on is what is the allowed mass scale for these heavy neutrino states? If one neglects cosmological bounds, for the aforesaid reasons, the answer is any - from the sub-eV region to the GUT scale. An important test to the number of active neutrino species with masses below the Z boson mass was the measurement of the Z invisible width at LEP [43] : $N_\nu = 2.984 \pm 0.008$. However, this measurement doesn't exclude sub-electroweak scale "heavy" neutrino states [147]. The SM predicts $N_\nu = 3$ and the νSM predicts $N_\nu = \sum_{i,j} |(K^\dagger K)_{ij}|^2 = 3$, since K is unitary. The $SI\nu SM$ predicts $N_\nu = \sum_{i,j} |(K^\dagger K)_{ij}|^2 < 3$, if there are no "heavy" states with masses below the Z boson. The prediction is below three because of the deviations from unitarity. If there are p "heavy" states with masses below the Z boson the prediction is:

$$N_\nu = \sum_{i,j} |(K^\dagger K)_{ij}|^2 + \sum_{i,k} |(K^\dagger R)_{ik}|^2 + \sum_{k,j} |(R^\dagger K)_{kj}|^2 + \sum_{k,l} |(R^\dagger R)_{kl}|^2 \leq 3 \; , \tag{2.121}$$

where $i,j = 1,2,3$ and $k,l = 1,...,p$, [Tab. B.4] was used and the prediction for N_ν is 3 if $p = 3$, otherwise is below 3. Thus, there can be more than three neutrino mass states with masses below the Z boson mass, since N_ν is always ≤ 3, even in the $SI\nu SM$, because of the unitarity of the full mixing matrix V and the fact that K is a contraction [149]. Summarizing, only states which interact weakly contribute to this quantity, N_ν, these states are given by [eq. 2.49], and are only three, the maximum value N_ν can have.

Chapter 3

Appealing Models within the Seesaw Type I Framework

From section 2.2, namely [Eqs. 2.66, 2.67, 2.68], one understands that heavy neutrino states (N) interact with SM particles, via mixing. For negative chiral states, this mixing is controlled by the matrix R [eq. 2.49], defined in [eq. 2.48]. Thus, their entries are the relevant coefficients for electroweak processes involving heavy neutrinos. This matrix is, naturally, as big as the deviations from unitary of K allow (this statement will be proved in a few lines), since V is unitary and K is almost unitary.

The production of these heavy neutrino states in electroweak processes is possible in machines like the LHC[1], and its production rates are controlled by the matrix R and their mass M_i. Knowing present unitary bounds [eq. 2.107], heavy neutrino masses around 100 GeV are not ruled out [150]. Nor are heavy neutrino masses in the eV and KeV scale [147, 151], since the smallness of the R matrix highly suppresses their interaction with SM neutrinos, making them invisible to measurements performed at LEP or at the present LHC.

Thus, an important question is: Is it possible to have heavy neutrinos with masses around the electroweak scale[2] or lower, while satisfying all present phenomenological bounds, especially deviations from unitarity?

This preoccupation with deviations from unitarity stems from the fact that, as stated above [eq. 2.58], in the usual seesaw[3], the deviations from unitarity of K are proportional to $(M^\nu M_R^{-1})^2$. With M_R assuming values of the order 10^{16} GeV, being below the experimental bound is not a problem. However, one could naively conclude that by taking the masses of the heavy states to be around the electroweak scale or lower, one would decrease the scale of M_R in such a way that deviations from unitarity would become well above the experimental bounds.

In other words, the question this work tries to answer is: Can one have a seesaw, consistent with experimental data, with all the benefits stated in Chapter 2, when the scale of M_R is close to the

[1] If they have a kinematically allowed mass, i.e., not above the LHC's maximum collision energy which at the moment lies around 13 TeV.

[2] Let's define this as the range from from $m_t \approx 170$ GeV to $10 \times mt \approx 1700$ GeV.

[3] Under the effective treatment, with M_R assuming values of the order 10^{16} GeV

electroweak scale or lower? Sizable deviations from unitarity along with masses in this scale would mean that the detection of heavy neutrinos might be just around the corner.

3.1 Seesaw Formulas Adequate to an Exact Treatment

Before proceeding, it is important to explore all the degrees of freedom in the Seesaw formulas [eq. 2.47]. This means finding suitable parametrizations for the sub-matrices of [eq. 2.48], as done in [1]:

$$V = \begin{pmatrix} K & R \\ S & Z \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & Z \end{pmatrix} \cdot \begin{pmatrix} I & Y \\ -X & I \end{pmatrix} \, , -X = Z^{-1}S \ , \ Y = K^{-1}R \ , \tag{3.1}$$

which is a completely general parametrization, valid for K and Z non-singular. I is the 3×3 identity matrix. The equations that stem from non-diagonal terms of $VV^\dagger = I_{6\times6}$ imply that:

$$Y = X^\dagger \ . \tag{3.2}$$

[eq. 3.1] can then be written as:

$$V = \begin{pmatrix} K & KX^\dagger \\ -ZX & Z \end{pmatrix} \ . \tag{3.3}$$

The equations that stem from the diagonal terms of the unitarity relations of V yield:

$$VV^\dagger = I_{6\times6} \rightarrow K(K^\dagger + X^\dagger X K^\dagger) = I \ , \ Z(Z^\dagger + XX^\dagger Z^\dagger) = I \ , \tag{3.4}$$

$$V^\dagger V = I_{6\times6} \rightarrow (K^\dagger + X^\dagger X K^\dagger)K = I \ , \ (Z^\dagger + XX^\dagger Z^\dagger)Z = I \ , \tag{3.5}$$

which, at first sight, one might argue that they have redundant information: The right and left inverse of K are equal and given by $K^\dagger + X^\dagger X K^\dagger$, or equivalent for Z. However, these are 4 different equations and will be useful later. From them it is also clear that the matrix X parametrizes the deviations from unitarity of K (and Z) and, as stated before, the matrix $R = KX^\dagger$, that controls the rate of electroweak processes involving heavy neutrinos, is as big as X allows - if X is zero there are no deviations from unitarity and R is zero.

With the importance of X established, it is useful to write it in terms of other known matrices, with physical meaning. To try that, it is mandatory to write [eq. 2.47], using V given by [eq. 3.3]:

$$-X^\dagger Z^\dagger M^{\nu T} = dK^T \ , \tag{3.6}$$

$$K^\dagger M^\nu - X^\dagger Z^\dagger M_R = -dX^T Z^T \ , \tag{3.7}$$

$$Z^\dagger M^{\nu T} = d_R X^* K^T \ , \tag{3.8}$$

$$XK^\dagger M^\nu + Z^\dagger M_R = d_R Z^T \ . \tag{3.9}$$

Replacing $Z^\dagger M^{\nu T}$ from [eq. 3.8] into [eq. 3.6] yields

$$d = -X^T \; d_R \; X \; . \tag{3.10}$$

The only X that satisfies this equation is given by:

$$X = \pm i \sqrt{d_R^{-1}} O_c \sqrt{d} \; , \tag{3.11}$$

where O_c is a complex orthogonal matrix, i.e., $O_c^T O_c = I$, or explicitly:

$$|X_{ij}| = \left| (O_c)_{ij} \sqrt{\frac{m_j}{M_i}} \right| \; . \tag{3.12}$$

This, by itself, proves that the deviations from unitarity are not only controlled by the mass scales involved, but also by this matrix O_c. Thanks to this degree of freedom, in principle, it is possible to generate deviations from unitarity of any order of magnitude, independently of the mass scales one is working with.

With X defined, and R and S written in terms of K, Z and X, it is also relevant to parametrize K and Z in terms of other matrices, using the Polar Decomposition theorem, and compare this result with the definitions given in [Eqs. 2.74, 2.76]. Using the Singular Value Decomposition of X one obtains:

$$X = W \; d_X \; U^\dagger \; , \; X X^\dagger = W \; d_X^2 \; W^\dagger \; , \; X^\dagger X = U \; d_X^2 \; U^\dagger \; , \tag{3.13}$$

where U and W are unitary matrices, and d_X is a real diagonal matrix. Using this, one can write the hermitian matrices $\left(I + X^\dagger X\right)$ and $\left(I + X \; X^\dagger\right)$ as,

$$\begin{aligned} I + X^\dagger X &= U \; \left(I + d_X^2\right) \; U^\dagger \; , \\ I + X \; X^\dagger &= W \; \left(I + d_X^2\right) \; W^\dagger \; . \end{aligned} \tag{3.14}$$

Inserting [eq. 3.14] into [eq. 3.4], one obtains:

$$\begin{aligned} K \; U \; \left(I + d_X^2\right) \; U^\dagger \; K^\dagger &= K \; U \; \sqrt{(I + d_X^2)} \; \cdot \sqrt{(I + d_X^2)} U^\dagger \; K^\dagger = I \; , \\ Z \; W \; \left(I + d_X^2\right) \; W^\dagger \; Z^\dagger &= Z \; W \; \sqrt{(I + d_X^2)} \; \cdot \sqrt{(I + d_X^2)} \; W^\dagger \; Z^\dagger = I \; . \end{aligned} \tag{3.15}$$

Therefore, one concludes that

$$K \; U \; \sqrt{(I + d_X^2)} = U_K \; , \; Z \; W \; \sqrt{(I + d_X^2)} = W_Z \; , \tag{3.16}$$

are unitary matrices, with:

$$\left(\sqrt{I+d_X^2}\right)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{1+d_{X}{}_1^2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+d_{X}{}_2^2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1+d_{X}{}_3^2}} \end{pmatrix} , \tag{3.17}$$

where the labels $i = 1, 2, 3$ are given in ascending order, such that, for instance, $d_{X}{}_2^2 > d_{X}{}_1^2$. Which leads to [4]

$$\begin{aligned} K &= U_K \, \left(\sqrt{(I+d_X^2)}\,\right)^{-1} U^\dagger = U_K U^\dagger \left(U \left(\sqrt{(I+d_X^2)}\,\right)^{-1} U^\dagger\right) = U_K U^\dagger \left(\sqrt{I+X^\dagger X}\,\right)^{-1} , \\ Z &= W_Z \, \left(\sqrt{(I+d_X^2)}\,\right)^{-1} W^\dagger = W_Z \, W^\dagger \left(W \left(\sqrt{(I+d_X^2)}\,\right)^{-1} W^\dagger\right) = W_Z W^\dagger \left(\sqrt{I+XX^\dagger}\,\right)^{-1} . \end{aligned} \tag{3.18}$$

Putting everything together:

$$V = \begin{pmatrix} K & R \\ S & Z \end{pmatrix} = \begin{pmatrix} U_K U^\dagger \left(\sqrt{I+X^\dagger X}\,\right)^{-1} & U_K U^\dagger \left(\sqrt{I+X^\dagger X}\,\right)^{-1} X^\dagger \\ -W_Z W^\dagger \left(\sqrt{I+X\,X^\dagger}\,\right)^{-1} X & W_Z W^\dagger \left(\sqrt{I+X\,X^\dagger}\,\right)^{-1} \end{pmatrix} . \tag{3.19}$$

One can now analyze [Eqs. 2.74, 2.76] and identify:

$$U_{PMNS} \cdot F = U_K U^\dagger \ , \ H_R = \left(\sqrt{I+X^\dagger X}\,\right)^{-1} = U \left(\sqrt{(I+d_X^2)}\,\right)^{-1} U^\dagger \ . \tag{3.20}$$

In the literature, one usually finds the following definition for a K with deviations from unitarity [143, 144, 152, 153]:

$$K = (I - \eta)\ V \ , \tag{3.21}$$

where $(I - \eta)$ is a hermitian matrix and V a unitary matrix, usually associated with U_{PMNS}. Using the unitarity of U and U_K together with the first equation of [eq. 3.18], one finds:

$$K = \left(U_K \ \left(\sqrt{I+d_X^2}\right)^{-1} \ U_K^\dagger\right) \ (U_K.U^\dagger) = \left(U_K \ U^\dagger \ \left(\sqrt{I+X^\dagger X}\,\right)^{-1} U \ U_K^\dagger\right) \ (U_K.U^\dagger) \ , \tag{3.22}$$

where

$$H_L = U_K \ \left(\sqrt{I+d_X^2}\right)^{-1} \ U_K^\dagger = U_K \ U^\dagger \ \left(\sqrt{I+X^\dagger X}\,\right)^{-1} U \ U_K^\dagger \ , \tag{3.23}$$

is an hermitian matrix and $U_K U^\dagger$ is unitary and equal to $U_{PMNS} \cdot F$ by [eq. 3.20]. This is consistent with the polar decomposition theorem, which states that if K is non-singular then:

$$K = H_L \ \left(U_K \ U^\dagger\right) = \left(U_K \ U^\dagger\right) \ H_R \ , \tag{3.24}$$

[4]To obtain these relations, one needs to use [eq. 3.14] and $\sqrt{I+d_X^2} = U^\dagger \ (\sqrt{I+X^\dagger X})\ U$

the unitary part on both decompositions is the same and the hermitian parts are related by the equation:

$$H_R = \left(U_K\ U^\dagger\right)\ H_L (U_K U^\dagger)^\dagger \tag{3.25}$$

To be in line with the literature, from now onward $H_L = I - \eta$, using η to describe the deviations from unitarity. Using [eq. 3.24] and the unitarity of $U_K\ U^\dagger$:

$$K\ K^\dagger = H_L^2 = (I-\eta)^2 = I - 2\eta + \eta^2\ , \tag{3.26}$$

where

$$\eta = I - H_L = I - U_K\ \left(\sqrt{I + d_X^2}\right)^{-1}\ U_K^\dagger\ . \tag{3.27}$$

Since U_K is unitary, η is highly dependent on the eigenvalues of X, as it should be. Thus, to explain the deviations from unitarity of K, it is important to understand what range of values one can find for $d_{X\,i}$.

3.2 On the Size of Deviations from Unitarity

Using [eq. 3.11], it is straightforward to conclude that the eigenvalues of $X^\dagger X$, $d_{X\,i}^2$, will be related to the eigenvalues of $O_c^\dagger O_c$. However, there's no general analytical expression that relates both. Nevertheless, one can find special cases where this relation exists, thus providing important insights. Since O_c is an orthogonal complex matrix, its eigenvalues are constrained by the usual equations:

$$O_c^T\ O_c = I\ ,\ Det(O_c) = x_1\ x_2\ x_3 = \pm 1\ ,\ Det(O_c^\dagger O_c) = |x_1|^2\ |x_2|^2\ |x_3|^2 = 1\ . \tag{3.28}$$

From the previous equation:

$$O_c^\dagger O_c O_c^T O_c^* = I \to O_c^\dagger O_c = \left(O_c^T O_c^*\right)^{-1}\ , \tag{3.29}$$

and, taking the trace on both sides of the last equation, gives[5]:

$$|x_1|^2 + |x_2|^2 + |x_3|^2 = |x_1|^{-2} + |x_2|^{-2} + |x_3|^{-2}\ , \tag{3.30}$$

[5] Note that $O_c^\dagger O_c = \left(O_c^T O_c^*\right)^T$. This yields that $O_c^\dagger O_c$ and $O_c^T O_c^*$ have the same eigenvalues. Finally, the eigenvalues of $\left(O_c^T O_c^*\right)^{-1}$ are just the inverses of the eigenvalues of $O_c^T O_c^*$

where $|x_i|^2$ is the i^{th} eigenvalue[6] of $O_c^\dagger O_c$. Using the last equation of [eq. 3.28] , $|x_3|^2 = |x_1|^{-2}|x_2|^{-2}$, and the previous equations:

$$\begin{aligned}
&|x_1|^2 + |x_2|^2 + |x_1|^{-2}|x_2|^{-2} = |x_1|^{-2} + |x_2|^{-2} + |x_1|^2|x_2|^2 \\
\Leftrightarrow\ &|x_1|^2|x_2|^2 - |x_1|^2 - |x_2|^2 = |x_1|^{-2}|x_2|^{-2} - |x_1|^{-2} - |x_2|^{-2} \\
\Leftrightarrow\ &|x_1|^2|x_2|^2 - |x_1|^2 - |x_2|^2 + 1 = |x_1|^{-2}|x_2|^{-2} - |x_1|^{-2} - |x_2|^{-2} + 1 \\
\Leftrightarrow\ &(1 - |x_1|^2)(1 - |x_2|^2) = (1 - |x_1|^{-2})(1 - |x_2|^{-2})\ .
\end{aligned} \tag{3.31}$$

This last equation proves that the eigenvalues of $O_c^\dagger O_c$ are: 1, r^2, r^{-2}. This can be seen by nothing that, in the final equation of [eq. 3.31], if $|x_1|^2 = 1$, the above conclusion is trivial given $r^2 = |x_2|^2$. The same applies if one exchanges indices 1 and 2. If $|x_i|^2 \neq 1$, with $i = 1, 2$, then:

$$\begin{aligned}
&(1 - |x_1|^2)(1 - |x_2|^2) = (1 - |x_1|^{-2})(1 - |x_2|^{-2}) \\
\Leftrightarrow\ &|x_1|^2|x_2|^2(|x_1|^{-2} - 1)(|x_2|^{-2} - 1) = (1 - |x_1|^{-2})(1 - |x_2|^{-2}) \\
\Leftrightarrow\ &|x_1|^2|x_2|^2 = 1 \Leftrightarrow |x_1|^2 = |x_2|^{-2}\ ,
\end{aligned} \tag{3.32}$$

which gives that $|x_3|^2 = 1$ by [eq. 3.28] and $r^2 = |x_j|^2$, with $j = 1, 2$.

In conclusion, one can write:

$$Tr(O_c^\dagger O_c) = 1 + r^2 + r^{-2}\ . \tag{3.33}$$

Plugging [eq. 3.11] into the above equation leads to:

$$Tr(O_c^\dagger O_c) = Tr(\sqrt{d^{-1}}X^\dagger \sqrt{d_R}\sqrt{d_R}X\sqrt{d^{-1}}) = 1 + r^2 + r^{-2}\ . \tag{3.34}$$

Taking the degenerate limit, where the diagonal mass matrices are given by $d = m\ I$, $d_R = M\ I$, where m and M are real numbers[7] and I is the 3×3 identity matrix, one can generate an upper bound on $Tr(X^\dagger X)$:

$$\frac{M}{m}Tr(X^\dagger X) = \frac{M}{m}(d_{X}{}_1^2 + d_{X}{}_2^2 + d_{X}{}_3^2) \leq 1 + r^2 + r^{-2}\ . \tag{3.35}$$

One can now estimate the size of the eigenvalues of $X^\dagger X$, as a function of r, by going to the equality limit. In this limit:

$$\frac{M}{m}d_{X}{}_1^2 \sim r^{-2}\ ,\ \frac{M}{m}d_{X}{}_2^2 \sim 1\ ,\ \frac{M}{m}d_{X}{}_3^2 \sim r^2\ . \tag{3.36}$$

Taking $m \sim 1 \times 10^{-11}\ GeV$ and $M \sim 1 \times 10^2\ GeV$ yields $\frac{M}{m} = 10^{13}$. For instance, for $r^2 = 10^{10}$ one obtains:

$$d_{X}{}_1^2 \sim 10^{-23}\ ,\ d_{X}{}_2^2 \sim 10^{-13}\ ,\ d_{X}{}_3^2 \sim 10^{-3}\ . \tag{3.37}$$

The term by term identification of [eq. 3.35] in the equality limit, presented in [eq. 3.36], may seem like a crude approximation but, in fact, it isn't, due to the difference in orders of magnitude of the terms.

To support this, the next section will contain a toy model, with degenerate light neutrinos with mass

[6] Real, since the matrix is hermitian.)

[7] m is the heaviest light neutrino mass and M is the lightest heavy neutrino mass, such that the inequality is satisfied.

m, and degenerate heavy neutrinos with mass M, which can be analytically solved.

Another model, with a simple O_c, will provide plots where the evolution of the $d_{X}{}_i^2$ is given as a function of the masses and x^2.

The behaviour presented in [eq. 3.37], and in the results of the next section, a sizable eigenvalue, one nearly zero but tractable eigenvalue, and another practically zero eigenvalue, will turn out to be true not only for the degenerate limit, $m_1 = m_2 = m_3 = m$ and $M_1 = M_2 = M_3 = M$, but as long as $\frac{M_i}{m_j} >> 1$.

3.3 Toy Models

In this section, two toy models are presented to illustrate how the deviations from unitarity behave in terms of the neutrino masses.

For the first model, one sets O_c to one of the simplest possibilities:

$$O_c = \begin{pmatrix} 0 & \sqrt{x^2+1} & ix \\ 0 & ix & -\sqrt{x^2+1} \\ 1 & 0 & 0 \end{pmatrix} , \tag{3.38}$$

where the parameter x^2, when large, corresponds to r^2 in [eq. 3.35] but when $x^2 << 1$, then $r^2 = r^{-2} = 1$. Plugging this into [eq. 3.11], choosing the minus sign:

$$X = \begin{pmatrix} 0 & -i\sqrt{\frac{m_2}{M_1}}\sqrt{x^2+1} & \sqrt{\frac{m_3}{M_1}}x \\ 0 & \sqrt{\frac{m_2}{M_2}}x & i\sqrt{\frac{m_3}{M_2}}\sqrt{x^2+1} \\ -i\sqrt{\frac{m_1}{M_3}} & 0 & 0 \end{pmatrix} , \tag{3.39}$$

which gives:

$$I + X^\dagger X = \begin{pmatrix} 1+\frac{m_1}{M_3} & 0 & 0 \\ 0 & 1+\frac{m_2 x^2}{M_2}+\frac{m_2(x^2+1)}{M_1} & i\frac{(M_1+M_2)x\sqrt{m_2 m_3(x^2+1)}}{M_1 M_2} \\ 0 & -i\frac{(M_1+M_2)x\sqrt{m_2 m_3(x^2+1)}}{M_1 M_2} & 1+\frac{m_3 x^2}{M_1}+\frac{m_3(x^2+1)}{M_2} \end{pmatrix} . \tag{3.40}$$

The eigenvalues of [eq. 3.40] can be calculated analytically, yielding:

$$\begin{aligned} 1 + d_{X}{}_1^2 &= \frac{g(x, M_1, M_2, m_2, m_3) - \sqrt{f(x, M_1, M_2, m_2, m_3)}}{2M_1 M_2} , \\ 1 + d_{X}{}_2^2 &= 1 + \frac{m_1}{M_3} , \\ 1 + d_{X}{}_3^2 &= \frac{g(x, M_1, M_2, m_2, m_3) + \sqrt{f(x, M_1, M_2, m_2, m_3)}}{2M_1 M_2} , \end{aligned} \tag{3.41}$$

where

$$
\begin{aligned}
&g(x, M_1, M_2, m_2, m_3) = x^2(M_1 m_2 + M_1 m_3 + m_2 M_2 + M_2 m_3) + 2M_1 M_2 + M_1 m_3 + m_2 M_2 \ , \\
&f(x, M_1, M_2, m_2, m_3) = \left[M_1(m_2 x^2 + 2M_2 + m_3 x^2 + m_3) + M_2(m_2(x^2+1) + m_3 x^2)\right]^2 \\
&- M_1 M_2 \left[x^2(4M_1 m_2 + 4M_1 m_3 + 4m_2 M_2 + 4M_2 m_3) + 4M_1 M_2 + 4M_1 m_3 + 4m_2 M_2 + 4m_2 m_3\right] \ .
\end{aligned}
\tag{3.42}
$$

To understand how the ${d_X}_i^2$ evolve with the masses, it's instructive to plot the functions given in [eqs. 3.41]. In the following plots, the light neutrino masses were all set to m, for simplicity. The heavy neutrino masses are in units of m. x was set to $x = 10^5$ in all of them but the last.

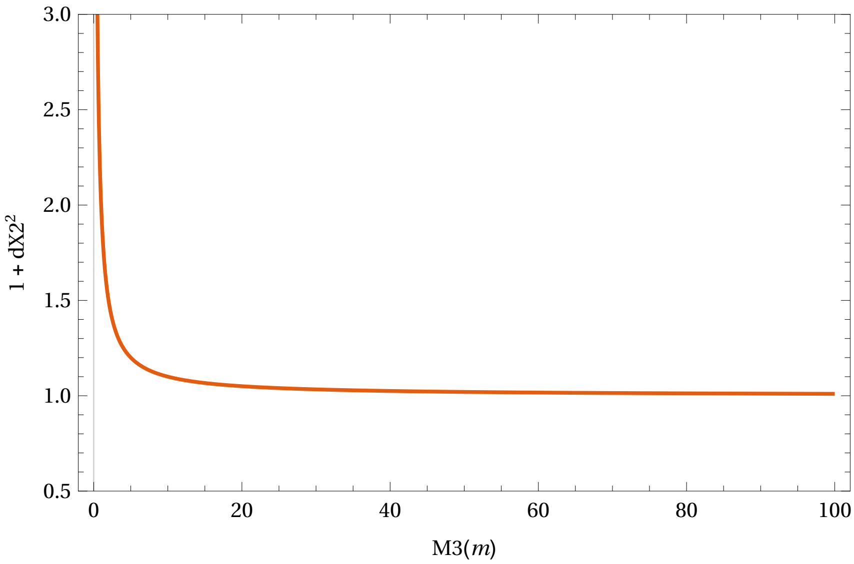
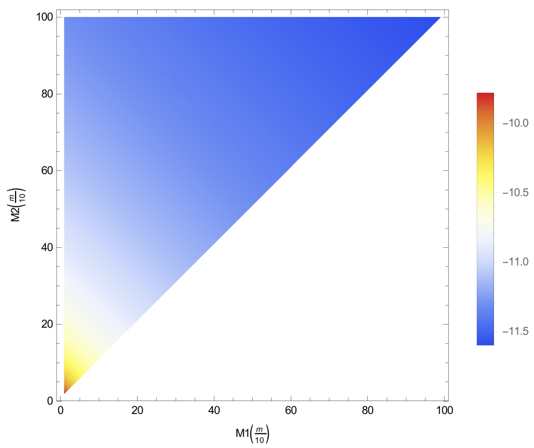
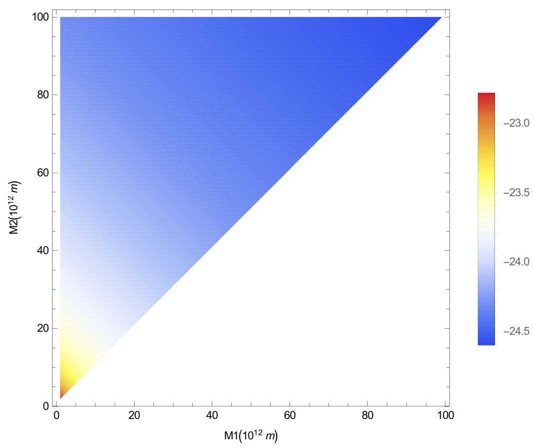


Figure 3.1: $1 + {d_X}_2^2$ as given in [eq. 3.41] as a function of M_3. M_3 is in units of the light neutrino mass $m_1 = m$. The behaviour is maintained for $m \to \infty$

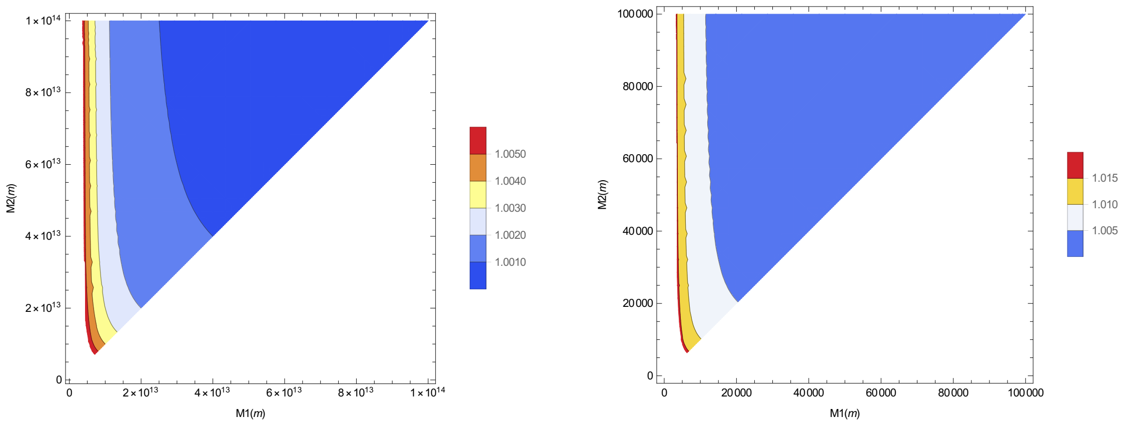


((a)) M_i are in units of the light neutrino mass $m_2 = m_3 = m$ divided by 10. In this plot, M_i ranges from 0 to $10m$.



((b)) M_i are in units of the light neutrino mass $m_2 = m_3 = m$ multiplied by 10^{12}. In this plot, M_i ranges from $10^{12}m$ to $10^{14}m$.

Figure 3.2: Order of magnitude of the eigenvalue ${d_X}_1^2$, using a density plot of $Log_{10}(Log(1 + {d_X}_1^2)) \approx Log_{10}({d_X}_1^2)$ as given in [eq. 3.41] as a function of M_1 and M_2. Colder colors indicate smaller values.



((a)) This is done for $x = 10^5$. In this plot, M_i ranges from $10^{12}m$ to $10^{14}m$.

((b)) This is done for $x = 5$.In this plot, M_i ranges from $10m$ to $10^6 m$.

Figure 3.3: Eigenvalue $d_{X}{}_3^2$, using a contour plot of $1 + d_{X}{}_3^2$ as given in [eq. 3.41] as a function of M_1 and M_2. M_i are in units of the light neutrino mass $m_2 = m_3 = m$. Colder colors indicate smaller values.

Starting by discussing the simplest, $1 + d_{X}{}_2^2$, [Fig. 3.1], one notes that for $M_3 >> m$ it is essentially 1 - for instance, $d_{X}{}_2^2$ has the value $\sim 10^{-13}$ for $M_3 = 10^{13}m$. For $M_3 < m$ $d_{X}{}_2^2$ can have arbitrarily large values. For $M_3 = m$ $d_{X}{}_2^2$ is 2. As for $d_{X}{}_1^2$, [Fig. 3.2], one notes that for $M_1, M_2 >> m$ it is essentially 0 - for instance, it has the value $\sim 10^{-23}$ for $M_1, M_2 = 10^{13}m$. For $M_1, M_2 < m$ it can have arbitrarily large values. For $M_1 = M_2 = m$ it is $\sim 10^{-10}$. Finally, for $1+d_{X}{}_3^2$, [Fig. 3.3], one notes that for $M_1, M_2 >> m$ it tends to 1, but much slower than the other two - for instance, it has the value $\sim 1 + 1 \times 10^{-3}$ for $M_1, M_2 = 10^{13}m$. For $M_1, M_2 \leq m$ it can have arbitrarily large values. It's important to note that all of these numerical values are dependent on the value of x. For instance, in [Fig. 3.3(b)] there is the same contour plot as [Fig. 3.3(a)] done for $x = 5$, where one can see that it is possible to achieve the same size of deviations from unitarity for much lower heavy neutrino masses.

The unitary matrix, U, that diagonalizes [eq. 3.40], can also be obtained analytically. However, its entries are too complicated and too long to give any insight on the eigenvectors. To obtain such thing, one needs to go to a parameter region where all expressions simplify.

This is done in the second model, where one considers the degenerate limit, where all light neutrinos have mass m, and all heavy neutrinos have mass M. Furthermore, one also needs to go to the region where $x >> 1 \to \sqrt{x^2+1} \approx x$, which turns out to be a very reasonable approximation for the case of heavy neutrino masses around the electroweak scale with sizable deviations from unitarity. Thus, in this model:

$$O_c \approx \begin{pmatrix} 0 & x & ix \\ 0 & ix & -x \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.43}$$

and, in approximation, one finds for X:

$$X = \begin{pmatrix} 0 & -i\sqrt{\frac{m}{M}}x & \sqrt{\frac{m}{M}}x \\ 0 & \sqrt{\frac{m}{M}}x & i\sqrt{\frac{m}{M}}x \\ -i\sqrt{\frac{m}{M}} & 0 & 0 \end{pmatrix}, \tag{3.44}$$

leading to:

$$I + X^\dagger X = \begin{pmatrix} 1+\frac{m}{M} & 0 & 0 \\ 0 & 1+\frac{2mx^2}{M} & i\frac{2mx^2}{M} \\ 0 & -i\frac{2mx^2}{M} & 1+\frac{2mx^2}{M} \end{pmatrix} . \tag{3.45}$$

The matrix that diagonalizes the matrix in [eq. 3.45] is given by:

$$U = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}}i & 0 & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{3.46}$$

Furthermore, the eigenvalues of $X^\dagger X$ are:

$$d_X^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{m}{M} & 0 \\ 0 & 0 & \frac{4mx^2}{M} \end{pmatrix} , \tag{3.47}$$

where making the substitutions $x = 10^5$ and $\frac{m}{M} = 10^{-13}$ gives values very close to the ones estimated in [eq. 3.37]. It's interesting to note that, in this parameter region, the functions $\sqrt{f}$ and g defined in [eq. 3.42] have the same value: $\sqrt{f} = g = 2\ m\ M\ x^2$. Let's now proceed and try to find an analytical formula for the matrix η, defined in [eq. 3.27]. One should be now convinced that, for $\frac{M_i}{m_j} >> 1$, $1 + {d_X}_1^2 \sim 1 + {d_X}_2^2 \approx 1$. Furthermore, in a good approximation:

$$\frac{1}{\sqrt{1+{d_X}_3^2}} \approx 1 - \frac{1}{2}{d_X}_3^2 \Longrightarrow \left(\sqrt{I + d_X^2}\right)^{-1} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\frac{1}{2}{d_X}_3^2 \end{pmatrix} , \tag{3.48}$$

Using this in [eq. 3.27], one obtains:

$$\eta = \frac{1}{2}{d_X}_3^2 \cdot \begin{pmatrix} |U_{K13}|^2 & U_{K13}\cdot U_{K23}^* & U_{K13}\cdot U_{K33}^* \\ U_{K23}\cdot U_{K13}^* & |U_{K23}|^2 & U_{K23}\cdot U_{K33}^* \\ U_{K33}\cdot U_{K13}^* & U_{K33}\cdot U_{K23}^* & |U_{K33}|^2 \end{pmatrix} . \tag{3.49}$$

With this one sees that when ${d_X}_3^2$ approaches 0, all entries of η will approach zero. Furthermore, if the entries of U_K are of same order of magnitude, such that every product of U_{Kij} yields ~ 1, η is a democratic matrix, dominated by ${d_X}_3^2$. The experimental bounds, [eq. 2.107], constrain much more the entries that are proportional to U_{K23} than the rest. Looking at [eq. 3.26], one concludes that $KK^\dagger - I \sim -2\eta$, for a small η. Therefore:

$$|\eta| \leq \begin{pmatrix} 1.25\times10^{-3} & 1.20\times10^{-5} & 1.35\times10^{-3} \\ 1.20\times10^{-5} & 2.00\times10^{-4} & 6.00\times10^{-4} \\ 1.35\times10^{-3} & 6.00\times10^{-4} & 2.8\times10^{-3} \end{pmatrix} . \tag{3.50}$$

Thus, to achieve such non-democratic deviations from unitarity like in [eq. 3.50], one will need a non-democratic U_K matrix. This suggests that, if one wants a model that has deviations from unitarity matching the experimental bounds, one will need to find a U_K with the 23 entry small enough such that the entries proportional to it are controlled by it and the rest is controlled by $d_{X}{}^2_3$.

One defines $U_{PMNS} \cdot F = U_K.U^\dagger$, as in [eq. 3.20], where F contains the Majorana phases, α_i, and U_{PMNS} is a unitary matrix with one Dirac phase like V_{CKM}. Therefore, to achieve a small U_{K23} one needs to control the quantity:

$$Line_2[U_{PMNS} \cdot F] \times Column_3[U] = U^{21}_{PMNS} \cdot U^{13} + U^{22}_{PMNS} \cdot e^{i\alpha_1} \cdot U^{23} + U^{23}_{PMNS} \cdot e^{i\alpha_2} \cdot U^{33} \ . \quad (3.51)$$

Thus, one can choose the α_i such that there is a cancellation and the above quantity is small.

For an O_c of the form:

$$O_c = \begin{pmatrix} 0 & \sqrt{x^2+1} & ix \\ 0 & ix & -\sqrt{x^2+1} \\ 1 & 0 & 0 \end{pmatrix} \ , \quad (3.52)$$

one gets $U^{13} = 0$. This puts too much strain on the process of controlling U_{K23}. Thus, the following O_c can be used:

$$O_c = O_c'.O = \begin{pmatrix} 0 & \sqrt{x^2+1} & ix \\ 0 & ix & -\sqrt{x^2+1} \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad (3.53)$$

This angle θ will generate a controllable non-zero U^{13} without changing the eigenvalues of X. This procedure proves that that the Majorana phases may have a crucial role on the size of a given entry of η.

The main conclusion of the last two sections is that, for $\frac{M_i}{m_j} >> 1$, the only eigenvalue of $X^\dagger X$ that contributes to the deviations from unitarity is $d_{X}{}^2_3$. For a fixed light and heavy mass scale, this variable depends on the parameter x, which is totally free. Thus, the conclusion seems to be that one can generate any size of deviations from unitarity, independently of the masses involved. However, there is a catch. In this general approach to the seesaw mechanism, the Dirac mass matrix, M^ν, is proportional to the matrix X, and, thus, also depends on the parameter x. In conclusion, the desirable size of the Yukawa couplings constrains the parameter space.

3.4 Constraining the Deviations from Unitarity using the entries of the Dirac Mass Matrix

In this section, it will become clear that the entries of the Dirac mass matrix constrain the possible deviations from unitarity for a given value of the lightest heavy neutrino mass. In other words, it is shown that there is a correlation among:

- The size of deviations from 3×3 unitarity of the leptonic mixing matrix K
- The mass of the lightest heavy neutrino, M_1.

From [eq. 3.8] one obtains:

$$M^\nu = K X^\dagger d_R \left(Z^*\right)^{-1} \ . \tag{3.54}$$

As said before, the experimental fact that K is almost unitary implies that Z is also almost unitary. Therefore the Dirac mass matrix M^ν is of the same order as X times d_R. Notice that the scale of d_R may be of the order of the top quark mass, so that indeed the Yukawa couplings need not be extremely small.

The elements of the neutrino Dirac mass matrix, M^ν, are connected to the deviations from unitarity of the leptonic mixing matrix, K, in the following way:

$$M^\nu = U_K \left(\sqrt{(I + d_X^2)}\right)^{-1} d_X \ W^\dagger \ d_R \ W^* \left(\sqrt{(I + d_X^2)}\right) W_Z^T \ , \tag{3.55}$$

where [Eqs. 3.16, 3.18, 3.13] were used. An interesting quantity that gives an insight on the order of the entries of M^ν is:

$$Tr\left[M^\nu M^{\nu\dagger}\right] = Tr\left[\left(\sqrt{(I + d_X^2)}\right)^{-1} d_X \ W^\dagger \ d_R \ W^* \ \left(I + d_X^2\right) \ W^T \ d_R \ W \ d_X \ \left(\sqrt{(I + d_X^2)}\right)^{-1}\right] . \tag{3.56}$$

As previously emphasized, deviations from 3×3 unitarity in the leptonic mixing matrix, K, are controlled by the matrix X. For $X = 0$, there are no deviations from unitarity. Small deviations from unitarity correspond to d_X small and, in that case, one has, in a very good approximation:

$$Tr\left[M^\nu M^{\nu\dagger}\right] \approx Tr\left[d_X \ W^\dagger d_R^2 W \ d_X\right] = Tr\left[d_X^2 \ W^\dagger d_R^2 W\right] \ , \tag{3.57}$$

where the terms with powers higher than 2 of d_X were neglected. This can be written as:

$$\begin{aligned} & Tr\left[M^\nu M^{\nu\dagger}\right] = d_{X_1}^2 \left(M_1^2 \left|W_{11}\right|^2 + M_2^2 \left|W_{21}\right|^2 + M_3^2 \left|W_{31}\right|^2\right) + \\ & d_{X_2}^2 \left(M_1^2 \left|W_{12}\right|^2 + M_2^2 \left|W_{22}\right|^2 + M_3^2 \left|W_{32}\right|^2\right) + \\ & d_{X_3}^2 \left(M_1^2 \left|W_{13}\right|^2 + M_2^2 \left|W_{23}\right|^2 + M_3^2 \left|W_{33}\right|^2\right) \ . \end{aligned} \tag{3.58}$$

Using the conclusion from the previous sections - that only one of the d_{X_i}, corresponding to d_{X_3}, can have a significant value (e.g. $d_{X_3} \approx 10^{-3}$), while the other two are negligible - one finds in good approximation:

$$Tr\left[M^\nu M^{\nu\dagger}\right] \approx d_{X_3}^2 \left(M_1^2 \left|W_{13}\right|^2 + M_2^2 \left|W_{23}\right|^2 + M_3^2 \left|W_{33}\right|^2\right) \ . \tag{3.59}$$

Using the unitary of W:

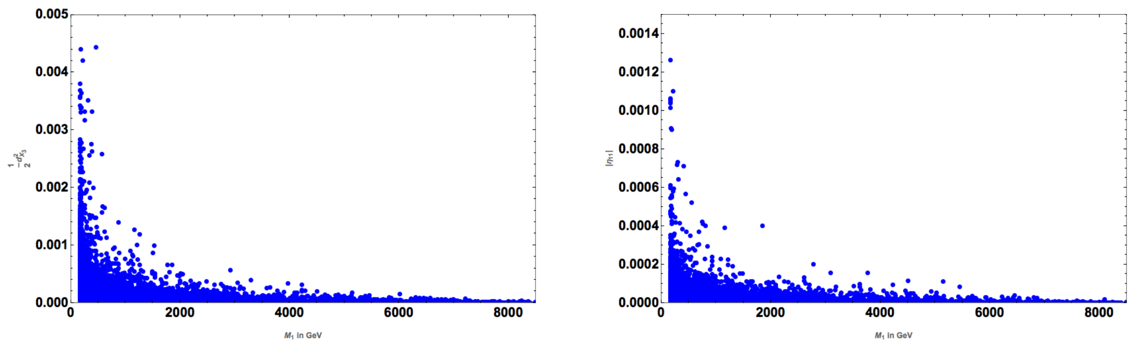
$$Tr\left[M^\nu M^{\nu\dagger}\right] \approx d_{X_3}^2 M_1^2 \left(1 + \left(\frac{M_2^2}{M_1^2} - 1\right) \left|W_{23}\right|^2 + \left(\frac{M_3^2}{M_1^2} - 1\right) \left|W_{33}\right|^2\right) \ , \tag{3.60}$$

which, with the choice $M_3 \geq M_2 \geq M_1$, leads to:

$$d_{X_3}^2 M_1^2 \leq Tr\left[M^\nu M^{\nu\dagger}\right] = \sum_{i,j} |M_{ij}^\nu|^2 \ . \tag{3.61}$$

From [eq. 3.61], it is clear that for significant values of d_{X_3}, M_1 cannot be too large in order to avoid a too large value of $Tr\left[M^\nu M^{\nu\dagger}\right]$, which in turn would imply that at least one of the $\left|M_{ij}^\nu\right|^2$ is too large. This can be seen in [Fig. 3.4(a)], where the plot of $\frac{1}{2}d_{X_3}^2$ versus M_1 is presented. This is done for a large x^2, the parameter of the matrix O_c. In this case, x^2 corresponds to r^2 in [eq. 3.35]. For the case when $x^2 << 1$, then $r^2 = r^{-2} = 1$, and the deviations from unitarity are totally controlled by the heavy mass scale. Both cases yield similar plots. Significant values of $d_{X_3}^2$ can only be obtained for $M_1 \leq 1\ TeV$, in the large x region. Of course that for a very small M_1, to obtain deviations from unitarity of this order ($\sim 10^{-3}$), $Tr\left[M^\nu M^{\nu\dagger}\right]$ would yield a very small result and this is also not wanted.

Thus, the quantity $Tr\left[M^\nu M^{\nu\dagger}\right]$ constrains the lightest heavy neutrino mass by giving a lower and an upper bound, for a given quantity of $d_{X_3}^2$. In the following plots, it is required that $Tr\left[M^\nu M^{\nu\dagger}\right] \leq m_t^2$. To create them, the case of normal ordering was considered, and the values of light neutrinos masses m_i, were varied up to $m_3 = 0.5$ eV. Concerning the heavy Majorana masses M_i, M_3 was allowed to reach values of the order of $10^4 m_t$ and the O_c were randomly generated with a large x. In [Fig. 3.4], the condition $|\eta_{12}| \leq 2 \times 10^{-5}$ is imposed. In [Fig. 3.4(b)], the absolute value of the 11 entry of the matrix η is plotted.



((a)) Maximum deviations from unitarity as a function of M_1, generated under the condition that $\mathrm{Tr}(M^\nu M^{\nu\dagger}) \leq m_t^2$ and $|\eta_{12}| \leq 2 \times 10^{-5}$.

((b)) $|\eta_{11}|$ deviations from unitarity as a function of M_1, generated under the condition that $\mathrm{Tr}(M^\nu M^{\nu\dagger}) \leq m_t^2$ and $|\eta_{12}| \leq 2 \times 10^{-5}$.

Figure 3.4: Evolution of the deviations from unitarity as a function of M_1.

3.5 The Importance of Loop Corrections

Loop corrections can be of two kinds: renormalizable and intrinsically finite. The renormalizable pieces consist of corrections to the tree level parameters already present in the Lagrangian. In the case of corrections to the masses, these are suppressed with respect to the tree level ones by the loop factor $\frac{1}{16\pi^2}$ and by being proportional to leptonic Yukawa couplings [154]. The intrinsically finite corrections are terms which need to be finite since there are no counterterms that could be used to absorb possible divergences arising from them. They are only suppressed by the loop factor, and, thus, can be potentially large. At one-loop level, the generalized mass matrix, M from [eq. 2.46] turns into:

$$M = M^{tree} + M^{loop} \ , \ M^{tree} = \begin{pmatrix} 0 & M^{\nu} \\ M^{\nu T} & M_R \end{pmatrix} \ , \ M^{loop} = \begin{pmatrix} \delta M_L & \delta M^{\nu} \\ (\delta M^{\nu})^T & \delta M_R \ . \end{pmatrix} \tag{3.62}$$

By observation of the previous equation one can conclude that δM_L will be the potentially dangerous correction, since it is the one without a tree level counterpart. The renormalizable and suppressed corrections are given by δM^{ν}. Discussing δM_R is cumbersome since M_R and M_i are free parameters of the theory.

The corrections stem from the two point function known as neutrino self energy, $\Sigma(p)$, where p is the neutrino momentum. This is calculated in the mass basis, then [eq. 2.47] is used to transform it to the interaction basis:

$$M^{loop} = V\Sigma(p)V^T \ . \tag{3.63}$$

The diagrams one should consider in order to calculate $\Sigma(p)$ at one-loop are:

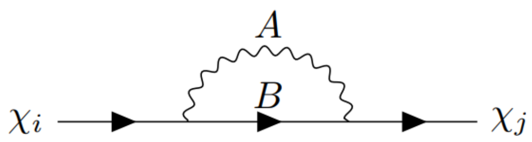


Figure 3.5: Loop Diagrams used to calculate the neutrino self energy.

Where if $A = Z, H, \phi_Z$ then $B = \chi_K$ or else if $A = \phi^{\pm}, W^{\pm}$, then $B = l^{\mp}$ and

$$\chi = \begin{pmatrix} n \\ N \end{pmatrix} \ , \ \nu_L = (K \ R)P_L\chi \ , \ \nu'_L = (S \ Z)P_L\chi \ , \tag{3.64}$$

as in [eq. 2.49]. $\Sigma(p)$ can be decomposed as:

$$\Sigma(p) = A_L(p^2)\not{p}P_L + A_R(p^2)\not{p}P_R + B_L(p^2)P_L + B_R(p^2)P_R \tag{3.65}$$

Writing the new term explicitly, using [Eqs. 2.46, 3.62]:

$$\nu_L^T C^{-1}\delta M_L^* \nu_L = \chi^T (K \ R)^T P_L^T C^{-1}\delta M_L^* P_L (K \ R)\chi = \chi^T C^{-1}(K \ R)^T P_L \delta M_L^* P_L (K \ R)\chi \ , \tag{3.66}$$

and using [Eqs. 3.63, 3.65] one can conclude:

$$(K \ R)^T P_L \delta M_L^* P_L (K \ R) = P_L \Sigma(p) P_L = P_L B_L(p^2) P_L \Longrightarrow \delta M_L = (K \ R) B_L^*(p^2)(K \ R)^T \ . \tag{3.67}$$

This reduces significantly the complexity of the expressions to calculate[8], as only the ones with $A =$

[8]The argument is general for any piece δM_i. M^{loop} only depends on $B_L(p^2)$ and it can be proven that diagrams with $W^{\pm}$ never contribute to $B_L(p^2)$.

Z, H, ϕ_Z contribute to δM_L [154]. The calculation of δM_L is done in Appendix C, and was performed using the Higgs, gauge and Goldstone boson Feynman Rules for an arbitrary gauge, ξ_i, given at [92], the textbook [155] and closely following [154, 156].

The result is finite (the infinities cancel), depends only on parameters of the theory and is gauge invariant[9], as expected:

$$\delta M_L = \delta M_L^Z + \delta M_L^H \ , \tag{3.68}$$

with

$$\begin{aligned} \delta M_L^Z &= \frac{3g^2}{64\pi^2 m_W^2}(K\ R)D^3\left(\frac{1}{m_Z^2}D^2 - I\right)^{-1}\log\left(D^2\frac{1}{m_Z^2}\right)(K\ R)^T \\ \delta M_L^H &= \frac{g^2}{64\pi^2 m_W^2}(K\ R)D^3\left(\frac{1}{m_H^2}D^2 - I\right)^{-1}\log\left(D^2\frac{1}{m_H^2}\right)(K\ R)^T \ , \end{aligned} \tag{3.69}$$

where the expressions resemble the finite part of Passarino-Veltman functions B_0 [155][10]. D is the diagonal mass matrix defined in [eq. 2.47].

As predicted, the corrections can be dangerously large due to the direct dependence on the heavy neutrino masses M_i.

It's interesting to note that even if the light neutrinos are massless at tree level the loop corrections are non-zero [157]. This happens due to the non-zero heavy neutrino masses M_i along with the fact that $L(m_B) = D^3\left(\frac{1}{m_B^2}D^2 - I\right)^{-1}\log\left(D^2\frac{1}{m_B^2}\right)$ is a diagonal matrix:

$$L(m_B) = \begin{pmatrix} L^m(m_B) & 0 \\ 0 & L^M(m_B) \end{pmatrix} \ , \tag{3.70}$$

where

$$L^m(m_B) = \begin{pmatrix} m_1^3\frac{\log(m_1^2/m_B^2)}{m_1^2/m_B^2-1} & 0 & 0 \\ 0 & m_2^3\frac{\log(m_2^2/m_B^2)}{m_2^2/m_B^2-1} & 0 \\ 0 & 0 & m_3^3\frac{\log(m_3^2/m_B^2)}{m_3^2/m_B^2-1} \end{pmatrix} \ , \tag{3.71}$$

and

$$L^M(m_B) = \begin{pmatrix} M_1^3\frac{\log(M_1^2/m_B^2)}{M_1^2/m_B^2-1} & 0 & 0 \\ 0 & M_2^3\frac{\log(M_2^2/m_B^2)}{M_2^2/m_B^2-1} & 0 \\ 0 & 0 & M_3^3\frac{\log(M_3^2/m_B^2)}{M_3^2/m_B^2-1} \end{pmatrix} \ , \tag{3.72}$$

with $m_B = m_Z, m_H$ and the entries vanish when $m_i, M_i \to 0$, $i = 1, 2, 3$.

3.6 One-Loop Seesaw equations in Exact Formalism

A relevant question is if the new δM_L term introduces any constraint in the matrix X defined in [eq. 3.11], since the existence of the zero block was fundamental in its derivation. With M given by [eq. 3.62],

[9]The term that stems from ϕ_Z makes the gauge dependent terms cancel out.

[10]The logarithm of a diagonal matrix is a matrix with the logarithm of its entries in the diagonal.

neglecting δM_R and δM^ν, but considering every other quantity as its one-loop version[11], [Eqs. 3.6, 3.8] change to:

$$K^\dagger \delta M_L - X^\dagger Z^\dagger M^{\nu T} = dK^T \ , \tag{3.73}$$

$$X K^\dagger \delta M_L + Z^\dagger M^{\nu T} = d_R X^* K^T \ , \tag{3.74}$$

and [Eqs. 3.7, 3.9] stay the same. Substituting $K^\dagger \delta M_L$ on [eq. 3.74] using [eq. 3.73] and recognizing Z^{-1} from [eq. 3.4] one obtains:

$$M^\nu = K\left(X^\dagger d_R - dX^T\right) Z^T \ , \tag{3.75}$$

which is the new form of [eq. 3.54]. One could transform from one to another using [eq. 3.10], however one cannot assume that [eq. 3.10] still applies at one-loop. Now taking $Z^\dagger M_R$ from [eq. 3.9]] onto [eq. 3.7] and using the new definition for M^ν [eq. 3.75] yields:

$$\left(K^\dagger K + X^\dagger X K^\dagger K - I\right)\left(X^\dagger d_R Z^T - dX^T Z^T\right) = 0 \ , \tag{3.76}$$

which is the one-loop equivalent of [eq. 3.10]. The difference is that the above equation is always true, due to [eq. 3.5]. Thus, X has no extra constraints at one-loop, and is still defined by [eq. 3.11], with O_c and d their one-loop versions. Since it's not possible to obtain an analytical formula for the one-loop corrections of O_c, a better definition for X at one-loop is given by:

$$X^{loop} = \left(K^{-1} R\right)^\dagger \ . \tag{3.77}$$

From [eq. 3.77] it is clear that if the one-loop corrections to the light neutrino masses are small, then the one-loop version of matrices K and R won't be much different from their tree level counterparts, implying the same fate for X and the deviations from unitarity.

3.7 How To Control Light Neutrino Loop Mass Corrections

Controlling light neutrino loop mass corrections, [eq. 3.68], reduces to control the quantity:

$$K\left(L^m(m_B)\right)K^T + R\left(L^M(m_B)\right)R^T \approx R\left(L^M(m_B)\right)R^T \ , \tag{3.78}$$

where [Eqs. 3.70, 3.71, 3.72] were used on [eq. 3.68]. The approximation is valid since, as said before, the entries of $L^m(m_B)$ vanish when $m_i \to 0$ and K is an almost unitary matrix. Thus, in very good approximation, [eq. 3.68] becomes:

$$\delta M_L \approx \frac{g^2}{64\pi^2 m_W^2} R\left[3L^M(m_Z) + L^M(m_H)\right] R^T \ . \tag{3.79}$$

Since $L^M(m_B)$ are, in general, large there are only three possibilities in order to generate a small δM_L:

[11] Assuming that δM_L was calculated using tree level quantities and M was diagonalized under the aforesaid assumptions, d, X, K, S, R and Z are now different from their tree level versions.

A Having a very small R, such that $R\left(L^M(m_B)\right)R^T$ is suppressed.

B Having a R with entries of arbitrary order of magnitude but with a given structure such that combined with a proper choice of $L^M(m_B)$ it yields a small $R\left(L^M(m_B)\right)R^T$ due to cancellations.

C Having two small heavy neutrino masses (of the order of the eV or KeV, for example), such that two of the columns of $L^M(m_B)$ are small while the remaining heavy neutrino has a large mass. Along with the choice of a special type of O_c such that one of the rows of X has small entries, leading to a column with small entries in R. This column should match the one column of $L^M(m_B)$ that is not small, i.e., the one that corresponds to the heavy neutrino with large mass. This way, $R\left(L^M(m_B)\right)R^T$ is suppressed. [12]

For case A, a small R means a small X, since $R = KX^\dagger$ given [Eqs. 3.1, 3.3], and K is almost unitary, agreeing with experimental data. This implies that, in this case, small deviations from unitarity suppress the loop corrections.

As for models of case B, the cancellation is trivial to achieve when all the heavy neutrinos are degenerate $M_1 = M_2 = M_3$, which implies $L^M(m_B) = M \times I$, where $M = M_1^3 \frac{\log(M_1^2/m_B^2)}{M_1^2/m_B^2 - 1}$ is a real number. In this parameter region, a small δM_L only requires $RR^T = KX^\dagger X^* K^T << 1$. For the same reason as before, this translates into $X^\dagger X^* << 1$. Using [eq. 3.11] in the parameter region where the heavy neutrinos are degenerate, $d_R = M_1 \times I$, gives:

$$X^\dagger X^* = -\sqrt{d} O_c^\dagger \sqrt{d_R^{-1}} \sqrt{d_R^{-1}} O_c^* \sqrt{d} = -M_1^{-1} \times d << 1 \ . \tag{3.80}$$

Thus, in this parameter region, the matrix X already satisfies the necessary condition to achieve a cancellation and obtain a small δM_L. Nevertheless, one can slightly break the degeneracy, while still having $X^\dagger X^* << 1$, resulting in a still small δM_L. This corresponds to having an almost conserved lepton-number-like charge [158, 159].

Finally, for case C models, the cancellation happens due to the matching of big entries of R with small entries of $L^M(m_B)$ and vice-versa. For this, it is fundamental to have sizable deviations from unitarity, such that one of the rows of X is very small and the other two are sizable.

It is relevant to cover the special case where the light neutrino masses are generated only at loop level. This is done in Appendix D.

3.8 Numerical Examples and the Effect of Deviations from Unitarity on Loop Corrections

This section is organized as follows. First, examples for the three types of models in which light neutrino loop mass corrections are controlled are presented: For case A - small R, and, thus, small deviations from unitarity, case B - sizable deviations from unitarity with quasi-degenerate heavy neutrinos and case C - sizable deviations from unitarity with two light heavy neutrinos.

[12] A scenario with 3 light heavy neutrino would also work but it is disfavoured due to always leading to unnaturally small neutrino Yukawa couplings, independently of the chosen deviations from unitarity.

The case A example is given for normal ordering and includes an analysis of the effect of the deviations from unitarity on the variation of the heaviest light neutrino mass after loop corrections.

The case B examples are given for normal and inverted ordering, each for two different patterns of deviations from unitarity. For the normal ordering scenario, an analysis of the effect of the deviations from unitarity on the variation of the heaviest light neutrino mass after loop corrections is also given.

A final case C example is given for normal ordering with M_1 of the order of the eV, M_2 of the order of the KeV and a large M_3, with sizable deviations from unitarity.

The numerical examples are given in the following tables, where the deviations from unitarity are expressed by the hermitian matrix η, defined in [eq. 3.27]. The first row contains quantities that are the same at tree and loop level - heavy neutrino masses, Dirac mass matrix, M^ν and Heavy neutrino mass matrix, M_R. The second row contains relevant quantities - the light neutrino masses, the matrix X and the mixing matrix that connects light and heavy neutrinos through electroweak processes, R - at tree level. The third row contains the same quantities as the second row, but at one-loop level. The mixing matrix K has entries in the U_{PMNS} 1σ allowed range, both for tree and one loop level. The differences of the squared light neutrino masses, Δm^2_{ij}, at one loop level are in the 1σ range of the values given in [Tab. 2.2]. All quantities with units of mass, except the light neutrino masses which are in eV, are expressed in terms of the top quark mass m_t. The matrix W_Z, defined in [eq. 3.16], was chosen to be

$$W_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{3.81}$$

since there are no experimental bounds for the Z matrix.

3.8.1 Case A: Small Deviations from Unitarity

The used O_c is of the type given in [eq. 3.53] with $\theta = \frac{\pi}{3}$ and $x = 4.8$. The Majorana phases were taken to be $\alpha_1 = \alpha_2 = 0$.

Table 3.1: Example for case A, with Normal Ordering of light neutrino masses. This example gives the following phenomenological important quantities: $|m_{\beta\beta}| = 5.97 \times 10^{-3}\ eV$, defined in [eq. 2.119], $m_\beta = 9.67 \times 10^{-3}\ eV$, defined in [eq. 2.113] and $N_\nu = 2.999$, defined in [eq. 2.121] with $p = 0$.

Heavy Neutrino Masses (m_t)	$\lvert M^\nu\rvert\ (m_t)$	$Tr\left[M^\nu M^{\nu\dagger}\right]\ (m_t^2)$	$\lvert M_R\rvert\ (m_t)$
$\begin{bmatrix} M_1 = 30 \\ M_2 = 60 \\ M_3 = 100 \end{bmatrix}$	$\begin{pmatrix} 1.02\times10^{-5} & 6.08\times10^{-8} & 1.10\times10^{-6} \\ 1.19\times10^{-5} & 3.52\times10^{-7} & 3.41\times10^{-6} \\ 7.97\times10^{-6} & 3.72\times10^{-7} & 4.32\times10^{-6} \end{pmatrix}$	3.41×10^{-10}	$\begin{pmatrix} 3.03\times10^{1} & 4.20\times10^{1} & 6.77 \\ 4.20\times10^{1} & 7.64\times10^{-1} & 9.38 \\ 6.77 & 9.38 & 9.90\times10^{1} \end{pmatrix}$
Tree Level Light Neutrino Masses (eV)	$\lvert\eta\rvert^{tree}$	X^{tree}	R^{tree}
$\begin{bmatrix} m_1 = 0.0062 \\ m_2 = 0.00902 \\ m_3 = 0.0542 \end{bmatrix}$	$\begin{pmatrix} 2.89\times10^{-14} & 3.40\times10^{-14} & 2.29\times10^{-14} \\ 3.40\times10^{-14} & 4.06\times10^{-14} & 2.72\times10^{-14} \\ 2.29\times10^{-14} & 2.72\times10^{-14} & 1.91\times10^{-14} \end{pmatrix}$	$\begin{pmatrix} -1.46\times10^{-7} & \left(-2.04\times10^{-7}\right)i & 2.30\times10^{-7} \\ \left(-1.06\times10^{-7}\right)i & 1.41\times10^{-7} & \left(1.66\times10^{-7}\right)i \\ \left(-8.85\times10^{-9}\right)i & 0 & \left(-4.94\times10^{-8}\right)i \end{pmatrix}$	$\begin{pmatrix} 1.3\times10^{-7}-\left(1.47\times10^{-7}\right)i & 1.02\times10^{-7}+\left(9.37\times10^{-8}\right)i & 6.9\times10^{-9}-\left(4.97\times10^{-9}\right)i \\ -1.97\times10^{-7}-\left(1.19\times10^{-7}\right)i & 8.24\times10^{-8}-\left(1.43\times10^{-7}\right)i & 6.68\times10^{-10}-\left(2.85\times10^{-8}\right)i \\ -1.01\times10^{-7}+\left(1.19\times10^{-7}\right)i & -8.29\times10^{-8}-\left(7.32\times10^{-8}\right)i & 7.69\times10^{-10}-\left(4.04\times10^{-8}\right)i \end{pmatrix}$
One Loop Light Neutrino Masses (eV)	$\lvert\eta\rvert^{loop}$	X^{loop}	R^{loop}
$\begin{bmatrix} m_1 = 0.00543 \\ m_2 = 0.0102 \\ m_3 = 0.0505 \end{bmatrix}$	$\begin{pmatrix} 2.92\times10^{-14} & 3.41\times10^{-14} & 2.27\times10^{-14} \\ 3.41\times10^{-14} & 4.09\times10^{-14} & 2.72\times10^{-14} \\ 2.27\times10^{-14} & 2.72\times10^{-14} & 1.88\times10^{-14} \end{pmatrix}$	$\begin{pmatrix} 1.39\times10^{-7}+\left(6.56\times10^{-9}\right)i & 1.98\times10^{-9}-\left(2.16\times10^{-7}\right)i & -2.24\times10^{-7}+\left(4.54\times10^{-10}\right)i \\ 4.60\times10^{-9}-\left(1.00\times10^{-7}\right)i & -1.50\times10^{-7}-\left(1.44\times10^{-9}\right)i & 2.90\times10^{-10}+\left(1.62\times10^{-7}\right)i \\ 2.90\times10^{-10}+\left(8.15\times10^{-9}\right)i & -4.07\times10^{-10}+\left(4.82\times10^{-10}\right)i & 5.04\times10^{-11}+\left(4.95\times10^{-8}\right)i \end{pmatrix}$	$\begin{pmatrix} 1.3\times10^{-7}-\left(1.47\times10^{-7}\right)i & 1.02\times10^{-7}+\left(9.37\times10^{-8}\right)i & 6.9\times10^{-9}-\left(4.97\times10^{-9}\right)i \\ -1.97\times10^{-7}-\left(1.19\times10^{-7}\right)i & 8.24\times10^{-8}-\left(1.43\times10^{-7}\right)i & 6.68\times10^{-10}-\left(2.85\times10^{-8}\right)i \\ -1.01\times10^{-7}+\left(1.19\times10^{-7}\right)i & -8.29\times10^{-8}-\left(7.32\times10^{-8}\right)i & 7.69\times10^{-10}-\left(4.04\times10^{-8}\right)i \end{pmatrix}$

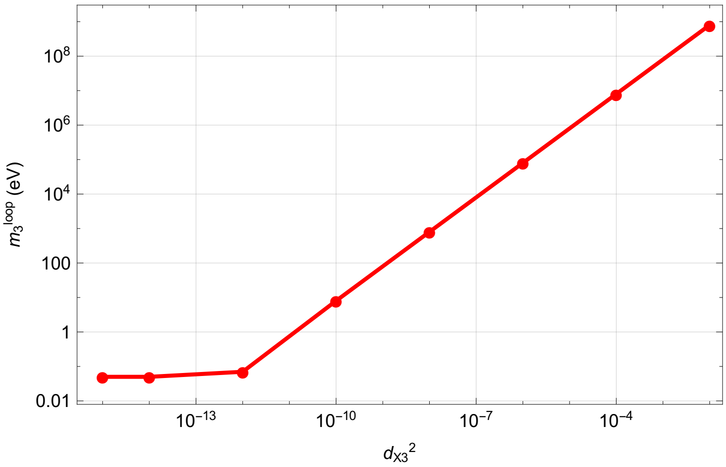


Figure 3.6: m_3 after loop corrections as a function of d_{X3}^2, generated in the example given in [Tab. 3.1], varying the value of the parameter x, while $Tr[M^\nu M^{\nu\dagger}] \leq m_t^2$ and with everything else kept constant.

The loop corrections become controlled near the minimum possible value for the deviations from unitarity $d_{X3}^2 \sim \frac{m}{M}$, as for small x one has $X \approx -i\sqrt{\frac{d}{d_R}}$. Higher level loop corrections on the example given in [Tab. 3.1] are not expected to be very big due to the smallness of the entries of the R matrix.

3.8.2 Case B: Sizable Deviations from Unitarity with two Quasi-degenerate Heavy Neutrinos

The used O_c is of the type given in [eq. 3.53] with $\theta = \frac{\pi}{3}$ and $x = 2.36 \times 10^5$. The used Majorana phases were $\alpha_1 = \frac{53}{58}\pi$, $\alpha_2 = \frac{19}{34}\pi$.

Table 3.2: Example for case B, with Normal Ordering of light neutrino masses. This example gives the following phenomenological important quantities: $|m_{\beta\beta}| = 6.58 \times 10^{-3}\ eV$, defined in [eq. 2.119], $m_\beta = 1.01 \times 10^{-2}\ eV$, defined in [eq. 2.113], and $N_\nu = 2.989$, defined in [eq. 2.121] with $p = 0$.

Heavy Neutrino Masses (m_t)	$\lvert M^\nu \rvert\ (m_t)$	$Tr\left[M^\nu M^{\nu\dagger}\right]\ (m_t^2)$	$\lvert M_R \rvert\ (m_t)$
$\begin{bmatrix} M_1 = 3 \\ M_2 = 3 + 1 \times 10^{-10} \\ M_3 = 50 \end{bmatrix}$	$\begin{pmatrix} 0.140 & 4.12 \times 10^{-13} & 6.49 \times 10^{-7} \\ 0.000876 & 2.06 \times 10^{-12} & 2.32 \times 10^{-6} \\ 0.171 & 1.84 \times 10^{-12} & 3.17 \times 10^{-6} \end{pmatrix}$	0.0488	$\begin{pmatrix} 7.15 \times 10^{-10} & 2.99 & 1.76 \times 10^{-4} \\ 2.99 & 2.14 \times 10^{-11} & 3.85 \times 10^{-5} \\ 1.76 \times 10^{-4} & 3.85 \times 10^{-5} & 5.00 \times 10^{1} \end{pmatrix}$
Tree Level Light Neutrino Masses (eV)	$\lvert \eta \rvert^{tree}$	X^{tree}	R^{tree}
$\begin{bmatrix} m_1 = 0.00507 \\ m_2 = 0.0100 \\ m_3 = 0.0522 \end{bmatrix}$	$\begin{pmatrix} 1.09 \times 10^{-3} & 6.82 \times 10^{-6} & 1.33 \times 10^{-3} \\ 6.82 \times 10^{-6} & 4.27 \times 10^{-8} & 8.34 \times 10^{-6} \\ 1.33 \times 10^{-3} & 8.34 \times 10^{-6} & 1.63 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -0.0206 & -0.0328i & 0.0351 \\ -0.0206i & 0.0328 & 0.0351i \\ \left(-1.13 \times 10^{-8}\right) i & 0 & \left(-6.85 \times 10^{-8}\right) i \end{pmatrix}$	$\begin{pmatrix} -0.0262 - 0.0201i & -0.0201 + 0.0262i & 4.78 \times 10^{-9} + \left(4.53 \times 10^{-10}\right) i \\ 0.000137 + 0.000154i & 0.000154 - 0.000137i & -4.46 \times 10^{-8} - \left(1.28 \times 10^{-8}\right) i \\ -0.0066 + 0.0398i & 0.0398 + 0.0066i & -5.12 \times 10^{-8} - \left(4.92 \times 10^{-9}\right) i \end{pmatrix}$
One Loop Light Neutrino Masses (eV)	$\lvert \eta \rvert^{loop}$	X^{loop}	R^{loop}
$\begin{bmatrix} m_1 = 0.00491 \\ m_2 = 0.0100 \\ m_3 = 0.0504 \end{bmatrix}$	$\begin{pmatrix} 1.09 \times 10^{-3} & 6.82 \times 10^{-6} & 1.33 \times 10^{-3} \\ 6.82 \times 10^{-6} & 4.27 \times 10^{-8} & 8.33 \times 10^{-6} \\ 1.33 \times 10^{-3} & 8.33 \times 10^{-6} & 1.63 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -0.0203 + \left(9.27 \times 10^{-6}\right) i & 6.29 \times 10^{-6} + 0.0330i & -0.0350 + \left(5.33 \times 10^{-7}\right) i \\ -9.27 \times 10^{-6} - 0.0203i & -0.0330 + \left(6.29 \times 10^{-6}\right) i & -5.33 \times 10^{-7} - 0.0350i \\ 2.77 \times 10^{-14} - \left(1.14 \times 10^{-8}\right) i & 9.27 \times 10^{-11} + \left(2.11 \times 10^{-12}\right) i & 2.68 \times 10^{-15} + \left(6.85 \times 10^{-8}\right) i \end{pmatrix}$	$\begin{pmatrix} -0.0262 - 0.0201i & -0.0201 + 0.0262i & 4.78 \times 10^{-9} + \left(4.53 \times 10^{-10}\right) i \\ 0.000137 + 0.000154i & 0.000154 - 0.000137i & -4.46 \times 10^{-8} - \left(1.28 \times 10^{-8}\right) i \\ -0.0066 + 0.0398i & 0.0398 + 0.0066i & -5.12 \times 10^{-8} - \left(4.92 \times 10^{-9}\right) i \end{pmatrix}$

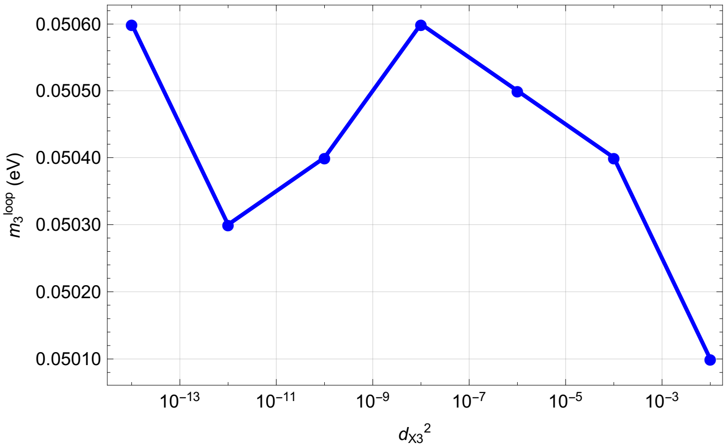


Figure 3.7: m_3 after loop corrections as a function of d_{X3}^2, generated in the example given in [Tab. 3.2], varying the value of the parameter x, while $Tr[M^\nu M^{\nu\dagger}] \leq m_t^2$ and with everything else kept constant.

The loop corrections are essentially constant, independently of the size of the deviations from unitarity d_{X3}^2. This happens due to the cancelling structure of R and because of the quasi-degeneracy of M_1 and M_2. Higher level loop corrections on the example given in [Tab. 3.2] are not expected to be very big due to the persistence of structure of R after loop corrections.

For a different heavy neutrino mass hierarchy, the used O_c is of the type given in [eq. 3.53] multiplied on the left by a matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.82}$$

with $\theta = \frac{\pi}{3}$ and $x = 3.60 \times 10^5$. The used Majorana phases were $\alpha_1 = \frac{35}{22}\pi$, $\alpha_2 = \frac{97}{107}\pi$.

Table 3.3: Example for case B, with Normal Ordering of light neutrino masses. This example gives the following phenomenological important quantities: $|m_{\beta\beta}| = 1.78 \times 10^{-2}\ eV$, defined in [eq. 2.119], $m_\beta = 1.00 \times 10^{-2}\ eV$, defined in [eq. 2.113], and $N_\nu = 2.992$, defined in [eq. 2.121] with $p = 0$.

Heavy Neutrino Masses (m_t)	$\lvert M^\nu\rvert\ (m_t)$	$Tr\left[M^\nu M^{\nu\dagger}\right]\ (m_t^2)$	$\lvert M_R\rvert\ (m_t)$
$\begin{pmatrix} M_1 = 3 \\ M_2 = 9 \\ M_3 = 9 + 1 \times 10^{-10} \end{pmatrix}$	$\begin{pmatrix} 0.00956 & 7.11 \times 10^{-13} & 1.64 \times 10^{-7} \\ 0.162 & 2.19 \times 10^{-12} & 5.89 \times 10^{-7} \\ 0.540 & 2.03 \times 10^{-12} & 7.44 \times 10^{-7} \end{pmatrix}$	0.318	$\begin{pmatrix} 1.09 \times 10^{-10} & 8.98 & 2.42 \times 10^{-5} \\ 8.98 & 2.54 \times 10^{-11} & 1.05 \times 10^{-5} \\ 2.42 \times 10^{-5} & 1.05 \times 10^{-5} & 3.00 \end{pmatrix}$
Tree Level Light Neutrino Masses (eV)	$\lvert\eta\rvert^{tree}$	X^{tree}	R^{tree}
$\begin{pmatrix} m_1 = 0.00500 \\ m_2 = 0.0100 \\ m_3 = 0.0502 \end{pmatrix}$	$\begin{pmatrix} 5.65 \times 10^{-7} & 9.55 \times 10^{-6} & 3.19 \times 10^{-5} \\ 9.55 \times 10^{-6} & 1.62 \times 10^{-4} & 5.40 \times 10^{-4} \\ 3.19 \times 10^{-5} & 5.40 \times 10^{-4} & 1.80 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} \left(-4.25 \times 10^{-8}\right) i & 0 & \left(-2.83 \times 10^{-7}\right) i \\ -0.0184 & -0.0289i & 0.0283 \\ -0.0184i & 0.0289 & 0.0283i \end{pmatrix}$	$\begin{pmatrix} 4.15 \times 10^{-8} + \left(3.55 \times 10^{-8}\right) i & 0.000636 + 0.000401i & 0.000401 - 0.000636i \\ -5.62 \times 10^{-8} - \left(1.93 \times 10^{-7}\right) i & 0.00709 + 0.0105i & 0.0105 - 0.00709i \\ -6.48 \times 10^{-8} - \left(1.85 \times 10^{-7}\right) i & -0.0424 + 0.00194i & 0.00194 + 0.0424i \end{pmatrix}$
One Loop Light Neutrino Masses (eV)	$\lvert\eta\rvert^{loop}$	X^{loop}	R^{loop}
$\begin{pmatrix} m_1 = 0.00484 \\ m_2 = 0.00992 \\ m_3 = 0.0503 \end{pmatrix}$	$\begin{pmatrix} 5.65 \times 10^{-7} & 9.54 \times 10^{-6} & 3.19 \times 10^{-5} \\ 9.54 \times 10^{-6} & 1.61 \times 10^{-4} & 5.39 \times 10^{-4} \\ 3.19 \times 10^{-5} & 5.39 \times 10^{-4} & 1.80 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -2.09 \times 10^{-12} + \left(4.17 \times 10^{-8}\right) i & -3.12 \times 10^{-10} + \left(1.09 \times 10^{-12}\right) i & 3.28 \times 10^{-13} - \left(2.83 \times 10^{-7}\right) i \\ -0.0183 + \left(2.27 \times 10^{-6}\right) i & 1.70 \times 10^{-6} + 0.0290i & -0.0282 + \left(5.48 \times 10^{-7}\right) i \\ 2.27 \times 10^{-6} + 0.0183i & 0.0290 - \left(1.70 \times 10^{-6}\right) i & 5.48 \times 10^{-7} + 0.0282i \end{pmatrix}$	$\begin{pmatrix} -4.15 \times 10^{-8} - \left(3.55 \times 10^{-8}\right) i & 0.000636 + 0.000401i & -0.000401 + 0.000636i \\ 5.62 \times 10^{-8} + \left(1.93 \times 10^{-7}\right) i & 0.00709 + 0.0105i & -0.0105 + 0.00709i \\ 6.48 \times 10^{-8} + \left(1.85 \times 10^{-7}\right) i & -0.0424 + 0.00194i & -0.00194 - 0.0424i \end{pmatrix}$

For the same heavy neutrino mass hierarchy as the first example of this subsection, [Tab. 3.2] but for inverted ordering, the used O_c is of the type given in [eq. 3.53] with $\theta = \frac{\pi}{4}$ and $x = 8.14 \times 10^4$. The used Majorana phases were $\alpha_1 = \frac{229}{150}\pi$, $\alpha_2 = \frac{23}{22}\pi$.

Table 3.4: Example for case B, with Inverted Ordering of light neutrino masses. This example gives the following phenomenological important quantities: $|m_{\beta\beta}| = 1.78 \times 10^{-2}\ eV$, defined in [eq. 2.119], $m_\beta = 5.00 \times 10^{-2}\ eV$, defined in [eq. 2.113],and $N_\nu = 2.996$, defined in [eq. 2.121] with $p = 0$.

Heavy Neutrino Masses (m_t)	$\lvert M^\nu \rvert\ (m_t)$	$Tr\left[M^\nu M^{\nu\dagger}\right]\ (m_t^2)$	$\lvert M_R \rvert\ (m_t)$
$\begin{pmatrix} M_1 = 3 \\ M_2 = 3 + 1 \times 10^{-10} \\ M_3 = 50 \end{pmatrix}$	$\begin{pmatrix} 0.00184 & 5.87 \times 10^{-12} & 2.27 \times 10^{-6} \\ 0.0605 & 1.88 \times 10^{-12} & 2.98 \times 10^{-6} \\ 0.124 & 1.42 \times 10^{-12} & 2.91 \times 10^{-6} \end{pmatrix}$	0.0189	$\begin{pmatrix} 1.95 \times 10^{-8} & 3.00 & 9.83 \times 10^{-4} \\ 3.00 & 4.09 \times 10^{-11} & 8.61 \times 10^{-5} \\ 9.83 \times 10^{-4} & 8.61 \times 10^{-5} & 5.00 \times 10^{1} \end{pmatrix}$
Tree Level Light Neutrino Masses (eV)	$\lvert\eta\rvert^{tree}$	X^{tree}	R^{tree}
$\begin{pmatrix} m_1 = 0.0516 \\ m_2 = 0.0517 \\ m_3 = 0.009 \end{pmatrix}$	$\begin{pmatrix} 1.87 \times 10^{-7} & 6.18 \times 10^{-6} & 1.26 \times 10^{-5} \\ 6.18 \times 10^{-6} & 2.04 \times 10^{-4} & 4.16 \times 10^{-4} \\ 1.26 \times 10^{-5} & 4.16 \times 10^{-4} & 8.49 \times 10^{-4} \end{pmatrix}$	$\begin{pmatrix} -0.0185 & -0.0256i & 0.00742 \\ -0.0185i & 0.0256 & 0.00742i \\ \left(-5.34 \times 10^{-8}\right) i & 0 & \left(-2.32 \times 10^{-8}\right) i \end{pmatrix}$	$\begin{pmatrix} -0.000395 + 0.000176i & 0.000176 + 0.000395i & 3.22 \times 10^{-9} + \left(4.47 \times 10^{-8}\right) i \\ 0.0143 + 0.000259i & 0.000259 - 0.0143i & -2.36 \times 10^{-9} - \left(3.59 \times 10^{-8}\right) i \\ -0.0291 - 0.00207i & -0.00207 + 0.0291i & -1.94 \times 10^{-9} + \left(9.65 \times 10^{-9}\right) i \end{pmatrix}$
One Loop Light Neutrino Masses (eV)	$\lvert\eta\rvert^{loop}$	X^{loop}	R^{loop}
$\begin{pmatrix} m_1 = 0.0503 \\ m_2 = 0.0510 \\ m_3 = 0.00875 \end{pmatrix}$	$\begin{pmatrix} 1.87 \times 10^{-7} & 6.17 \times 10^{-6} & 1.26 \times 10^{-5} \\ 6.17 \times 10^{-6} & 2.04 \times 10^{-4} & 4.16 \times 10^{-4} \\ 1.26 \times 10^{-5} & 4.16 \times 10^{-4} & 8.48 \times 10^{-4} \end{pmatrix}$	$\begin{pmatrix} 0.0185 - \left(9.70 \times 10^{-6}\right) i & -5.70 \times 10^{-6} - 0.0257i & 0.00731 + \left(4.43 \times 10^{-6}\right) i \\ 9.70 \times 10^{-6} + 0.0185i & 0.0257 - \left(5.70 \times 10^{-6}\right) i & -4.43 \times 10^{-6} + 0.00731i \\ 1.22 \times 10^{-13} - \left(5.33 \times 10^{-8}\right) i & -7.85 \times 10^{-11} + \left(2.42 \times 10^{-11}\right) i & -2.52 \times 10^{-13} + \left(2.36 \times 10^{-8}\right) i \end{pmatrix}$	$\begin{pmatrix} 0.000395 - 0.000176i & -0.000176 - 0.000395i & 3.22 \times 10^{-9} + \left(4.47 \times 10^{-8}\right) i \\ -0.0143 - 0.000259i & -0.000259 + 0.0143i & -2.36 \times 10^{-9} - \left(3.59 \times 10^{-8}\right) i \\ 0.029 + 0.00207i & 0.00208 - 0.0291i & -1.94 \times 10^{-9} + \left(9.65 \times 10^{-9}\right) i \end{pmatrix}$

For the same heavy neutrino mass hierarchy as the second example of this subsection, [Tab. 3.3], but for inverted ordering, the used O_c is of the type

$$O_c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{x^2+1} & 0 & ix \\ ix & 0 & -\sqrt{x^2+1} \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.83}$$

with $\theta = \frac{\pi}{10}$ and $x = 2.44 \times 10^5$. The used Majorana phases were $\alpha_1 = \frac{3}{2}\pi$, $\alpha_2 = \frac{47}{80}\pi$.

Table 3.5: Example for case B, with Inverted Ordering of light neutrino masses. This example gives the following phenomenological important quantities: $|m_{\beta\beta}| = 1.76 \times 10^{-2}\ eV$, defined in [eq. 2.119], $m_\beta = 4.97 \times 10^{-2}\ eV$, defined in [eq. 2.113] and $N_\nu = 2.991$, defined in [eq. 2.121] with $p = 0$.

Heavy Neutrino Masses (m_t)	$\lvert M^\nu \rvert\ (m_t)$	$Tr\left[M^\nu M^{\nu\dagger}\right]\ (m_t^2)$	$\lvert M_R \rvert\ (m_t)$
$\begin{pmatrix} M_1 = 3 \\ M_2 = 9 \\ M_3 = 9 + 1 \times 10^{-10} \end{pmatrix}$	$\begin{pmatrix} 0.425 & 3.23 \times 10^{-12} & 5.56 \times 10^{-7} \\ 0.00434 & 1.49 \times 10^{-12} & 5.31 \times 10^{-7} \\ 0.432 & 1.93 \times 10^{-12} & 5.47 \times 10^{-7} \end{pmatrix}$	0.367	$\begin{pmatrix} 1.54 \times 10^{-10} & 8.98 & 1.48 \times 10^{-7} \\ 8.98 & 5.35 \times 10^{-11} & 5.61 \times 10^{-8} \\ 1.48 \times 10^{-7} & 5.61 \times 10^{-8} & 3.00 \end{pmatrix}$
Tree Level Light Neutrino Masses (eV)	$\lvert\eta\rvert^{tree}$	X^{tree}	R^{tree}
$\begin{pmatrix} m_1 = 0.0509 \\ m_2 = 0.0516 \\ m_3 = 0.00852 \end{pmatrix}$	$\begin{pmatrix} 1.12 \times 10^{-3} & 1.14 \times 10^{-5} & 1.13 \times 10^{-3} \\ 1.14 \times 10^{-5} & 1.16 \times 10^{-7} & 1.16 \times 10^{-5} \\ 1.13 \times 10^{-3} & 1.16 \times 10^{-5} & 1.15 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} \left(9.66 \times 10^{-8}\right) i & \left(-2.99 \times 10^{-7}\right) i & 0 \\ -0.0420i & -0.0137i & 0.0181 \\ 0.0420 & 0.0137 & 0.0181i \end{pmatrix}$	$\begin{pmatrix} 1.68 \times 10^{-7} - \left(7.89 \times 10^{-8}\right) i & 0.00532 + 0.0330i & 0.0330 - 0.00532i \\ 1.69 \times 10^{-7} + \left(5.18 \times 10^{-8}\right) i & -0.000225 - 0.000256i & -0.000256 + 0.000225i \\ -1.80 \times 10^{-7} - \left(2.80 \times 10^{-8}\right) i & -0.0148 + 0.0305i & 0.0305 + 0.0148i \end{pmatrix}$
One Loop Light Neutrino Masses (eV)	$\lvert\eta\rvert^{loop}$	X^{loop}	R^{loop}
$\begin{pmatrix} m_1 = 0.0501 \\ m_2 = 0.0508 \\ m_3 = 0.00828 \end{pmatrix}$	$\begin{pmatrix} 1.11 \times 10^{-3} & 1.14 \times 10^{-5} & 1.13 \times 10^{-3} \\ 1.14 \times 10^{-5} & 1.16 \times 10^{-7} & 1.16 \times 10^{-5} \\ 1.13 \times 10^{-3} & 1.16 \times 10^{-5} & 1.15 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} 1.10 \times 10^{-9} + \left(7.73 \times 10^{-8}\right) i & 2.78 \times 10^{-10} + \left(3.05 \times 10^{-7}\right) i & 1.94 \times 10^{-11} - \left(1.74 \times 10^{-11}\right) i \\ -0.0000374 + 0.0428i & 0.000154 - 0.0110i & 0.0180 - \left(5.06 \times 10^{-6}\right) i \\ 0.0428 + 0.0000374i & -0.0110 - 0.000154i & -5.06 \times 10^{-6} - 0.0180i \end{pmatrix}$	$\begin{pmatrix} 1.68 \times 10^{-7} - \left(7.89 \times 10^{-8}\right) i & -0.00532 - 0.033i & 0.033 - 0.00532i \\ 1.69 \times 10^{-7} + \left(5.18 \times 10^{-8}\right) i & 0.000225 + 0.000256i & -0.000256 + 0.000225i \\ -1.8 \times 10^{-7} - \left(2.8 \times 10^{-8}\right) i & 0.0148 - 0.0305i & 0.0305 + 0.0148i \end{pmatrix}$

3.8.3 Case C: Sizable Deviations from Unitarity with two Light Heavy Neutrinos

The used O_c is of the type given in [eq. 3.53] with $\theta = \frac{\pi}{3}$ and $x = 0.78$. The used Majorana phases were $\alpha_1 = \frac{52}{125}\pi$, $\alpha_2 = \frac{389}{200}\pi$.

Table 3.6: Example for case C, with Normal Ordering of light neutrino masses. This example gives the following phenomenological important quantities: $|m_{\beta\beta}| = 1.13 \times 10^{-3}\ eV$, defined in [eq. 2.119], $m_\beta = 1.03 \times 10^{-2}\ eV$, defined in [eq. 2.113], and $N_\nu = 2.999$, defined in [eq. 2.121] with $p = 2$.

Heavy Neutrino Masses (m_t)	$\lvert M^\nu \rvert\ (m_t)$	$Tr\left[M^\nu M^{\nu\dagger}\right]\ (m_t^2)$	$\lvert M_R \rvert\ (m_t)$
$\begin{pmatrix} M_1 = 2.88 \times 10^{-11} \\ M_2 = 1.44 \times 10^{-8} \\ M_3 = 5.76 \times 10^{12} \end{pmatrix}$	$\begin{pmatrix} 0.226 & 3.62 \times 10^{-13} & 3.24 \times 10^{-11} \\ 0.749 & 2.25 \times 10^{-12} & 8.69 \times 10^{-11} \\ 1.02 & 1.73 \times 10^{-12} & 1.22 \times 10^{-10} \end{pmatrix}$	1.66	$\begin{pmatrix} 5.76 \times 10^{12} & 9.01 & 4.93 \times 10^{2} \\ 9.01 & 1.54 \times 10^{-11} & 1.43 \times 10^{-9} \\ 4.93 \times 10^{2} & 1.43 \times 10^{-9} & 5.66 \times 10^{-8} \end{pmatrix}$
Tree Level Light Neutrino Masses (eV)	$\lvert \eta \rvert^{tree}$	X^{tree}	R^{tree}
$\begin{pmatrix} m_1 = 0.00500 \\ m_2 = 0.00987 \\ m_3 = 0.0627 \end{pmatrix}$	$\begin{pmatrix} 1.30 \times 10^{-3} & 1.86 \times 10^{-5} & 1.32 \times 10^{-3} \\ 1.86 \times 10^{-5} & 1.58 \times 10^{-6} & 1.88 \times 10^{-5} \\ 1.32 \times 10^{-3} & 1.88 \times 10^{-5} & 1.35 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} -0.0217 & -0.0562i & 0.0407 \\ -0.00158i & 0.00154 & 0.00297i \\ \left(-3.31 \times 10^{-14}\right) i & 0 & \left(-2.21 \times 10^{-13}\right) i \end{pmatrix}$	$\begin{pmatrix} -0.0488 + 0.0141i & 0.000655 + 0.00218i & -3.21 \times 10^{-14} + \left(2.26 \times 10^{-14}\right) i \\ -0.000699 + 0.000381i & -0.000271 - 0.00156i & 2.22 \times 10^{-14} + \left(1.28 \times 10^{-13}\right) i \\ 0.0486 - 0.0179i & -0.000815 - 0.00230i & 2.55 \times 10^{-14} + \left(1.76 \times 10^{-13}\right) i \end{pmatrix}$
One Loop Light Neutrino Masses (eV)	$\lvert \eta \rvert^{loop}$	X^{loop}	R^{loop}
$\begin{pmatrix} m_1 = 0.00467 \\ m_2 = 0.00986 \\ m_3 = 0.0504 \end{pmatrix}$	$\begin{pmatrix} 1.23 \times 10^{-3} & 6.91 \times 10^{-6} & 1.32 \times 10^{-3} \\ 6.91 \times 10^{-6} & 1.35 \times 10^{-6} & 6.92 \times 10^{-6} \\ 1.32 \times 10^{-3} & 6.92 \times 10^{-6} & 1.34 \times 10^{-3} \end{pmatrix}$	$\begin{pmatrix} 0.0202 - 0.0000320i & 0.0000754 + 0.0559i & -0.0418 - 0.000156i \\ 9.71 \times 10^{-7} + 0.00147i & -0.00153 + \left(2.86 \times 10^{-6}\right) i & 5.67 \times 10^{-6} - 0.00305i \\ 1.75 \times 10^{-16} - \left(4.22 \times 10^{-14}\right) i & 5.20 \times 10^{-18} - \left(3.55 \times 10^{-16}\right) i & -3.82 \times 10^{-17} - \left(2.20 \times 10^{-13}\right) i \end{pmatrix}$	$\begin{pmatrix} 0.0488 - 0.0141i & -0.000655 - 0.00218i & -3.21 \times 10^{-14} + \left(2.26 \times 10^{-14}\right) i \\ 0.000305 - 0.00015i & 0.000282 + 0.00158i & 2.21 \times 10^{-14} + \left(1.28 \times 10^{-13}\right) i \\ -0.0485 + 0.0179i & 0.000815 + 0.0023i & 2.55 \times 10^{-14} + \left(1.76 \times 10^{-13}\right) i \end{pmatrix}$

This situation is possible because of the interplay of three things. The order of magnitude of the masses M_1 and M_2, the big deviations from unitarity and the O_c chosen to be like in [eq. 3.53]. As the deviations from unitarity are sizable, and X is of this type, its third row is very small, thus cancelling the effect of a very large M_3 on [eq. 3.79]. Furthermore, because of the smallness of M_1 and M_2, the first two entries of [eq. 3.72] are small, thus controlling the loop generated mass matrix δM_L.

Studying the effect of the variation of the parameter x in these type of models, as done in [Figs. 3.6, 3.7], is cumbersome, since x doesn't control the deviations from unitarity for these type of models. This happens because these models achieve sizable deviations from unitarity for $x \sim 1$. Thus, reducing x doesn't reduce the deviations from unitarity, because, as explained below [eq. 3.38], one enters the regime where $r = 1$, with r defined on [eq. 3.35]. The only way to decrease the deviations from unitarity is increasing the heavy neutrino masses, as one can infer from analyzing [eq. 3.36] with $r = 1$, and this, of course, has a big effect on the loop corrections. Summarizing, it's not possible to isolate the effect of the deviations of unitarity on the loop corrections for these type of models.

Chapter 4

Conclusions

In this work, a novel parametrization, adequate for the exact treatment of Seesaw Type I models independent of the scale of M_R was exploited. This revealed a matrix, X, defined in [eq. 3.11], responsible for the deviations from unitarity of the leptonic mixing matrix K. This parametrization clarifies the relation between heavy neutrino masses and deviations from unitarity which is explained in subsections 3.2, 3.3 and 3.4 and can be summarized in [eq. 3.61], which means that to achieve natural values for the Yukawa couplings one needs to take both the size of the deviations from unitarity and the scale of the heavy neutrino masses into account. The possibly dangerously large one-loop corrections were studied, and from that, three types of models with controlled loop corrections were suggested.

Case A models, with small deviations from unitarity, without constraints on the heavy neutrinos masses and with possibly small Yukawa couplings.[1] These are very complicated to prove experimentally.

Case B models, with two quasi-degenerate heavy neutrino masses of the order of the top mass, sizable deviations from unitarity and without unnaturally small Yukawa couplings. These are appealing because they can be observed in the next round of experiments at the LHC. Moreover, it would be interesting to study if the existence of at least two quasi-degenerate heavy neutrinos enables the possibility of resonant Leptogenesis, providing an explanation to the observed matter-anti matter asymmetry [124, 128, 129].

Case C models, with two light heavy neutrinos, sizable deviations from unitarity and without unnaturally small Yukawa couplings. These are appealing because KATRIN will be able to explore the existence of at least one heavy (mostly sterile) neutrino in the mass range of $1 - 18.5\ KeV$, with a mixing to the active neutrino ν_e as $|R_{11}|^2 \geq 10^{-6}$ [160, 161]. Furthermore, they can explain the MiniBooNE excess [69] and other anomalies [147] and can give explanations to other Physics puzzles like dark matter (when M_2 has a mass on the KeV scale like in the example given in [Tab. 3.6]) as pointed out in [151].

The question of the possibility of Thermal Leptogenesis for case A and case C is highly relevant, and requires further study. All models explain the smallness of light neutrino masses and case C models have a dark matter candidate.

Experimental input from KATRIN, the LHC and neutrino oscillation experiments will be fundamental to discern which, if any, of these models might match with Nature.

[1]Unnaturally small for heavy neutrino masses close to the electroweak scale like in the example in [Tab. 3.1]. For heavy neutrino masses near the GUT scale one retrieves the standard seesaw, which can have order 1 Yukawa couplings.

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Appendix A

νSM extra Feynman Rules

All momenta point towards the vertex, arrows mean fermion flow (flow of $-e$ charge), for outgoing states flip the vertex while maintaining all the momenta and fermion flow arrows but the momenta of the outgoing states, which one should also flip. A Dirac fermion is a particle if its momenta points in the same direction as the fermion flow, otherwise it's an anti-particle. If one flips the momenta of a charged boson then one is considering the vertex with the charge conjugate of it, i.e., diagrammatically $B^-(p) = B^+(-p)$.

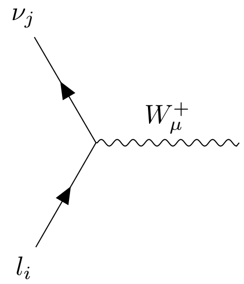
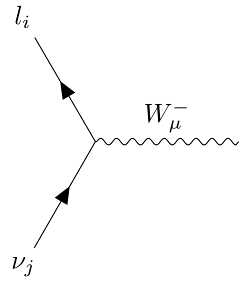
<table>
<tr><th>Vertex</th><th>Rule</th></tr>
<tr><td>
</td><td>$\frac{-ig}{\sqrt{2}}\gamma_\mu P_L U^*_{ij}$</td></tr>
<tr><td>
</td><td>$\frac{-ig}{\sqrt{2}}\gamma_\mu P_L U_{ij}$</td></tr>
</table>

Table A.1: νSM Charged Currents

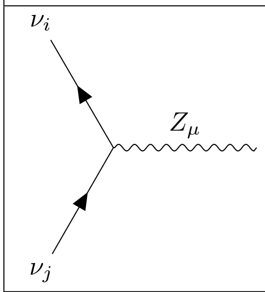
Vertex	Rule
	$\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L \delta_{ij}$

Table A.2: νSM Neutral Currents

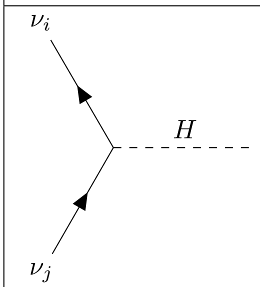
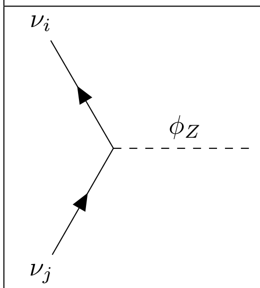
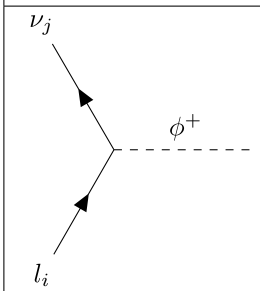
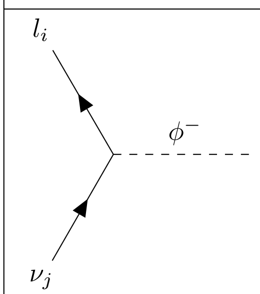
Vertex	Rule
	$\frac{-ig}{2m_W} m_{\nu_{ii}} \delta_{ij}$
	$\frac{-g}{2m_W}\gamma_5 m_{\nu_{ii}} \delta_{ij}$
	$\frac{ig}{\sqrt{2}m_W}\left(m_{\nu_{jj}} P_L - m_{l_{ii}} P_R\right) U_{ij}^*$
	$\frac{ig}{\sqrt{2}m_W}\left(m_{\nu_{jj}} P_R - m_{l_{ii}} P_L\right) U_{ij}$

Table A.3: νSM Lepton-Higgs and Lepton-Goldstone Bosons Interactions

Appendix B

$SI\nu SM$ extra Feynman Rules

Taking the reality condition on [eq. 2.29] and applies it to the 1/2 spin field expansion on [Tab. 1.2], while using the standard transformation of fields under charge conjugation:

$$\Psi^c = C\overline{\Psi}^T \rightarrow \, , v_s(p) = C\gamma_0^T u_s^*(p) \, , u_s(p) = C\gamma_0^T v_s^*(p) \, , \tag{B.1}$$

one gets the following relation:

$$a_p^s = b_p^s \, . \tag{B.2}$$

This means that a Majorana fermion is its own anti-particle, and that there are more combinations of field operators that can create and annihilate a particle than the usual $\Psi\overline{\Psi}$. Using [eq. 2.29] one gets:

$$\begin{aligned} S_p(\nu_a \nu_b) &= S_p(\nu_a \overline{\nu}_d) C^T_{db} = -\left(\frac{i(\gamma_\mu p^\mu + m)}{p^2 - m^2} C\right)_{ab} \, , \\ S_p(\overline{\nu}_a \overline{\nu}_b) &= C^{-1}_{ad} S_p(\nu_d \overline{\nu}_b) = \left(C^{-1} \frac{i(\gamma_\mu p^\mu + m)}{p^2 - m^2}\right)_{ab} \, , \\ S_p(\overline{\nu}_a \nu_b) &= C^{-1}_{ad} S_p(\nu_d \overline{\nu}_c) C^T_{cb} = -\left(C^{-1} \frac{i(\gamma_\mu p^\mu + m)}{p^2 - m^2} C\right)_{ab} \, . \end{aligned} \tag{B.3}$$

However, it is possible to write the Lagrangian of every chiral theory with Majorana fermions in a form independent of C - using [eq. 2.34]. If one is consistent and extracts the theory's Feynman Rules from that C independent Lagrangian, one only needs to use the usual Dirac propagator [162, 163]. If one sticks with a C dependent Lagrangian, it is always possible to cancel every C using a C dependent propagator and [Eqs. B.2] , leading to a C independent result. As for the external lines, if one has 2 Majorana fields in the incoming and/or outgoing, the result is independent of the labeling of particle or anti-particle to them in the vertex. The standard procedure is to consider that one is a particle and the other is an anti-particle, choosing the spinors accordingly [163]. If there's only one Majorana field in the incoming and/or outgoing lines of a vertex, two interesting cases are possible. There is a lepton number conserving vertex, given in [eq. 2.67] and lepton number violating vertex, given by transforming [eq. 2.67] like in [eq. 2.69]. For the lepton number conserving case, the label of particle or anti-particle given to the Majorana particle is fixed by the labeling of the Dirac particle [162]. Nevertheless, if one is dealing with a Dirac fermion number violating process involving Dirac fermions one should use the lepton number violating vertex and take special care on using the adequate spinor for the conjugated charged particle - if one uses the vertex factor on [Tables B.3, B.8] one should use the spinor given in [Table B.1] under "Conjugated Dirac Fermion" and "Anti-Conjugated Dirac fermion"[1]. Furthermore, a factor of $\frac{-1}{2}$ must be associated with each closed Majorana fermion loop [162]. This can be summarized in the following Feynman rules for external lines:

Type of Particle	Incoming	Outgoing
Scalar	1	1
Dirac Fermion/Anti - Conjugated Dirac Fermion	$u_s(p)$	$\overline{u}_s(p)$
Dirac Anti-Fermion/Conjugated Dirac Fermion	$\overline{v}_s(p)$	$v_s(p)$
Majorana Fermion	$u_s(p)$, $\overline{v}_s(p)$	$\overline{u}_s(p)$, $v_s(p)$
Vector Boson	$\epsilon^{\mu}_r(p)$	$\epsilon^{*\mu}_r(p)$

Table B.1: Feynman Rules for external lines

For Majorana neutrinos, the direction of the arrow (fermion flow) is meaningless - the same vertex with a reverted direction of the arrow exists. It might be useful to use, in some cases:

$$u(\pm q) = v(\mp q) \ . \tag{B.4}$$

In the cases where the arrow doesn't matter (vertices involving two Majorana particles) all the direction possibilities are drawn. Note that, including the lepton number violating cases, these amount to four (the four non-repeating combinations of the arrow's direction). However it can be proven that the lepton number violating cases are actually the same vertices as the lepton number conserving ones [114] (for Majorana particles). To avoid overcounting, one should only consider one fermion flow and use or a lepton number conserving vertex or a lepton number violating vertex, with the proper choice of spinors for each case choice. In this work, one uses the former - see first equality of example given in [eq. B.5]. Choosing the other flow gives an extra minus sign for vector-like vertices, scalar-like vertices don't change. Thus, when dealing with multiple diagrams it's fundamental to stick with the choice of fermion flow (label of particle/anti-particle) for all diagrams, such that the one gets the relative signs correctly. To aid the interpretation of the following Feynman rules, one presents the amplitude of the process $Z_\mu(p_1) + \nu_i(p_2) \to \nu_j(p_3)$:

$$iM = \left[\overline{u(p_3)}\left(\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L (K^\dagger K)_{ij}\right) u(p_2)\right]\epsilon_\mu(p_1) = -\left[\overline{v(p_2)}\left(\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L (K^\dagger K)_{ij}\right) v(p_3)\right]\epsilon_\mu(p_1) \ , \tag{B.5}$$

of the process $e^-(p_1) + \nu_j(p_2) \to \phi^-(p_3)$:

$$\begin{aligned} iM &= \left[\overline{v(p_2)}\left(\frac{ig}{\sqrt{2}m_W}\cdot\left(P_L S_{jk} M^{\nu *}_{1k} - P_R K^*_{1j} m_{l_{11}}\right)\right) u(p_1)\right] \\ &= \left[\overline{v(p_1)}\left(\frac{ig}{\sqrt{2}m_W}\cdot\left(P_L S_{jk} M^{\nu *}_{1k} - P_R K^*_{1j} m_{l_{11}}\right)\right) u(p_2)\right] \ , \end{aligned} \tag{B.6}$$

and of the t-channel amplitude of the Dirac fermion number violating process $e^+(p_1)+e^+(p_2) \to W^+_\mu(p_3)+$

[1]Note that Conjugated Dirac Fermion and Dirac Fermion have opposite charges.

$W^+_\nu(p_4)$, with only the three light neutrinos in the intermediate state:

$$iM_t = \sum_j \left[\overline{v(p_1)} \left(\frac{-ig}{\sqrt{2}} \gamma_\mu P_L (K)_{1j} \right) \frac{i(\gamma_\delta p^\delta + m_j)}{p^2 - m_j^2} \left(-\frac{-ig}{\sqrt{2}} \gamma_\nu P_L (K)_{1j} \right) u(p_2) \right] \epsilon^{\mu *}(p_3) \epsilon^{\nu *}(p_4) \ , \qquad \text{(B.7)}$$

where $p = p_3 - p_1 = p_2 - p_4$, and $e^+(p_2) = (e^-(p_2))^c$ has the role of Anti Conjugated Dirac fermion , which means that the vertex connecting the momenta p_2 and p_4 is the lepton number violating one. Furthermore, M^ν presented in these tables is the the neutrino Dirac mass matrix in the basis where the charged lepton Dirac mass matrix is diagonal (same meaning as in [eq. 2.41]). The following tables should be compared with [Tables A.1, A.2, A.3]. All momenta point towards the vertex, arrows mean fermion flow (flow of -e charge), for outgoing states flip the vertex while maintaining all the momenta and fermion flow arrows but the momenta of the outgoing states, which one should also flip. A Dirac fermion is a particle if its momenta points in the same direction as the fermion flow, otherwise it's an anti-particle. If one flips the momenta of a charged boson then one is considering the vertex with the charge conjugate of it, i.e., diagrammatically $B^-(p) = B^+(-p)$.

The equation [Eq. C.3] is needed to write the Higgs and Goldstone Boson Feynman Rules in the way they are presented in tables [Tabs. B.5, B.6 , B.7, B.8]

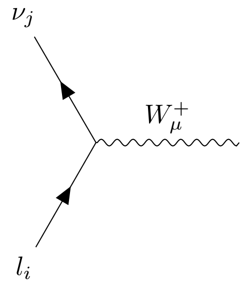
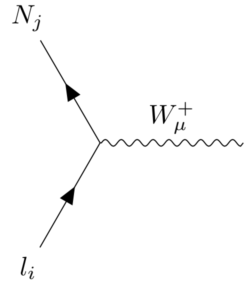
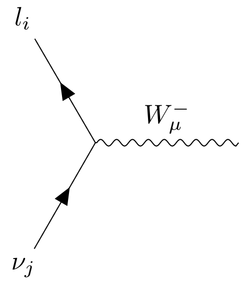
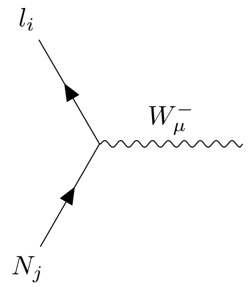
Vertex	Rule
	$\frac{-ig}{\sqrt{2}}\gamma_\mu P_L K_{ij}^*$
	$\frac{-ig}{\sqrt{2}}\gamma_\mu P_L R_{ij}^*$
	$\frac{-ig}{\sqrt{2}}\gamma_\mu P_L K_{ij}$
	$\frac{-ig}{\sqrt{2}}\gamma_\mu P_L R_{ij}$

Table B.2: $SI\nu SM$ Charged Currents

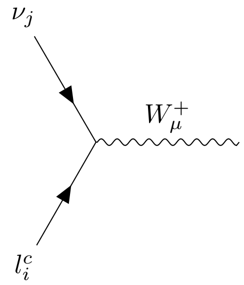
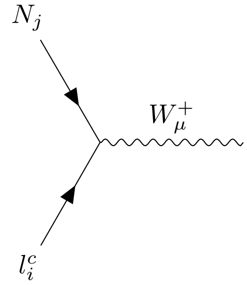
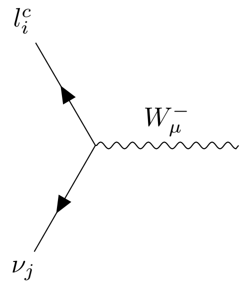
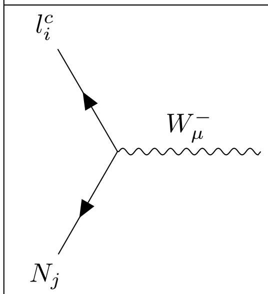
Vertex	Rule
	$-\frac{-ig}{\sqrt{2}}\gamma_\mu P_L K_{ij}^*$
	$-\frac{-ig}{\sqrt{2}}\gamma_\mu P_L R_{ij}^*$
	$-\frac{-ig}{\sqrt{2}}\gamma_\mu P_L K_{ij}$
	$-\frac{-ig}{\sqrt{2}}\gamma_\mu P_L R_{ij}$

Table B.3: $SI\nu SM$ Lepton Number Violating Charged Currents

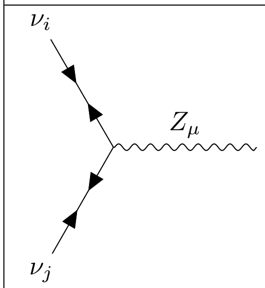
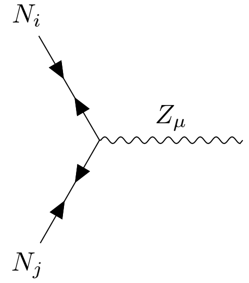
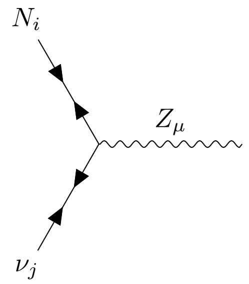
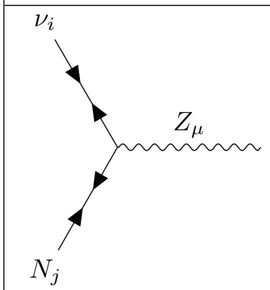
Vertex	Rule
	$\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L(K^\dagger K)_{ij}$
	$\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L(R^\dagger R)_{ij}$
	$\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L(R^\dagger K)_{ij}$
	$\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L(K^\dagger R)_{ij}$

Table B.4: $SI\nu SM$ Neutral Currents

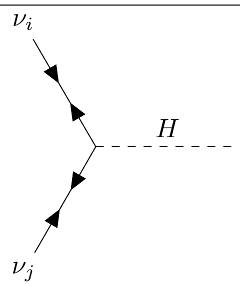
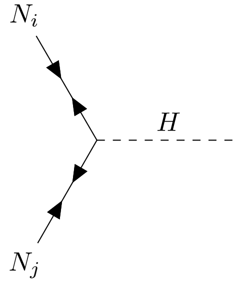
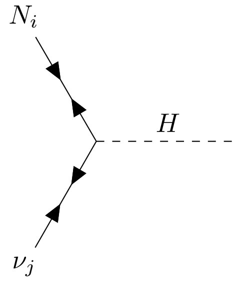
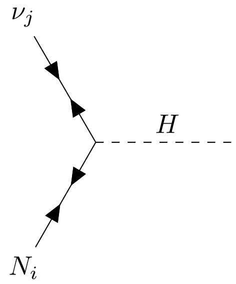
Vertex	Rule
	$\frac{-ig}{2m_W} \cdot \big((K^\dagger K\ d)_{ij} P_R + (d\ K^\dagger K)_{ij} P_L\big)$
	$\frac{-ig}{2m_W} \cdot \big((R^\dagger R\ d_R)_{ij} P_R + (d_R\ R^\dagger R)_{ij} P_L\big)$
	$\frac{-ig}{2m_W} \quad \cdot \quad \big((R^\dagger K\ d)_{ij} P_R + (d\ R^\dagger K)_{ij} P_L\big)$
	$\frac{-ig}{2m_W} \cdot \big((K^\dagger R\ d_R)_{ij} P_R + (d_R\ K^\dagger R)_{ij} P_L\big)$

Table B.5: $SI\nu SM$ Neutral Lepton-Higgs Interactions

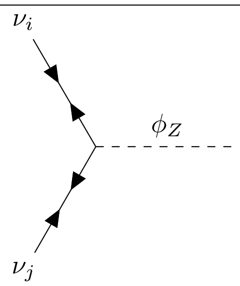
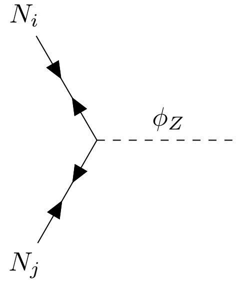
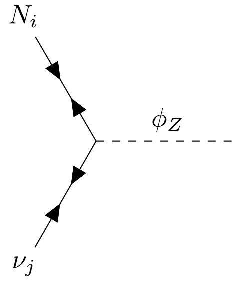
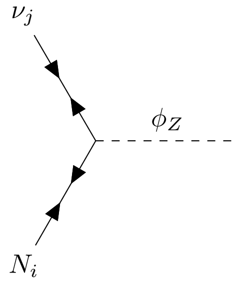
Vertex	Rule
	$\frac{-g}{2m_W} \cdot \big((K^\dagger K\ d)_{ij} P_R - (d\ K^\dagger K)_{ij} P_L\big)$
	$\frac{-g}{2m_W} \cdot \big((R^\dagger R\ d_R)_{ij} P_R - (d_R\ R^\dagger R)_{ij} P_L\big)$
	$\frac{-g}{2m_W} \cdot \big((R^\dagger K\ d)_{ij} P_R + (d\ R^\dagger K)_{ij} P_L\big)$
	$\frac{-g}{2m_W} \cdot \big((K^\dagger R\ d_R)_{ij} P_R + (d_R\ K^\dagger R)_{ij} P_L\big)$

Table B.6: $SI\nu SM$ Lepton-Neutral Goldstone Boson Interactions

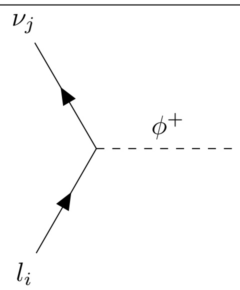
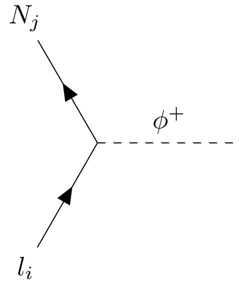
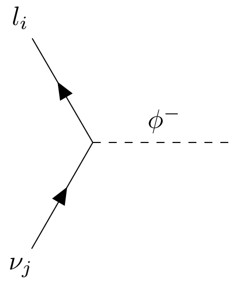
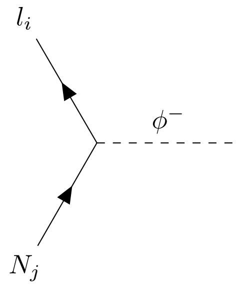
<table>
<tr><th>Vertex</th><th>Rule</th></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{jj}P_L - m_{l_{ii}}P_R)\,K^*_{ij}$$</td></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{Rjj}P_L - m_{l_{ii}}P_R)\,R^*_{ij}$$</td></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{jj}P_R - m_{l_{ii}}P_L)\,K_{ij}$$</td></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{Rjj}P_R - m_{l_{ii}}P_L)\,R_{ij}$$</td></tr>
</table>

Table B.7: $SI\nu SM$ Lepton-Charged Goldstone Boson Interactions

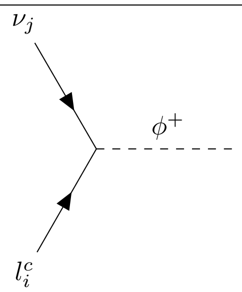
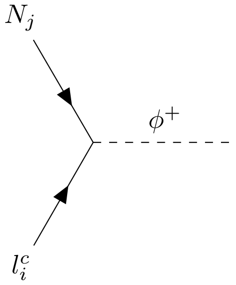
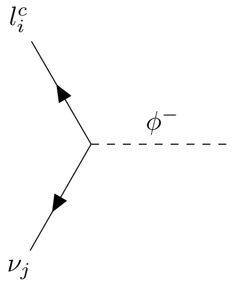
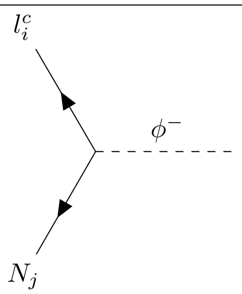
<table>
<tr><th>Vertex</th><th>Rule</th></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{jj} P_L - m_{l_{ii}} P_R)\, K_{ij}^*$$</td></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{Rjj} P_L - m_{l_{ii}} P_R)\, R_{ij}^*$$</td></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{jj} P_R - m_{l_{ii}} P_L)\, K_{ij}$$</td></tr>
<tr><td>
</td><td>$$\frac{ig}{\sqrt{2}m_W} \cdot (d_{Rjj} P_R - m_{l_{ii}} P_L)\, R_{ij}$$</td></tr>
</table>

Table B.8: *$SI\nu SM$* Lepton Number Violating Lepton-Charged Goldstone Boson Interactions

Appendix C

One-Loop Calculation of δM_L

It is useful to treat all 6 neutrinos all at once, using χ, defined in [Eq. 3.64]. To ease this, one needs to extract flavour equations in terms of the 3×6 mixing matrices $(K\ R)$ and $(S\ Z)$. From the diagonalization equation [Eq. 2.47] one gets the following equations:

$$(K\ R)D(K\ R)^T = 0_{3\times3}\ ,\ (K\ R)D(S\ Z)^T = M^\nu\ ,\ (S\ Z)D(S\ Z)^T = M_R\ , \tag{C.1}$$

and from the unitarity equations [Eqs. 3.4, 3.5] one obtains:

$$(K\ R)(K\ R)^\dagger = I_{3\times3}\ ,\ (S\ Z)(S\ Z)^\dagger = I_{3\times3}\ ,\ (K\ R)(S\ Z)^\dagger = 0_{3\times3}\ ,\ (K\ R)^\dagger(K\ R)+(S\ Z)^\dagger(S\ Z) = I_{6\times6}\ . \tag{C.2}$$

From using the transpose of the second equation of [Eq. C.1] on $(S\ Z)^\dagger M^{\nu T}$, the last equation of [Eq. C.2] and the first equation of [Eq. C.1] one obtains:

$$(S\ Z)^\dagger M^{\nu T} = D(K\ R)^T\ . \tag{C.3}$$

The following equation, which is derived using the first equation of [Eq. C.1], will also be useful:

$$\begin{aligned}&(K\ R)(k^2 D)(k^2\cdot I_{6\times6} - D^2)^{-1}(K\ R)^T = (K\ R)\left[D^3 + D(k^2\cdot I_{6\times6} - D^2)\right](k^2\cdot I_{6\times6} - D^2)^{-1}(K\ R)^T\\ &= (K\ R)(D^3)(k^2\cdot I_{6\times6} - D^2)^{-1}(K\ R)^T\ .\end{aligned} \tag{C.4}$$

One will also need to rewrite the vertices given in [Tabs. B.4, B.5, B.6] such that the terms have a P_L and P_R part and $B_{L,R} = B^T_{L,R}$, needed to fulfill the Majorana consistency condition [154] for the self-energy:

$$\Sigma(p) = C\Sigma(-p)C^{-1}\ , \tag{C.5}$$

where the decomposition that was used for the self-energy is defined in [Eq. 3.65]. In order to achieve this, one needs to rewrite the Lagrangian parts which correspond to the vertices given in [Tabs. B.4, B.5, B.6], in terms of χ, using $\chi = C\overline{\chi}^T$ to get them in the correct form. These are the parts of the Lagrangian which correspond to neutrino neutral currents and to the interaction of ϕ_Z and H with the neutrinos. This is because, as correctly stated in [154], the diagrams in [Fig. 3.5] with $W^{\pm}$ only contribute to A_R and A_L and the ones with $\phi^{\pm}$ which contribute to B_L and which would contribute to δM_L give zero due to the first equation of [Eq. C.1].

After using properties of the C matrix and of Majorana neutrinos, given in the beginning of section 2.2 of chapter 2, one obtains:

$$L_Z = \frac{-g}{4\cos\theta_W} Z_\mu \overline{\chi}\gamma^\mu (P_L F - P_R F^T)\chi \ , \tag{C.6}$$

where $F = [(K\ R)^\dagger (K\ R)]$, and

$$L_H = \frac{-g}{4m_W} H\overline{\chi}[(A + A^T)P_R + (B + B^T)P_L]\chi \ , \tag{C.7}$$

$$L_{\phi_Z} = \frac{ig}{4m_W} \phi_Z \overline{\chi}[(A + A^T)P_R - (B + B^T)P_L]\chi \ , \tag{C.8}$$

where $A = (K\ R)^\dagger M^\nu (S\ Z)^* = (K\ R)^\dagger (K\ R)\ D$ and $B = (S\ Z)^T M^{\nu\dagger} (K\ R) = D\ (K\ R)^\dagger (K\ R)$ and [Eq. C.3] was used.

For the diagram of [Fig. 3.5] with the Z boson, one defines the momenta in a clockwise direction, being k the neutrino momentum and $p - k$ the boson momentum. Using the Feynman Rules:

$$\begin{aligned} &- i\Sigma_Z = 4 \times \left(\frac{-ig}{4\cos\theta_W}\right)^2 \gamma^\mu (P_L F - P_R F^T) \\ &\int \frac{d^4k}{(2\pi)^4} \sum_l \frac{i(\not{k} + m_l)}{k^2 - m_l^2} \left(\frac{-i}{(p-k)^2 - m_Z^2}\left(g_{\mu\nu} - (1-\xi_Z)\frac{(p-k)_\mu (p-k)_\nu}{(p-k)^2 - \xi_Z m_Z^2}\right)\right)\gamma^\nu (P_L F - P_R F^T) \ , \end{aligned} \tag{C.9}$$

where the 4 is a symmetry factor due to the Majorana character of the neutrinos. After some Dirac algebra, it is possible to conclude that only $m_l \gamma^\mu \gamma^\nu (-F^T F)$ contributes to B_L^*. Then, one needs to go to $d = 4 - \epsilon$ dimensions, contract this with the Z propagator and use $\gamma_\mu \gamma^\mu = d$ and $(\not{p} - \not{k})(\not{p} - \not{k}) = (p-k)^2$. Finally, after simplifying the denominators[1]:

$$\begin{aligned} B_{LZ}(p)^* = &-\frac{4}{2^d \pi^{d-2}} \times \left(\frac{g}{4\cos\theta_W}\right)^2 F^\dagger D \\ &\left[\frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\ \left(k^2 I_{6\times 6} - D^2\right)^{-1}\left[\frac{d}{(p-k)^2 - m_Z^2} + \frac{1}{m_Z^2}\left(\frac{(p-k)^2}{(p-k)^2 - \xi_Z m_Z^2} - \frac{(p-k)^2}{(p-k)^2 - \xi_Z m_Z^2}\right)\right]\right] F^* \ . \end{aligned} \tag{C.10}$$

For the diagram of [Fig. 3.5] with the H boson, one defines the momenta in a clockwise direction, being k the neutrino momentum and $p - k$ the boson momentum. Using the Feynman Rules, the contribution

[1] B_L is conjugated to be consistent with [Eqs. 2.47, 3.67] and the definition of the mixing matrix as V, such that the Feynman Rules given above are correct.

to the self-energy is:

$$
\begin{aligned}
&- i\Sigma_H = 4 \times \left(\frac{-ig}{4m_W}\right)^2 \left(P_R(A + A^T) + P_L(B + B^T)\right) \\
&\int \frac{d^4k}{(2\pi)^4} \sum_l \frac{i(\not{k} + m_l)}{k^2 - m_l^2} \frac{-i}{(p-k)^2 - m_H^2} \left(P_R(A + A^T) + P_L(B + B^T)\right) \ ,
\end{aligned}
\tag{C.11}
$$

where the 4 is a symmetry factor due to the Majorana character of the neutrinos. After some simplifications it is possible to conclude that only $(B + B^T)m_l(B + B^T)$ contributes to B_L^*. Going to $d = 4 - \epsilon$ dimensions:

$$
B_{LH}(p)^* = -\frac{4}{2^d \pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (B^* + B^\dagger)D \left[\frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \ \left(k^2 I_{6\times 6} - D^2\right)^{-1} \left[\frac{1}{(p-k)^2 - m_H^2}\right]\right] (B^* + B^\dagger) \ .
\tag{C.12}
$$

For the diagram of [Fig. 3.5] with the ϕ_Z Goldstone boson, one defines the momenta in a clockwise direction, being k the neutrino momentum and $p-k$ the Goldstone boson momentum. Using the Feynman Rules, the contribution to the self-energy is:

$$
\begin{aligned}
&- i\Sigma_{\phi_Z} = 4 \times \left(\frac{-g}{4m_W}\right)^2 \left(P_R(A + A^T) - P_L(B + B^T)\right) \\
&\int \frac{d^4k}{(2\pi)^4} \sum_l \frac{i(\not{k} + m_l)}{k^2 - m_l^2} \frac{-i}{(p-k)^2 - \xi_Z m_Z^2} \left(P_R(A + A^T) - P_L(B + B^T)\right) \ ,
\end{aligned}
\tag{C.13}
$$

where the 4 is a symmetry factor due to the Majorana character of the neutrinos. After some simplifications it is possible to conclude that only $(B + B^T)m_l(B + B^T)$ contributes to B_L^*. Going to $d = 4 - \epsilon$ dimensions:

$$
B_{L\phi_Z}(p)^* = \frac{4}{2^d \pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (B^* + B^\dagger)D \left[\frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \ \left(k^2 I_{6\times 6} - D^2\right)^{-1} \left[\frac{1}{(p-k)^2 - \xi_Z m_Z^2}\right]\right] (B^* + B^\dagger) \ .
\tag{C.14}
$$

Using the fact that B_L can be evaluated at $p = 0$ [154] and [Eq. 3.67]:

$$
\begin{aligned}
\delta M_L^{Zin} = &-\frac{4}{2^d \pi^{d-2}} \times \left(\frac{g}{4\cos\theta_W}\right)^2 (K\ R)F^\dagger D \\
&\left[\frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \ \left(k^2 I_{6\times 6} - D^2\right)^{-1} \left[\frac{d}{k^2 - m_Z^2} + \frac{1}{m_Z^2}\left(\frac{k^2}{k^2 - \xi_Z m_Z^2} - \frac{k^2}{(k)^2 - \xi_Z m_Z^2}\right)\right]\right] F^*(K\ R)^T \ .
\end{aligned}
\tag{C.15}
$$

Using the first equation of [Eq. C.2] and [Eq. C.4]:

$$\delta M_L^{Zin} = -\frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4\cos\theta_W}\right)^2 (K\ R)D$$
$$\left[\frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\ \left(k^2 I_{6\times 6} - D^2\right)^{-1}\left[\frac{d}{k^2-m_Z^2} + \frac{1}{m_Z^2}\left(\frac{D^2}{k^2-\xi_Z m_Z^2} - \frac{D^2}{k^2-\xi_Z m_Z^2}\right)\right]\right](K\ R)^T\ , \tag{C.16}$$

and the definition of Passarino-Veltman function B_0 from [155]:

$$\delta M_L^{Zin} = -\frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4\cos\theta_W}\right)^2 (K\ R)D\left[dB_0(0,m_Z^2,D^2) + \frac{D^2}{m_Z^2}\left[B_0(0,\xi_Z m_Z^2,D^2) - B_0(0,m_Z^2,D^2)\right]\right](K\ R)^T\ . \tag{C.17}$$

For H and ϕ_Z:

$$\delta M_L^{Hin} = -\frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (K\ R)(B^* + B^\dagger)D\left[\frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\ \left(k^2 I_{6\times 6} - D^2\right)^{-1}\left[\frac{1}{k^2-m_H^2}\right]\right](B^* + B^\dagger)(K\ R)^T\ , \tag{C.18}$$

$$\delta M_L^{\phi_Z in} = \frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (K\ R)(B^* + B^\dagger)D\left[\frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\ \left(k^2 I_{6\times 6} - D^2\right)^{-1}\left[\frac{1}{k^2-\xi_Z m_Z^2}\right]\right](B^* + B^\dagger)(K\ R)^T\ . \tag{C.19}$$

Using the first equation of [Eq. C.1] and the definition of Passarino-Veltman function B_0 from [155]:

$$\delta M_L^{Hin} = -\frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (K\ R)D^2\left[B_0(0,m_H^2,D^2)\right]D(K\ R)^T\ , \tag{C.20}$$

$$\delta M_L^{\phi_Z in} = \frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (K\ R)D^2\left[B_0(0,\xi_Z m_Z^2,D^2)\right]D(K\ R)^T\ . \tag{C.21}$$

Putting all together, since $\delta M_L = \delta M_L^{\phi_Z in} + \delta M_L^{Hin} + \delta M_L^{\phi_Z in}$ and using $m_W^2 = \cos\theta_W{}^2 m_Z^2$, it is clear that the ϕ_Z contribution cancels the gauge dependent Z contribution, turning the result gauge invariant:

$$\begin{aligned}
\delta M_L =& -\frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4\cos\theta_W}\right)^2 (K\ R)D\left[dB_0(0,m_Z^2,D^2) - \frac{D^2}{m_Z^2}\left[B_0(0,m_Z^2,D^2)\right]\right](K\ R)^T - \\
& \frac{4}{2^d\pi^{d-2}} \times \left(\frac{g}{4m_W}\right)^2 (K\ R)D^2\left[B_0(0,m_H^2,D^2)\right]D(K\ R)^T \\
=& -\frac{g^2}{2^{d+2}\pi^{d-2}m_W^2}(K\ R)\left[d\ Dm_Z^2 B_0(0,m_Z^2,D^2) - D^3 B_0(0,m_Z^2,D^2) + D^3 B_0(0,m_H^2,D^2)\right](K\ R)^T \\
=& -\frac{g^2}{2^{d+2}\pi^{d-2}m_W^2}(K\ R)\left[T^{Z1} + T^{Z2} + T^H\right](K\ R)^T\ .
\end{aligned} \tag{C.22}$$

Using the definition from [154, 155]:

$$B_0(0,m_B^2,M^2) = -\left(\Delta_\epsilon\ + \log M^2 + \frac{\log\frac{M^2}{m_B^2}}{\frac{M^2}{m_B^2} - 1}\right)\ , \tag{C.23}$$

where Δ_ϵ is the term that is divergent for $\epsilon = 0$, yields:

$$\begin{aligned}
-(K\ R)T^{Z1}(K\ R)^T &= (K\ R)dDm_Z^2\left[\Delta_\epsilon + \log D^2 + \frac{\log\frac{D^2}{m_Z^2}}{\frac{D^2}{m_Z^2}-I}\right](K\ R)^T \\
&= (K\ R)dDm_Z^2\left[\log D^2 + \frac{\log\frac{D^2}{m_Z^2}}{\frac{D^2}{m_Z^2}-I}\right](K\ R)^T \\
&= (K\ R)dm_Z^2\left[\left(\frac{D^3}{m_Z^2}\log D^2 - D\log D^2 + D\log\frac{D^2}{m_Z^2}\right)\left(\frac{D^2}{m_Z^2}-I\right)^{-1}\right](K\ R)^T \\
&= d(K\ R)\left[\left(D^3\log D^2 - m_Z^2 D\log m_Z^2\right)\left(\frac{D^2}{m_Z^2}-I\right)^{-1}\right](K\ R)^T \\
&= d(K\ R)\left[\left(D^3\log\frac{D^2}{m_Z^2}\right)\left(\frac{D^2}{m_Z^2}-I\right)^{-1}\right](K\ R)^T\ ,
\end{aligned} \tag{C.24}$$

where the first equation of [Eq. C.1] was used to eliminate the divergent term[2] and it is again used in a different form, in the final equality:

$$\begin{aligned}
&-(K\ R)\ \left(m_Z^2 D\log m_Z^2\right)\left(\frac{D^2}{m_Z^2}-I\right)^{-1}(K\ R)^T \\
=&-(K\ R)\ \left[\left(D^3\log m_Z^2\right)-\left(m_Z^2 D\log m_Z^2\right)\left(\frac{D^2}{m_Z^2}-I\right)\right]\left(\frac{D^2}{m_Z^2}-I\right)^{-1}(K\ R)^T \\
=&-(K\ R)\ \left(D^3\log m_Z^2\right)\left(\frac{D^2}{m_Z^2}-I\right)^{-1}(K\ R)^T\ .
\end{aligned} \tag{C.25}$$

As for the other terms:

$$\begin{aligned}
-(K\ R)\left[T^{Z2}+T^H\right](K\ R)^T &= (K\ R)\left[-D^3\left(\Delta_\epsilon + \log D^2 + \frac{\log\frac{D^2}{m_Z^2}}{\frac{D^2}{m_Z^2}-I}\right) + D^3\left(\Delta_\epsilon + \log D^2 + \frac{\log\frac{D^2}{m_H^2}}{\frac{D^2}{m_H^2}-I}\right)\right](K\ R)^T \\
&= (K\ R)\left[-D^3\left(\frac{\log\frac{D^2}{m_Z^2}}{\frac{D^2}{m_Z^2}-I}\right) + D^3\left(\frac{\log\frac{D^2}{m_H^2}}{\frac{D^2}{m_H^2}-I}\right)\right](K\ R)^T\ ,
\end{aligned} \tag{C.26}$$

where it is clear that the divergent terms, and also the $\log(D^2)$ ones, cancel among them.

Retrieving the factors from [Eq. C.22] and taking $d = 4$, one can now split the result into two, the Z boson mass dependent terms, combining [Eq. C.24] with part of [Eq. C.26] and the Higgs boson mass dependent terms, part of [Eq. C.26]:

$$\begin{aligned}
\delta M_L^Z &= \frac{3g^2}{64\pi^2 m_W^2}(K\ R)D^3\left(\frac{1}{m_Z^2}D^2 - I\right)^{-1}\log\left(D^2\frac{1}{m_Z^2}\right)(K\ R)^T\ , \\
\delta M_L^H &= \frac{g^2}{64\pi^2 m_W^2}(K\ R)D^3\left(\frac{1}{m_H^2}D^2 - I\right)^{-1}\log\left(D^2\frac{1}{m_H^2}\right)(K\ R)^T\ ,
\end{aligned} \tag{C.27}$$

[2] This proves that if M_L existed at tree level, the loop corrections would be infinite.

as given in the text in [Eq. 3.69]. Finally, δM_L is given by :

$$\delta M_L = \delta M_L^Z + \delta M_L^H \ . \tag{C.28}$$

Appendix D

Light Neutrino Masses generated at Loop Level in Exact Formalism

For massless light neutrinos at tree level, one cannot use [Eq. 3.11] for X, as this would give the trivial case with $M^\nu = X = R = S = 0$. Not even at loop level the neutrino masses would be generated as $\delta M_L = 0$ in this case. One has to find a new solution for [Eq. 3.10] with $d = 0$. This solution is given by:

$$X^{d=0} = \pm i\sqrt{d_R^{-1}}\; O_c\; Q\; \sqrt{J}, \tag{D.1}$$

where $Q^T\; Q = 0$ and J is any matrix with units of mass, that can be chosen to be $J = d_R$. One can easily check that this solves the equation $0 = -X^T\; d_R\; X$. The combination $Oc\; Q$ only has one non-zero eigenvalue, that will be responsible for the deviations from unitarity. Q is a matrix of complex entries. A possible type of Q is:

$$Q = \begin{pmatrix} a & b & c \\ ia & ib & ic \\ 0 & 0 & 0 \end{pmatrix} . \tag{D.2}$$

All the other previously derived equations are valid for this scenario, as long as one takes $d = 0$. As for the loop corrections, in this case [Eq. 3.79] is exact since $L^m(m_B) = 0$. All the previous discussion remains unchanged but the fact that, in the parameter region where the heavy neutrinos are degenerate, $d_R = M_1 \times I$,:

$$(X^{d=0})^\dagger (X^{d=0})^* = -Q^\dagger \sqrt{d_R^{-1}}\sqrt{d_R^{-1}} Q^* = -M_1^{-1}\left(Q^T Q\right)^* = 0\ . \tag{D.3}$$

Which proves that the loop corrections are identically zero for degenerate heavy neutrinos. Thus, in this case, the light neutrino masses arise from breaking the degeneracy of heavy neutrinos, yielding light neutrino masses related to their mass difference. However, this effect is non-trivial as Q is singular which makes X and R also singular, and in turn δM_L singular as well. Combining the loop generated eigenvalue, proportional to the heavy neutrino mass differences with the possible eigenvalues of M^ν to generate the singular values of M that yield the light neutrino masses requires further investigation.