Bimodules of Nest Algebras on Banach Spaces

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May 28, 2018

Abstract

We generalize results concerning nests, nest algebras and bimodules over a nest algebra to the setting of Banach spaces. We investigate bimodules by studying the relation between the bimodules and the support functions and the essential support functions on the nest. A particular focus is given to determining the maximal and the minimal bimodule with a given support function, essential support function or support function pair. We succeed in every case except in what concerns the minimal bimodule.

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1 Introduction

The terms nest and nest algebra were introduced in 1963 by J. R. Ringrose as a generalization of a class of operator algebras including, for example, the algebra of upper triangular matrices and some of the maximal triangular algebras appearing in the work of Kadison and Singer (cf. [8],[14]).

In what concerns bimodules, the weakly closed bimodules of a given nest algebra on a Hilbert space were completely characterized by Erdos and Power as early as 1982 (cf. [6]). Those were characterized in terms of order homomorphisms on the corresponding nest, later to be called support functions (see [3]).

Davidson, Donsig and Hudson, [3], went further to introduce the essential support function of a given norm closed bimodule, which allowed for finding the maximal and the minimal norm closed bimodule having the same essential support function.

A natural question is, of course, to what extent results in the Hilbert space setting can be seen to hold for Banach spaces. Although many authors have tackled this problem, it still is the case that nest algebras on Banach spaces are far less explored in the literature. One of the possible reasons for this being the difficulty presented by the lack of orthogonal projections, a key tool used in the investigation of Hilbert space nest algebras.

We present a generalization of a substantial part of these concepts and results to the setting of nest algebras on Banach spaces. More precisely, we generalize the main results concerning the relation between support functions and bimodules. In what concerns essential support functions, we give an analogous definition to that afore mentioned, which is not an exact generalization but, nevertheless, allows to show that essentially the same results hold in the case of Banach spaces.

We start by fixing some notation.

Throughout let X be a Banach space and let $\mathcal{B}(X)$ denote the algebra of the linear bounded operators $T : X \to X$.

Given a subset M of X we denote by $[M]$ the closure of the span of M in X.

Given x in X and f in the dual X^* of X, we denote by $f \otimes x$ the rank one operator given by $(f \otimes x)(w) = f(w)x$ for every $w \in X$.

Given subspaces M_i of X, $i \in I$, where I is an indexing set, we will denote by

$$\bigwedge_{i\in I} M_i \quad \text{and} \quad \bigvee_{i\in I} M_i$$

the intersection of the M_i and the smallest closed subspace containing all the M_i, respectively, i.e., $\bigwedge_{i\in I} M_i = \bigcap_{i\in I} M_i$ and $\bigvee_{i\in I} M_i = \overline{\operatorname{span}\left(\bigcup_{i\in I} M_i\right)}$.

For a vector space S, we write $\dim(S) < \infty$ if S possesses a finite basis. Conversely, we write $\dim(S) = \infty$ if S contains an infinite linearly independent set.

Given subspaces N and M of X, we will also consider the quotient $M/N := \{m + N : m \in M\}$, which we know to be a vector space with $(m + N) + (m' + N) = (m + m') + N$ and $a(m + N) = am + N$ for every $m, m' \in M$, $a \in \mathbb{C}$.

Throughout the text we will make a careful use of the symbols $\subset$ and $\subseteq$. The symbol $\subset$ will be used exclusively for strict inclusions.

We now introduce the concept of nest of subspaces of X.

Definition 1.1. *A nest of subspaces of X is a family $\mathcal{E}$ of closed subspaces of X totally ordered by inclusion and such that $\{0\}, X \in \mathcal{E}$. We say that a nest is* complete *if for any family $\{E_i : i \in I\}$ of elements of $\mathcal{E}$ we have that $\bigwedge_{i\in I} E_i$ and $\bigvee_{i\in I} E_i$ also belong to $\mathcal{E}$.*

The elements of a nest form then a subspace lattice.

Definition 1.2. *For a complete nest $\mathcal{E}$ and $E \in \mathcal{E}$ we define*

$$E_- = \bigvee\{F \in \mathcal{E} : F \subset E\} \quad \text{and} \quad E_+ = \bigwedge\{F \in \mathcal{E} : E \subset F\}.$$

If $E_- \neq E$, that is, $E_- \subset E$, then this means that there is no subspace E' of the nest such that $E_- \subset E' \subset E$. Hence E_- is the immediate predecessor of E in the nest and also $E = (E_-)_+$. In the same way, if $E \subset E_+$, then E_+ is the immediate successor of E and $E = (E_+)_-$.

Notation 1.3. *In the sequel, even when not specifically mentioned, we assume every intersection and closed span is taken over subspaces of the nest $\mathcal{E}$ in cause.*

Definition 1.4. *When $E_- \subset E$, we call the subspace E/E_- an* atom *of $\mathcal{E}$. We call a nest* continuous *when there are no atoms, that is, if $E_- = E$ for all $E \in \mathcal{E}$.*

Like in any subspace lattice, we can associate to $\mathcal{E}$ an algebra.

Definition 1.5. *Let $\mathcal{E}$ be a nest of subspaces of X. The* nest algebra *associated to $\mathcal{E}$, denoted $\mathcal{T}(\mathcal{E})$, is the subset of $\mathcal{B}(X)$ consisting of those operators that leave every subspace of $\mathcal{E}$ invariant, i.e.,*

$$\mathcal{T}(\mathcal{E}) = \{T \in \mathcal{B}(X) : TE \subseteq E \quad \forall E \in \mathcal{E}\}.$$

Proposition 1.6. *The nest algebra associated to a nest $\mathcal{E}$ is a weakly closed algebra.*

We now proceed to show that a complete nest is also reflexive. It turns out that the finite rank operators play a crucial role in the theory of nest algebras in general and, in particular, in the proof of the reflexivity of nests. The following lemmas characterize the rank one operators that belong to $\mathcal{T}(\mathcal{E})$.

Lemma 1.7. *Let $\mathcal{E}$ be a complete nest of subspaces of a Banach space X. For any $w \in X$ and $f \in X^*$ the rank one operator $f \otimes w$ belongs to $\mathcal{T}(\mathcal{E})$ if and only if there exists $E \in \mathcal{E}$ such that $w \in E$ and $f \in (E_-)^\perp$.*

Lemma 1.8. *Let $\mathcal{E}$ be a complete nest of subspaces of a Banach space X. For any $w \in X$ and $f \in X^*$ the rank one operator $f \otimes w$ belongs to $\mathcal{T}(\mathcal{E})$ if and only if there exists $E \in \mathcal{E}$ such that $w \in E_+$ and $f \in E^\perp$.*

The next result states the reflexivity of $\mathcal{E}$ as a subspace lattice. Latter, when dealing with bimodules, this result will enable us to find find elements in $\mathcal{E}$ and also enable us to define the support function associated to a bimodule.

Proposition 1.9. *Let $\mathcal{E}$ be a complete nest of subspaces of X and let E be a closed subspace of X such that $TE \subseteq E$ for every $T \in \mathcal{T}(\mathcal{E})$. Then $E \in \mathcal{E}$.*

The set of nests of subspaces of X is partially ordered by inclusion. Using Zorn's lemma, it can be shown that every nest is contained in a maximal nest. Also, it can be shown that every maximal nest is complete, hence the following definition makes sense.

Definition 1.10. *Given a nest $\mathcal{E}$ of subspaces of X we define $Co(\mathcal{E})$ to be the smallest complete nest containing $\mathcal{E}$.*

Thanks to the following result, from here on we will restrict ourselves to the realm of complete nests. Hence, from here on in the text by a nest we will always mean a complete nest.

Proposition 1.11. *Let $\mathcal{E}$ be a nest of subspaces of X. Then $\mathcal{T}(\mathcal{E}) = \mathcal{T}(Co(\mathcal{E}))$.*

The following result is a fundamental result relating nest algebras. It states that the set of finite rank operators in $\mathcal{T}(\mathcal{E})$ is dense in $\mathcal{T}(\mathcal{E})$ in the strong operator topology.

Theorem 1.12. *Let $\mathcal{E}$ be a nest of subspaces of a Banach space X. The closure of the set $\mathcal{T}(\mathcal{E})_0 = \{R \in \mathcal{T}(\mathcal{E}) : \text{rank } R < \infty\}$ in the strong operator topology is the whole $\mathcal{T}(\mathcal{E})$.*

2 Bimodules and support functions

In this section we describe certain $\mathcal{T}(\mathcal{E})$-bimodules of $\mathcal{B}(X)$, namely, the reflexive, the strongly closed and the weakly closed ones. In fact, by the end we will see that all these are the same. We will characterize them in terms of the *support functions* of the nest.

Definition 2.1. *A linear subspace $\mathcal{J}$ of $\mathcal{B}(X)$ is said to be a $\mathcal{T}(\mathcal{E})$-bimodule if $\mathcal{J}\mathcal{T}(\mathcal{E})$, $\mathcal{T}(\mathcal{E})\mathcal{J} \subseteq \mathcal{J}$. In what follows, for simplicity, $\mathcal{T}(\mathcal{E})$-bimodules may be referred to as bimodules.*

Definition 2.2. *A* support function *on a nest $\mathcal{E}$ is an order preserving map $\Phi : \mathcal{E} \to \mathcal{E}$, i.e., a map such that if $N \subseteq M$ then $\Phi(N) \subseteq \Phi(M)$. Φ is called an* admissible support function *if in addition Φ is* left continuous*, that is, for every $N \in \mathcal{E} \backslash \{\{0\}\}$ we have*

$$\bigvee_{E \subset N} \Phi(E) = \Phi(N_-).$$

Support functions on $\mathcal{E}$ form a partially ordered set. For two support functions Φ_1 and Φ_2 we write $\Phi_1 \leq \Phi_2$ when $\Phi_1(E) \subseteq \Phi_2(E)$ for every $E \in \mathcal{E}$.

Admissible support functions will be very useful to us. In the next proposition we characterize the greatest admissible support function not greater than a given support function.

Proposition 2.3. *Let Φ be a support function on $\mathcal{E}$. Then the greatest admissible support function Φ_- such that $\Phi_- \leq \Phi$ is given by $\Phi_-(\{0\}) = \Phi(\{0\})$ and, for $E \neq \{0\}$,*

$$\Phi_-(E) = \bigvee_{F_- \subset E} \Phi(F) = \begin{cases} \Phi(E) & \text{if } E_- \subset E \\ \bigvee_{F \subset E} \Phi(F) & \text{if } E_- = E \end{cases}.$$

Our main purpose is to show that each reflexive or weakly closed bimodule can be described as

$$\mathcal{J} = \{T \in \mathcal{B}(X) : TE \subseteq \Phi(E) \quad \forall E \in \mathcal{E}\}$$

for some admissible support function Φ on $\mathcal{E}$.

2.1 Reflexive bimodules

Definition 2.4. *Let $\mathcal{J}$ be a linear subspace of $\mathcal{B}(X)$. The* reflexive cover *Ref $\mathcal{J}$ of $\mathcal{J}$ is defined by*

$$\mathit{Ref}\,\mathcal{J} = \{T \in \mathcal{B}(X) : Tx \in [\mathcal{J}x], x \in X\}.$$

We say that $\mathcal{J}$ is reflexive *if $\mathcal{J} = \mathit{Ref}\,\mathcal{J}$.*

We will associate to each bimodule an admissible support function and to each support function we will associate a weakly closed and reflexive bimodule.

Definition 2.5. *Given a bimodule $\mathcal{J}$ of $\mathcal{B}(X)$, we associate to it the support function*

$$\begin{aligned} \Phi_{\mathcal{J}} : \mathcal{E} &\longrightarrow \mathcal{E} \\ E &\mapsto [\mathcal{J}E] \end{aligned}$$

We will sometimes refer to $\Phi_{\mathcal{J}}$ as the associated support function of *$\mathcal{J}$.*

Notice that, since $\mathcal{J}$ is a bimodule, we have that $\mathcal{T}(\mathcal{E})([\mathcal{J}E]) \subseteq [\mathcal{T}(\mathcal{E})\mathcal{J}E] \subseteq [\mathcal{J}E]$, so by the reflexivity of the nest $\mathcal{E}$ we have that $[\mathcal{J}E] \in \mathcal{E}$. Hence $\Phi_{\mathcal{J}}$ can be seen to be a support function on $\mathcal{E}$.

Definition 2.6. *Given a support function Φ on $\mathcal{E}$ we associate to it the subspace of $\mathcal{B}(X)$*

$$\mathcal{M}(\Phi) := \{T \in \mathcal{B}(X) : TE \subseteq \Phi(E) \quad \forall E \in \mathcal{E}\}.$$

$\mathcal{M}(\Phi)$ is shown to be a weakly closed and reflexive bimodule for every support function Φ. The following proposition characterizes the reflexive bimodules.

Proposition 2.7. *Let $\mathcal{J}$ be a bimodule. Then $\mathcal{J}$ is reflexive if and only if $\mathcal{J} = \mathcal{M}(\Phi_{\mathcal{J}})$. Also, $\Phi_{\mathcal{J}}$ is admissible.*

Hence, the correspondence $\Phi \mapsto \mathcal{M}(\Phi)$ from the set of admissible support functions to the set of reflexive bimodules is surjective. Now arises the question of whether this correspondence is a bijection. It can be seen that two support functions that differ only by the image of $\{0\}$ have the same associated bimodule. It turns out that, up to the image of $\{0\}$, the correspondence is indeed injective.

Proposition 2.8. *Let Φ and Θ be two left continuous support functions on $\mathcal{E}$ such that $\mathcal{M}(\Phi) = \mathcal{M}(\Theta) = \mathcal{M}$. Then $\Phi(E) = \Theta(E)$ for every $E \in \mathcal{E}\backslash\{\{0\}\}$.*

From this one can conclude that there is a bijection between reflexive bimodules and left continuous support functions Φ on $\mathcal{E}$ such that $\Phi(\{0\}) = \{0\}$.

The following result describes the associated support function of $\mathcal{M}(\Phi)$.

Proposition 2.9. *Let Φ be a support function on $\mathcal{E}$ with $\Phi(\{0\}) = \{0\}$. Then*

$$\Phi_{\mathcal{M}(\Phi)} = \Phi_{-}.$$

Corollary 2.10. *Let Φ be an admissible support function on $\mathcal{E}$ with $\Phi(\{0\}) = \{0\}$. Then*

$$\Phi_{\mathcal{M}(\Phi)} = \Phi.$$

Similarly to what we did for nest algebras, we characterize the rank one operators in $\mathcal{M}(\Phi)$, as these are particularly useful in the study of bimodules.

Definition 2.11. *Let Φ be a support function on $\mathcal{E}$. For $E \in \mathcal{E}$, we define*

$$E^{\sim} = \bigwedge_{E \subset F} \Phi(F) \quad \textit{and} \quad E_{\sim} = \bigvee_{\Phi(F) \subset E} F.$$

Lemma 2.12. *Let Φ be a support function on $\mathcal{E}$. For any $w \in X$ and $f \in X^*$ the rank one operator $f \otimes w$ belongs to $\mathcal{M}(\Phi)$ if and only if there exists $E \in \mathcal{E}$ such that $w \in E^{\sim}$ and $f \in E^{\perp}$.*

Lemma 2.13. *Let Φ be a support function on $\mathcal{E}$. For any $w \in X$ and $f \in X^*$ the rank one operator $f \otimes w$ belongs to $\mathcal{M}(\Phi)$ if and only if there exists $E \in \mathcal{E}$ such that $w \in E$ and $f \in (E_{\sim})^{\perp}$.*

Notice that when Φ is the identity in $\mathcal{E}$ we obtain Lemmas 1.7 and 1.8.

The following result is essentially [11], Theorem 3.1..

Proposition 2.14. *Let Φ be a support function on the nest $\mathcal{E}$. If T is an operator in $\mathcal{M}(\Phi)$ of rank n, then T can be written as the sum of n rank one operators in $\mathcal{M}(\Phi)$.*

2.2 Weakly closed bimodules

The next result relates a bimodule to the subspace of its finite rank operators. Its corollary will be particularly useful when we wish to prove that two weakly closed bimodules are equal.

Definition 2.15. *Given a bimodule $\mathcal{J}$ we define $\mathcal{J}_0$ as the bimodule of $\mathcal{J}$ formed by its finite rank operators.*

The following is a consequence of the finite rank operators in $\mathcal{T}(\mathcal{E})$ being strongly dense in $\mathcal{T}(\mathcal{E})$.

Proposition 2.16. *Let $\mathcal{J}$ be a bimodule of $\mathcal{B}(X)$. Then $\mathcal{J} \subseteq \overline{\mathcal{J}_0}^{SOT} \subseteq \overline{\mathcal{J}_0}^{WOT}$.*

Corollary 2.17. *Let $\mathcal{J}$ be a weakly closed bimodule. Then $\mathcal{J} = \overline{\mathcal{J}_0}^{WOT}$.*

Here we show that the weakly closed bimodules have the same characterization as the reflexive ones, in terms of support functions on $\mathcal{E}$. The proof for this is an adaptation of the proof for the Hilbert space case. For the Banach space case we need the following result.

Proposition 2.18. *Let $E \in \mathcal{E}$. Then the closure of span$\{N^{\perp} : N \in \mathcal{E}, N_+ \supset E\}$ in the w^*-topology is $E^{\perp}$.*

Proposition 2.19. *Let $\mathcal{J}$ be a weakly closed bimodule of $\mathcal{B}(X)$. Then $\mathcal{J}$ and $\mathcal{M}(\Phi_{\mathcal{J}})$ contain the same rank one operators.*

Proof. Clearly $\mathcal{J} \subseteq \mathcal{M}(\Phi_{\mathcal{J}})$. Let $f \otimes w$ be a rank one operator in $M(\Phi_{\mathcal{J}})$, so that, by Proposition 2.12, there exists $E \in \mathcal{E}$ such that $f \in E^{\perp}$ and $w \in E^{\sim}$.

Consider a net $\{f_\alpha\}$ with $f_\alpha \in N_\alpha^{\perp}$, $N_{\alpha+} \supset E$ such that $\{f_\alpha\}$ w^*- converges to f, which is possible by Proposition 2.18. Now choose $w_\alpha \in N_{\alpha+} \backslash E$. Notice that, by the reflexivity of $\mathcal{E}$, $[\mathcal{T}(\mathcal{E})w_\alpha] \in \mathcal{E}$. As $w_\alpha \in [\mathcal{T}(\mathcal{E})w_\alpha]$ we must have the strict inclusion $E \subset [\mathcal{T}(\mathcal{E})w_\alpha]$ and it follows from the definition of $E^{\sim}$ that $E^{\sim} \subseteq \Phi_{\mathcal{J}}([\mathcal{T}(\mathcal{E})w_\alpha]) = [\mathcal{J}[\mathcal{T}(\mathcal{E})w_\alpha]] \subseteq [\mathcal{J}w_\alpha]$. Hence we can find $J_\alpha \in \mathcal{J}$ such that $\{J_\alpha w_\alpha\}$ converges to w in norm.

Since $f_\alpha \in N_\alpha^{\perp}$ and $w_\alpha \in N_{\alpha+}$, it follows that $f_\alpha \otimes w_\alpha \in \mathcal{T}(\mathcal{E})$, by Lemma 1.7. So $J_\alpha(f_\alpha \otimes w_\alpha) = f_\alpha \otimes J_\alpha w_\alpha \in \mathcal{J}$. Given that $\{f_\alpha\}$ converges to f in the w^*-topology and $\{J_\alpha w_\alpha\}$ converges to w we have that $\{f_\alpha \otimes J_\alpha w_\alpha\}$ weakly converges to $f \otimes w$. Since $\mathcal{J}$ is weakly closed we conclude that $f \otimes w \in \mathcal{J}$, as we wanted to show. $\square$

As $\mathcal{J}$ and $\mathcal{M}(\Phi)$ are both weakly closed bimodules, the following theorem follows now from Corollary 2.17.

Theorem 2.20. *Let $\mathcal{J}$ be a weakly closed bimodule. Then*

$$\mathcal{J} = \mathcal{M}(\Phi_{\mathcal{J}}).$$

Remark 2.21. Hence we can say that the reflexive bimodules coincide with the weakly closed ones.

Proof. We know we have $\mathcal{J} \subseteq \mathcal{M}(\Phi_{\mathcal{J}})$. Now notice $\mathcal{J}$ and $\mathcal{M}(\Phi_{\mathcal{J}})$ have the same finite rank operators. Indeed, a finite rank operator in $\mathcal{J}$ is also in $\mathcal{M}(\Phi_{\mathcal{J}})$. Also, given a finite rank operator T in $\mathcal{M}(\Phi_{\mathcal{J}})$, it is the sum of rank one operators in $\mathcal{M}(\Phi_{\mathcal{J}})$, by Proposition 2.14. As, by Proposition 2.19, $\mathcal{J}$ and $\mathcal{M}(\Phi_{\mathcal{J}})$ have the same rank one operators, it follows that $T \in \mathcal{J}$.

Since $\mathcal{J}$ and $\mathcal{M}(\Phi_{\mathcal{J}})$ are both weakly closed bimodules, it now follows from Corollary 2.17 that they are equal. $\square$

Now we will present another way of viewing the support function of a weakly closed bimodule. We fix some notation.

Definition 2.22. *Let E be a nest of closed subspaces of X. For $x \in X$ denote by*

$$\widehat{N}_x = \bigwedge\{N \in \mathcal{E} : x \in N\},$$

the smallest subspace of the nest that contains x, and for $f \in X^*$ denote by

$$\check{N}_f = \bigvee\{N \in \mathcal{E} : f \in N^{\perp}\},$$

the largest subspace of the nest that is annihilated by f.

Definition 2.23. *Let $\mathcal{J}$ be a bimodule. Define $\Theta : \mathcal{E} \to \mathcal{E}$ by*

$$\Theta(N) = \bigvee\{\widehat{N}_x : f \otimes x \in \mathcal{J}, \check{N}_f \subset N\}.$$

Then we have the following.

Proposition 2.24. *Let $\mathcal{J}$ be a weakly closed bimodule. Then*

$$\mathcal{J} = \mathcal{M}(\Theta)$$

where Θ is the admissible support function on $\mathcal{E}$ of definition 2.23. Moreover, $\Theta = \Phi_{\mathcal{J}}$.

We finish this section with some results concerning bimodules in general, that is, not necessarily reflexive or weakly closed.

Proposition 2.25. *Let Φ be an admissible support function on $\mathcal{E}$ such that $\Phi(\{0\}) = \{0\}$. Then the maximal bimodule with support Φ is $\mathcal{M}(\Phi)$.*

Proposition 2.26. *Let Φ be an admissible support function on $\mathcal{E}$ and let $\mathcal{J}$ be a bimodule of $\mathcal{B}(X)$ with associated support function Φ. Then $\mathcal{J}_0$ and $\overline{\mathcal{J}_0}$ also have associated support function Φ.*

Remark 2.27. When X is a Hilbert space and Φ is admissible, it can actually be shown that $\overline{\mathcal{M}(\Phi)_0}$ is minimal among the closed bimodules with associated support function Φ, i.e., is contained in every closed bimodule with associated support function Φ. See [3], Proposition 1.2. The proof cannot be adapted to the more general context of Banach spaces.

3 Essential support functions and their associated bimodules

In this section we will characterize bimodules through another support function, the essential support function. For our main results ahead we will need to make some extra assumptions about the nest. We will require that the nest has either property P_* or property P_∞, which we now define.

Definition 3.1. *We say that a nest $\mathcal{E}$ has* property P_* *if*

i) given $N \in \mathcal{E}$ with $N = N_-$, there exists a strictly increasing sequence $\{N_k\}$ in $\mathcal{E}$ such that $\bigvee_{k=1}^{\infty} N_k = N$.

ii) given $N \in \mathcal{E}$ with $N = N_+$, there exists a strictly decreasing sequence $\{N_k\}$ in $\mathcal{E}$ such that $\bigwedge_{k=1}^{\infty} N_k = N$.

We say that $\mathcal{E}$ has property P_∞ *if, given $N \in \mathcal{E}$, $N_- \subset N$ implies that*

$$\dim(N/N_-) = \infty.$$

Remark 3.2. Notice that when assuming these properties we do not lose too much generality in the sense that this is the setting in a vast number of cases.

To define what we mean by an essential support function on $\mathcal{E}$ we will be interested if certain subspaces have a finite basis or not.

Definition 3.3. *Given a nest $\mathcal{E}$, we will denote by $\mathcal{E}_f$ the set of elements $N \in \mathcal{E}$ such that $0 < \dim(N/N_-) < \infty$. We will also denote $\mathcal{E}_\infty = \mathcal{E} \backslash \mathcal{E}_f$.*

Definition 3.4. *An* essential support function *on $\mathcal{E}$ is a support function Ψ on $\mathcal{E}$ satisfying the following properties.*

(A) *If $N_1, N_2 \in \mathcal{E}$ are such that $N_1 \subseteq N_2$ and $\dim(N_2/N_1) < \infty$, then $\Psi(N_2) = \Psi(N_1)$;*

(B) *$\Psi(N) \in \mathcal{E}_f$ implies that $\Psi(N) = \Psi(N)_+$.*

If in addition Φ is an admissible support function on $\mathcal{E}$ such that $\Phi(\{0\}) = \{0\}$, $\Psi \leq \Phi$ and the following property is satisfied

(B') *$\Psi(N) \in \mathcal{E}_f$ implies that $\Psi(N) = \Psi(N)_+ \subset \Phi(N)$,*

then we call (Φ, Ψ) an admissible pair of support functions *on $\mathcal{E}$.*

To each bimodule we associate an essential support function.

Definition 3.5. *Given a bimodule $\mathcal{J}$, let the function $\Phi^e_{\mathcal{J}} : \mathcal{E} \to \mathcal{E}$ be defined by*

$$\Phi^e_{\mathcal{J}}(N) = \bigwedge\{L \in \mathcal{E} : \dim(TN/L) < \infty \quad \forall T \in \mathcal{J}\}.$$

We will sometimes refer to $\Phi^e_{\mathcal{J}}$ as the associated essential support function of $\mathcal{J}$. *We will also refer to $(\Phi_{\mathcal{J}}, \Phi^e_{\mathcal{J}})$ as the* associated support function pair of $\mathcal{J}$.

Proposition 3.6. *Let $\mathcal{J}$ be a bimodule and let $N, L \in \mathcal{E}$. If $L \subset \Phi^e_{\mathcal{J}}(N)$, then there exists $T \in \mathcal{J}$ such that* $\dim(TN/L) = \infty$. *If $L_+ \supset \Phi^e_{\mathcal{J}}(N)$, then* $\dim(TN/L) < \infty$ *for every $T \in \mathcal{J}$.*

Proposition 3.7. *Let $\mathcal{J}$ be a bimodule. Then $(\Phi_{\mathcal{J}}, \Phi^e_{\mathcal{J}})$ is an admissible pair of support functions on $\mathcal{E}$.*

Now, to each essential support function we associate a bimodule.

Definition 3.8. *Given an essential support function Ψ on $\mathcal{E}$, let $\mathcal{M}^e(\Psi)$ be the subspace of $\mathcal{B}(X)$ defined by*

$$\mathcal{M}^e(\Psi) := \{T \in \mathcal{B}(X) : \dim(TN/L) < \infty \quad \forall N, L \in \mathcal{E} \text{ with } L_+ \supset \Psi(N)\}.$$

Proposition 3.9. *Let Ψ be an essential support function on $\mathcal{E}$. Then $\mathcal{M}^e(\Psi)$ is a bimodule which contains every bimodule with associated essential support function Ψ.*

Assuming that $\mathcal{E}$ has property P_* (cf. Definition 3.1 and Remark 3.2), it can be shown that $\mathcal{M}^e(\Psi)$ has indeed Ψ as its associated essential support function. This will allow us to conclude that $\mathcal{M}^e(\Psi)$ is the maximal bimodule with essential support function Ψ.

Proposition 3.10. *Let $\mathcal{E}$ be a nest with property P_*. If Ψ is an essential support function on $\mathcal{E}$, then*

$$\Phi^e_{\mathcal{M}^e(\Psi)} = \Psi.$$

We now describe the maximal bimodule with a given support function pair (Φ, Ψ). As we know, $\mathcal{M}(\Phi)$ is the maximal bimodule with associated support function Φ and $\mathcal{M}^e(\Psi)$ is the maximal bimodule with associated essential support function Ψ. Thus the cantidate bimodule should be $\mathcal{M}(\Phi) \cap \mathcal{M}^e(\Psi)$.

Theorem 3.11. *Let $\mathcal{E}$ be a nest with property P_* and let (Φ, Ψ) be an admissible pair of support functions on $\mathcal{E}$. Then $\mathcal{M}(\Phi, \Psi) = \mathcal{M}(\Phi) \cap \mathcal{M}^e(\Psi)$ is the largest $\mathcal{T}(\mathcal{E})$-bimodule with associated support function pair (Φ, Ψ).*

Having characterized the largest bimodule with a given support pair (Φ, Ψ), we now turn to the problem of finding the smallest bimodule that has support pair (Φ, Ψ). However, we will not give a complete answer to this problem. What we do is we will find a not necessarily closed bimodule with support function pair (Φ, Ψ), which we conjecture to be contained in every closed bimodule with support function pair (Φ, Ψ), though we will not be able to prove this conjecture. Moreover, we will do this only for nests with property P_∞ (cf. Definition 3.1).

Definition 3.12. *Given an essential support function Ψ on $\mathcal{E}$, let $\mathcal{M}^0(\Psi)$ be the subspace of $\mathcal{B}(X)$ defined by*

$$\mathcal{M}^0(\Psi) = \overline{\sum_{\substack{L,N \in \mathcal{E} \\ L_- \subset \Psi(N)}} \mathit{span}\{f \otimes x : f \in N_-^\perp, x \in L\}}.$$

Proposition 3.13. *$\mathcal{M}^0(\Psi)$ is a bimodule of $\mathcal{B}(X)$ which is contained in every norm closed bimodule with essential support function Ψ.*

The following proposition will then show that $\mathcal{M}^0(\Psi)$ is indeed minimal among the norm closed bimodules with left continuous essential support function Ψ.

Proposition 3.14. *Let $\mathcal{E}$ be a nest with property P_∞ and let Ψ be an essential support function on $\mathcal{E}$ with $\Psi(\{0\}) = \{0\}$. Then the associated support function and the associated essential support function of $\mathcal{M}^0(\Psi)$ both equal Ψ_-.*

We now construct the bimodule that we conjecture to be the minimal bimodule with a given associated support function pair. We know that adding finite rank operator to a bimodule does not change its essential support function, hence the following definition makes sense.

Definition 3.15. *Let (Φ, Ψ) be an an admissible pair of support functions on $\mathcal{E}$. Define*

$$\mathcal{M}^0(\Phi, \Psi) = \mathcal{M}^0(\Psi) + \mathcal{M}(\Phi)_0.$$

Proposition 3.16. *Let $\mathcal{E}$ be a nest with property P_∞ and let (Φ, Ψ) be an admissible support function pair on $\mathcal{E}$. Then $\mathcal{M}^0(\Phi, \Psi)$ is a bimodule with support function pair (Φ, Ψ_-).*

Corollary 3.17. *Let $\mathcal{E}$ be a nest with property P_∞ and let (Φ, Ψ) be an admissible support function pair on $\mathcal{E}$, with Ψ admissible. Then $\mathcal{M}^0(\Phi, \Psi)$ is a bimodule with support function pair (Φ, Ψ).*

Remark 3.18. As it has been mentioned before (cf. Remark 2.27) when X is a Hilbert space $\mathcal{M}(\Phi)_0$ is contained in every closed bimodule with associated support function Φ. Since, by Proposition 3.13, $\mathcal{M}^0(\Psi)$ is contained in every closed bimodule with essential support Ψ and, it follows that $\mathcal{M}^0(\Phi, \Psi)$ is contained in every closed bimodule with support function pair (Φ, Ψ).

We conjecture that in the context of Banach spaces, $\mathcal{M}^0(\Phi, \Psi)$ is also contained in every closed bimodule with support function pair (Φ, Ψ).

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