Frequency domain force identification problem and the relation between force and displacement multi-degrees-of-freedom transmissibility

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ABSTRACT

Force identification in dynamic systems presents itself as a problem of great importance in engineering. The identification of these dynamic forces that result, for example, from external applied forces, which are transmitted to the supports of a structure or machine, is made using the concept of transmissibility. For systems with one degree of freedom (SDOF), the transmissibility is defined as the ratio between transmitted and applied force amplitudes – Force Transmissibility. This relation can also be applied to Displacement Transmissibility, in which case, is related by ratio between imposed and transmitted displacement and whose value modulus will be equal as the one obtained with force transmissibility. In the case of transmissibility in systems with multiple degrees of freedom (MDOF) the relation between transmissibilities is not obtained so expeditiously.

In this context, this article focuses on the thesis main objective, i.e., to deepen the knowledge on transmissibility mechanisms in dynamic MDOF systems in order to develop and verify force identification methodologies based on force and displacement transmissibility, with the focus on force reconstruction and on the potential to extend existing transmissibility relations as well as transmissibility-based load identification location and reconstruction to rotordynamics.

Keywords: Rotordynamics; Frequency domain; Force identification; Force transmissibility; Displacement transmissibility; Multiple degrees of freedom.

1. INTRODUCTION

Force identification is important in various areas of structural dynamics. The process of estimating the localization and amplitude of applied loads from measured responses is here referred as force identification or inverse force determination. The direct force identification problem exists when one knows the applied forces and wants to estimate reactions at the supports. Several examples show the benefits of these inverse problems, such as the estimation of operational forces at inaccessible locations of installed machinery [1], or forces transmitted to the supports [2].

In the present work, the focus is on the transmissibility concept extended to MDOF systems by Maia et al. [3] employed to estimate the frequency domain dynamic loads acting on a structure based on response measurements. This concept was applied to force reconstruction in beams by Lage et al. [4] and Neves et al. [5]. Most recently Maia et al. [6], explored the transmissibility in conjunction with a two-step methodology for force identification based on transmissibility of motion.

Finally, on Lage et al. [7], the authors suggest it is possible to establish a relation between force and displacement transmissibility for MDOF systems. The authors observed that the needed link is present when the displacement transmissibility is obtained between the same coordinates where the applied and reaction forces are considered in the force transmissibility case. This implies that, boundary conditions are not exactly the same (as it happens to 1D Transmissibility) and instead follow some rules. The authors show numerically as well as experimentally, that it is possible to obtain the displacement transmissibility from measured forces, and the force transmissibility from measured displacements.

With the prior knowledge of the rotor model, the method uses frequency domain measured response at some DOF along the structure to locate and reconstruct the generated loads, here considered only due to a punctual mass unbalance. The aim is to extend existing transmissibility relations as well as transmissibility-based load identification location and reconstruction to rotordynamics.

2. FUNDAMENTALS

2.1. Rotordynamics Fundamentals

Rotordynamics is the branch of engineering that studies lateral and torsional vibrations of rotating shafts. For the purpose of this article, one will focus only on bending influence. Rotating machines differ according to their application in real situations, however, their principal components are the disk, the shaft, the bearings and the seals. Thus, a rotor can be represented as a series of disks attached to a beam that rotates along the bearings with an angular speed $\Omega\ [rad/s]$, the rotor spin speed also defined as $N\ [RPM]$. Also, loads are usually caused by residual unbalances that, although small, cannot be avoided. Nonetheless, loads may have various origins, such: rotor unbalance (synchronous excitation), bearing asymmetry (asynchronous excitation) and harmonic forces.

Classic rotordynamics analysis is generally linearized and the vast majority of parameters, like natural frequencies, critical speeds or instability threshold are applicable only among the linearity scope. Although rotors are often linear (nominal conditions-displacements remain within the limits allowable in operation), components like bearings, dampers, and seals can display a severe nonlinear behavior. If nonlinearities aren't neglected, many of the tools and concepts of rotordynamics (Campbell diagram and the critical speed analysis) lose their meaning and the attention must be drawn to the unbalance response and spectral analysis of the vibration records [8].

Rotors can be classified according to its symmetry based on factors like geometry, materials and supports, despite the focus given to supports by Lalanne and Ferraris [9]. The simplest model of this kind is a Jeffcott rotor [10] composed by a simple supported rotating beam with a disk at the middle length. An illustration of these systems is plotted in *Figure 2.1*.

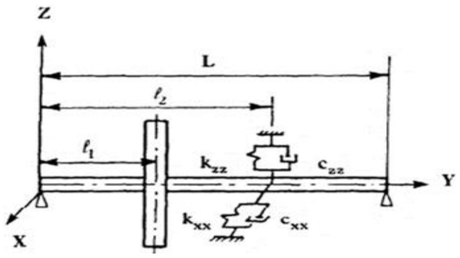


Figure 2.1 – Model of the rotor – Stationary reference frame $R_0(X, Y, Z)$ [9].

The rotor illustrated in *Figure 2.1* is composed by a flexible shaft of length L with a single rigid disk situated at $Y = l_1$ and a discrete bearing located at $Y = l_2$. The bearing characteristics, this is, stiffness and damping dictate if the rotor system is symmetric or asymmetric.

The shaft is modeled as a beam with a constant circular cross-section resorting to the Euler-Bernoulli beam theory with transverse shear correction. The element matrices were deduced following the procedure used by Genta [11] and Nelson [12], and include the effects of rotatory inertia, gyroscopic moments, shear deformation and axial load. Axial torque effects and internal damping are not considered. The disk is considered axisymmetric and modeled as a rigid body connected to one node, being characterized by its mass and inertial properties along the three Cartesian axes. The bearings are modelled as a linear spring-damper system in the $X - Z$ plane.

In order to obtain the governing equations, the structure is discretized into several elements characterized by a total of n degrees of freedom. Afterwards, the equations of motion are combined with these quantities to obtain the equations that characterize the discretized structure behavior. If a stationary reference frame is used, the linear or linearized structure behavior follows the equation form:

$$[M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = \{f(t)\} \tag{2.1}$$

where $[M]^{n\times n}$ is the mass matrix of the system, $[C]^{n\times n}$ is the damping matrix, $[K]^{n\times n}$ the stiffness matrix of the structure and the vector $\{q\} = \{q_1, q_2, \cdots, q_n\}^T$ is the generalized coordinates vector, being n the number of degrees-of-freedom (DOF) of model. The vector $\{f(t)\}^{n\times 1}$ contains the loads applied along the same generalized coordinates. Furthermore, t represents the time variable.

Consider a rotating system described by *(2.1)*. On the frequency domain, the force function is assumed to be harmonic, and takes the form:

$$\{f(t)\} = \{F(\omega)\}e^{j\omega t} \tag{2.2}$$

where $\{f\}$ and $\{F\}$ are vectors of the force function in time domain and the respective amplitude in frequency domain, respectively. For each frequency value, the response in the i^{th}DOF in steady-state conditions can be written as:

$$\{q_i(t)\} = \{Q_i(\omega)\}e^{j\omega t} \tag{2.3}$$

Applying the first and second derivatives to *(2.3)* with respect to time, one obtains respectively:

$$\begin{cases} \dot{q}_i = j\omega Q_i e^{j\omega t} \\ \ddot{q}_i = -\omega^2 Q_i e^{j\omega t} \end{cases} \tag{2.4}$$

Substituting *(2.3)* and *(2.4)* in *(2.2)* one obtains:

$$\left[-\omega^2[M] + j\omega[C] + [K]\right]\{Q\} = \{F(\omega)\} \leftrightarrow [Z(\omega)]\{Q(\omega)\} = \{F(\omega)\}, \{Q\} = \{Q_1, Q_2, \dots, Q_n\}^T \tag{2.5}$$

where $[Z(\omega)]$ is the dynamic stiffness matrix. Using a direct approach for force identification, the load vector $\{F\}$ can be computed knowing the complete dynamic response Q of the system at every DOF. Inverting *(2.5)*:

$$[Z(\omega)]^{-1}\{F(\omega)\} = \{Q(\omega)\} \leftrightarrow [H(\omega)]\{F(\omega)\} = \{Q(\omega)\} \tag{2.6}$$

Equation *(2.6)* introduces the *receptance matrix* or *frequency response function* (FRF) matrix of the system. $[H]$ relates the applied loads with the obtained displacements.

Along this document, the rotor is modelled using the Finite Element Model (FEM) and only examples in the stationary reference frame are considered. In this case, *nodal displacement vector* q will have the following stationary degrees of freedom:

$$q = [u, w, \theta, \psi]^T \tag{2.7}$$

where u, w are the transversal displacements and θ, ψ the angular displacements or rotations, in and around the X and Z directions, respectively.

2.2. Mass Unbalance Force Excitation

The mass unbalance is modeled as a punctual mass m_u situated at a given distance d from the geometric center of the shaft at an angular position α with respect to the Z axis. The resultant mass unbalance force vector in the frequency domain, and converted to the complex plane, $\{F(\omega)\}$ results:

$$\begin{bmatrix} F_u^{m_u} \\ F_w^{m_u} \end{bmatrix} = m_u d\Omega^2 \left[\begin{Bmatrix} sin(\alpha) \\ cos(\alpha) \end{Bmatrix} + j\begin{Bmatrix} -cos(\alpha) \\ sin(\alpha) \end{Bmatrix}\right] \tag{2.8}$$

where, $F_u^{m_u}$ and $F_w^{m_u}$ represent the frequency domain force amplitude. The unbalance is applied in the rigid disk, resulting in a synchronous harmonic force with u, w components applied at the shafts node where the disk is connected, which amplitude grows with the square of the spin speed $F_{unbalance} \propto \Omega^2$.

3. METHODOLOGIES

Force identification methods as the used here, require the prior knowledge of the rotor model, in this work described using FEM. Once the structure is discretized, characterized by a total of n degrees of freedom DOF, the dynamic behavior of the linear/ linearized rotating structure can be described resorting to *(2.5).* With this in mind, the concepts for both displacement and force transmissibility, are applied in conjunction with a two-step methodology, where first, the number of forces and their location are obtained, and second, the load vector is reconstructed using some of the responses obtained.

3.1. Force Identification based on Displacement Transmissibility

To analyse the displacement transmissibility, one is interested in the case where the structure is considered free (no supports). Due to this, and for the case of displacement transmissibility, the only non-zero forces are in set A.

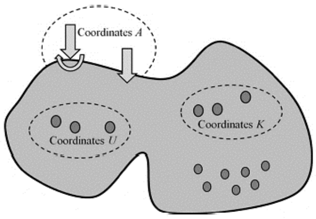


Figure 3.1 – Displacement transmissibility - coordinate sets U, K, A.

The force localization method is based in the search of displacement transmissibility matrix $[T_{UK}]^{(d)}$ associated with the correct number and position of external applied forces $\{F_A\}$. This starts by measuring the response amplitude of sets K and U at each frequency ω_i, $\{Y_K^{measured}(\omega_i)\}$ and $\{Y_U^{measured}(\omega_i)\}$ respectively. Knowing $\{Y_K^{measured}(\omega_i)\}$, one can calculate the estimated dynamic response $\{Y_U^{estimated}(\omega_i)\}$ resorting to *(3.1)*.

$$\{Y_U^{estimated}(\omega_i)\} = [H_{UA}(\omega_i)][H_{KA}(\omega_i)]^+\{Y_K^{measured}(\omega_i)\} = [T_{UK}]^{(d)}\{Y_K^{measured}(\omega_i)\} \tag{3.1}$$

For the pseudo-inverse operation to be possible it is required that $n_K \geq n_A$. Note that, for the case of force identification, the set U of n_U coordinates is known and $\{Y_U^{measured}(\omega_i)\}$ is used to compare with $\{Y_U^{estimated}(\omega_i)\}$, being the set A unknown. With this in mind, the set A has to be defined, and since one does not know how many loads are applied and their location (nodal and DOF) the algorithm considers $n_A = n_K$ and first calculates the nodal combinations of the possible sets A (see *Table 3.1*).

Table 3.1 – Combinations of possible sets A.

Number of Forces $\{\boldsymbol{F_A}\}$	**Number of Combinations**	**Nodal Combinations**	**Force Combinations**
1	$C_1^N = \dfrac{N!}{1!\,(N-1)!} = N$	$1,2,\dots,N$	$\{F_1\},\{F_2\},\dots,\{F_N\}$
2	$C_2^N = \dfrac{N!}{2!\,(N-2)!}$	$\begin{cases}\{1,2\},\cdots,\{1,N\}\\ \vdots \\ \{N-1,N\}\end{cases}$	$\begin{cases}\{F_1,F_2\},\cdots,\{F_1,F_N\}\\ \vdots \\ \{F_{N-1},F_N\}\end{cases}$
3	$C_3^N = \dfrac{N!}{3!\,(N-3)!}$	$\begin{cases}\{1,2,3\},\cdots,\{1,2,N\}\\ \vdots\end{cases}$	$\begin{cases}\{F_1,F_2,F_3\},\cdots,\{F_1,F_2,F_N\}\\ \vdots\end{cases}$
$n_A \leq n_K$	$C_A^N = \dfrac{N!}{A!\,(N-A)!}$	$\vdots$	$\vdots$

From *(3.1)* and for *each j* combination, the respective matrix $[T_{UK}]^{(d)}$ is calculated from:

$$[T_{UK}{}^{A_j}]^{(d)} = [H_{UA}(\omega_i)][H_{KA}(\omega_i)]^+, where \begin{cases}[H_{UA}] = [H(UDOF, ADOF_j, \omega_i)]\\ [H_{KA}] = [H(KDOF, ADOF_j, \omega_i)]\end{cases} \tag{3.2}$$

Having $[T_{UK}]^{(d)}$, one can finally calculate the estimated dynamic response $\{Y_U^{estimated}(\omega_i)\}$ for each j combination of possible sets A and i frequency, resorting to *(3.1)* in the following form:

$$\begin{Bmatrix} Y_{U_1}^{estimated}(\omega_i) \\ Y_{U_2}^{estimated}(\omega_i) \\ \vdots \\ Y_{U_j}^{estimated}(\omega_i) \end{Bmatrix} = [T_{UK}{}^{A_j}(\omega_i)]^{(d)} \begin{Bmatrix} Y_{K_1}^{measured}(\omega_i) \\ Y_{K_2}^{measured}(\omega_i) \\ \vdots \\ Y_{K_j}^{measured}(\omega_i) \end{Bmatrix}, j = 1,2,\dots, number\ of\ A\ combinations \tag{3.3}$$

The error between the calculated/estimated and measured responses at each DOF is calculated with the function defined in [6]:

$$error_{j,k} = \sum_{\omega_i} \left(log\left|Y_{U_{i,k}}^{measured}(\omega_i)\right| - log\left|Y_{U_{j,i,k}}^{estimated}(\omega_i)\right|\right)^2, k = 1,2,\dots,n_U \tag{3.4}$$

where n_U is the size of the set U DOF. Finally, for each j load combination, the computed error of the estimated amplitude in every DOF of set U is stored in the error vector:

$$\varepsilon = \{error_{j,k}\} \tag{3.5}$$

The accumulated error for all k DOF of set U is computed as the norm of the vector $\{\varepsilon\}$, for a j combination:

$$\varepsilon_j^{accumulated} = norm\{\varepsilon_k\} \tag{3.6}$$

The combination j of possible sets A that minimizes the accumulated error $\varepsilon_j^{accumulated}$ leads to the number and position of the external forces applied to the structure.

The forces can now be reconstructed by solving a simple inverse problem, applying the FRF matrix $[H_{KA}]$ in the entries corresponding to the correct set A and the measured dynamic responses Y_K.

$$\{F_A(\omega_i)\} = [H_{KA}]^+\{Y_K^{measured}(\omega_i)\}, where\ A\ is\ the\ set\ that\ minimizes\ \varepsilon \tag{3.7}$$

3.2. Force Identification based on Force Transmissibility - direct force identification (U=A)

For the reconstruction of the reaction forces (direct problem) one needs the numerical model for the transmissibility matrix $[T_{UK}]^{(f)}$ and to know the vector of applied forces $\{F_K^{measured}\}$. The force localization method is based in the search of $[T_{UK}]^{(f)}$ associated with the correct number and position of reaction forces $\{F_U\}$. This starts by measuring the response amplitude $\{Y_K^{measured}(\omega_i)\}$ at each frequency ω_i, and reconstructing the corresponding force $\{F_K^{measured}(\omega_i)\}$, using $\{F(\omega_i)\} = [Z(\omega_i)]\{Y(\omega_i)\}$. Note that, one could obtain $\{F_K^{measured}(\omega_i)\}$ directly from experimental values. Knowing $\{F_K^{measured}(\omega_i)\}$, one calculates the estimated reaction forces $\{F_U^{estimated}(\omega_i)\}$ with:

$$\{F_U^{estimated}(\omega_i)\} = -[H_{UU}]^{-1}[H_{UK}]\{F_K^{measured}(\omega_i)\} = [T_{UK}]^{(f)}\{F_K^{measured}(\omega_i)\} \tag{3.8}$$

Similar to what is done for the set A in displacement transmissibility, in this section, is the set U, of unknown reaction forces that has to be defined. The force identification problem for the direct force identification is illustrated in *Figure 3.2*.

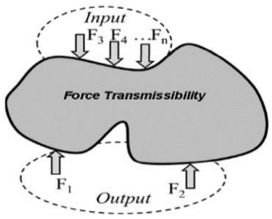


Figure 3.2 – Force transmissibility configuration – direct force identification.

Since one does not know how many reactions ($\{Y_U\} = 0$) exist, the algorithm considers $n_U = number\ of\ Boundary\ Conditions$, and calculate the nodal combinations of the possible sets U. These are obtained using the scheme of *Table 3.1* for $U = A$ and assuming that the number of possible reaction locations is N. From *(3.8)* and for *each j combination*, the respective matrix $[T_{UK}]^{(f)}$ is calculated from:

$$[T_{UK}{}^{U_j}]^{(f)} = -[H_{UU}(\omega_i)]^{-1}[H_{UK}(\omega_i)], where \begin{cases} [H_{UU}] = [H(UDOF_j, UDOF_j, \omega_i)] \\ [H_{UK}] = [H(UDOF_j, KDOF, \omega_i)] \end{cases} \tag{3.9}$$

Having $[T_{UK}]^{(f)}$, one can finally calculate the estimated applied force $\{F_K^{estimated}(\omega_i)\}$ for each j combination of possible sets U and i frequency, resorting to:

$$\begin{Bmatrix} F_{K_1}^{estimated}(\omega_i) \\ F_{K_2}^{estimated}(\omega_i) \\ \vdots \\ F_{K_j}^{estimated}(\omega_i) \end{Bmatrix} = \left([T_{UK}{}^{U_j}]^{(f)}\right)^+ \begin{Bmatrix} F_{U_1}^{measured}(UDOF_1, \omega_i) \\ F_{U_2}^{measured}(UDOF_2, \omega_i) \\ \vdots \\ F_{U_j}^{measured}(UDOF_j, \omega_i) \end{Bmatrix}, j = 1,2, \dots, n_U\ combinations \tag{3.10}$$

The error between the calculated/estimated and measured applied forces at each DOF is calculated with the function defined in [5]:

$$error_{j,k} = \sum_{\omega_i} \left(\left|F_{K_{i,k}}^{measured}(\omega_i)\right| - \left|F_{K_{j,i,k}}^{estimated}(\omega_i)\right|\right)^2, k = 1,2, \dots, n_K \tag{3.11}$$

where n_K is the size of the estimated set K DOF. Finally, for each j reactions combination, the computed error of the estimated applied forces in every DOF of set K is stored in the error vector of *(3.5)* and the accumulated error for all k DOF of set K is computed by the norm of the vector $\{\varepsilon\}$, as in *(3.6)*. Note that, for the pseudo-inverse operation in *(3.10)* to be possible it is required that $n_U \geq n_K$.

The reaction forces (direct problem) can now be reconstructed applying the FRF matrices $[H_{UU}]$ and $[H_{UK}]$ in the entries corresponding to the correct set U and the measured applied forces F_K:

$$\begin{aligned} \{F_U^{estimated}(\omega_i)\} &= -[H_{UU}]^{-1}[H_{UK}]\{F_K^{measured}(\omega_i)\} \\ &= [T_{UK}]^{(f)}\{F_K^{measured}(\omega_i)\}, where\ U\ is\ the\ set\ that\ minimizes\ \varepsilon \end{aligned} \tag{3.12}$$

3.3. Relation Between Force and Displacement Transmissibilities

One can now define the relation between force and displacement transmissibilities, extended do rotordynamics. For the purpose, the methodology presented on Lage et al. [7] and Lage [13] is here applied. Starting by recalling the displacement transmissibility equation *(3.2) for a* MDOF system*,* with free boundary conditions, as in *Figure 3.1*. Since there are no constraints/boundary conditions, one can assume that sets A and U coincide ($A = U$), and from this point relate $\{Y_U\}$ and $\{Y_K\}$, when forces are applied at U coordinates. In this case, the displacement transmissibility equation *(3.1)* becomes:

$$[T_{UK}]^{(d)} = [H_{UU}(\omega_i)][H_{KU}(\omega_i)]^{+} \tag{3.13}$$

For the pseudo-inverse operation to be possible it is required that $n_K \geq n_U$. Recalling to the force transmissibility equation *(3.9)* for the direct force identification problem, one can relate it with *(3.13)*:

$$[T_{UK}]^{(f)} = -[H_{UU}]^{-1}[H_{UK}] = -\left(\left([T_{UK}]^{(d)}\right)^{+}\right)^{T} \tag{3.14}$$

where $n_U \geq n_K$. Conversely, one may also establish the following:

$$[T_{UK}]^{(d)} = -\left(\left([T_{UK}]^{(f)}\right)^{T}\right)^{+} \tag{3.15}$$

where $n_K \geq n_U$. From *(3.14)* one can obtain the force transmissibility between two sets of coordinates U and K, from the displacement transmissibility between the same two sets of coordinates. From *(3.14)* and *(3.15)* one observes that there are limitations to the direct application of the relation between transmissibilities, which are limited by the dimension of the sets U and K.

Limitations of the transmissibility relations

As referred, the direct application of the relations between displacement and force transmissibilities is limited by the dimension of the sets U and K. Analyzing the limitations for each case:

1. $\boldsymbol{n_K = n_U}$ – The ideal case. It is possible to measure both transmissibilities and to apply the transmissibility relations *(3.14)* and *(3.15)* directly, given that at least one of them is known.
2. $\boldsymbol{n_K > n_U}$ – It is possible to measure displacement transmissibility but it is not possible to apply *(3.14)* since, for the pseudo-inverse operation in *(3.14)* to be possible, it is required that $n_U \geq n_K$. Also, it is possible to measure force transmissibility and one can apply directly *(3.15)*.
3. $\boldsymbol{n_K < n_U}$ – It is not possible to measure displacement transmissibility, because it does not fulfill the criterion $n_K \geq n_U$ imposed by *(3.13).* Consequently, although being possible to directly obtain the force transmissibility from displacement transmissibility one cannot apply *(3.14),* unless one adds fictitious coordinates to set K till $n_K = n_U$, where $\{F_K^{Fictitious}\} = 0$. Also, it is possible to measure force transmissibility but it isn't possible to apply *(3.15)* since, for the pseudo-inverse operation in *(3.15)* to be possible, it is required that $n_K \geq n_U$. Again, the solution is to add fictitious coordinates to set K till $n_K = n_U$, where $\{F_K^{Fictitious}\} = 0$.

Note that, these added fictitious coordinates are associated to dynamic responses at the set K on displacement transmissibility and to applied external forces at the set K on force transmissibility.

4. Applications

4.1. The Model

Consider a simply supported damped asymmetric rotor with a mass unbalance m_u placed at the outer radius d of the disk at an angular position α with respect to the Z axis. The damped asymmetric rotor model consists of fifteen Euler-Bernoulli beam elements, having sixteen nodes. Taking this into account, consider the boundary conditions are to be applied at nodes one and sixteen, the disk and mass unbalance excitation forces at node six as illustrated in *Figure 4.1*.

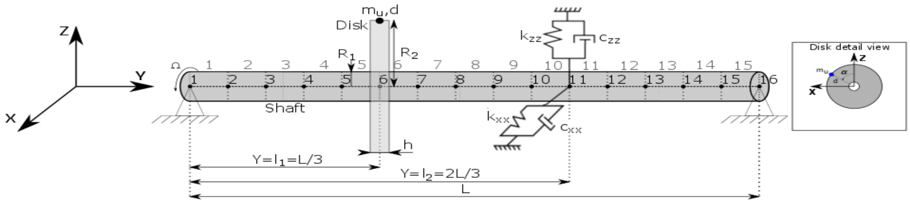


Figure 4.1 – Damped asymmetric FEM rotor model with a Mass Unbalance at the disk - $R_0(X, Y, Z)$.

For the force identification problem, the rotor FEM model is subjected to a Mass Unbalance excitation force defined by *(2.8)*. As far as the simulations are concerned, only u and w displacements and F_u and F_w forces along the X and Z axis are considered. One starts by verifying the relation *(3.15),* because the Force Identification based on Force Transmissibility is of direct application from *Figure 4.1*.

4.2. Displacement Transmissibility from Force Transmissibility

Consider the force identification problem based on force transmissibility *(section 3.2)*, with the set U of unknown reactions defined as $\boldsymbol{F_U} = \{\boldsymbol{F_1} \quad \boldsymbol{F_{16}}\}$, and the set K of known applied loads defined as $\boldsymbol{F_K} = \{\boldsymbol{F_6}\}$. Since the applicability of *(3.15),* that calculates displacement transmissibility from force transmissibility is limited by the dimension of sets U and K, one has to verify in section 3.3 if there are any limitations to it. Since $n_K = 2$ and $n_U = 4$ and consequently $\boldsymbol{n_K} < \boldsymbol{n_U}$, one cannot apply *(3.15)*. To solve this, one has to add new forces to set K to ensure that $\boldsymbol{n_K} = \boldsymbol{n_U}$, i.e., to add a fictitious force $\{F_K^{Fictitious}\} = 0$ to the force transmissibility matrix. Regarding this, a new set K of known applied loads is defined as $\boldsymbol{F_K} = \{\boldsymbol{F_6} \quad \boldsymbol{F_{11}}\}$, with $\boldsymbol{F_{11}} = \boldsymbol{0}$. A qualitative representation of the force transmissibility configuration for the rotor FEM model with the additional fictitious force, is illustrated in *Figure 4.2.*

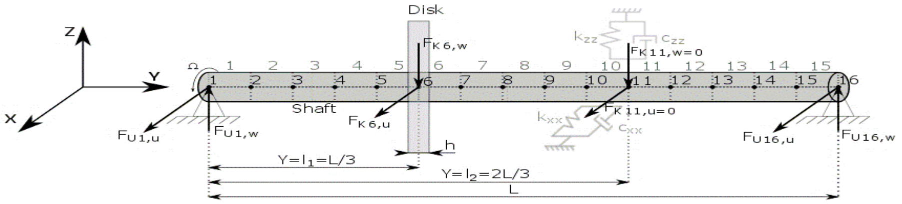


Figure 4.2 – Force transmissibility configuration for the reaction forces identification - force sets U, K.

The new force transmissibility matrix $[T_{UK}]^{(f)}$ for each frequency ω_i, can be obtained resorting to *(3.9)* applied to the global DOF of set U and the new set K with fictitious forces:

$$[T_{UK}]^{(f)} = -[H_{UU}(\omega_i)]^{-1}[H_{UK}(\omega_i)], where \begin{cases}[H_{UU}] = [H(UDOF, UDOF, \omega_i)] \\ [H_{UK}] = [H(UDOF, KDOF, \omega_i)]\end{cases}, \begin{cases} UDOF = (1 \quad 2 \quad 61 \quad 62) \\ KDOF = (21 \quad 22 \quad 41 \quad 42)\end{cases} \tag{4.1}$$

Developing $[H_{UU}]$ and $[H_{UK}]$ for the sets U and K, and substituting in *(4.1)* one obtains:

$$[T_{UK}]^{(f)} = [T(UDOF, KDOF, \omega_i)]^{(f)} = \begin{bmatrix} T_{1,21} & T_{1,22} & T_{1,41} & T_{1,42} \\ T_{2,21} & T_{2,22} & T_{2,41} & T_{2,42} \\ T_{61,21} & T_{61,22} & T_{61,41} & T_{61,42} \\ T_{62,21} & T_{62,22} & T_{62,41} & T_{62,42} \end{bmatrix}^{(f)} \tag{4.2}$$

Knowing $\{F_K\}$ from *(2.8)*, and since a mass unbalance in a stationary frame results in a synchronous load ($\omega = \Omega$), one can estimate the reaction forces $\{F_U(\omega_i)\}$ resorting to *(3.12)*:

$$\{F_U(\omega_i)\} = [T_{UK}]^{(f)}\{F_K(\omega_i)\} \leftrightarrow \begin{bmatrix} F_{U1,u}(\omega_i) \\ F_{U1,w}(\omega_i) \\ F_{U16,u}(\omega_i) \\ F_{U16,w}(\omega_i) \end{bmatrix} = \begin{bmatrix} F_1^U(\omega_i) \\ F_2^U(\omega_i) \\ F_{61}^U(\omega_i) \\ F_{62}^U(\omega_i) \end{bmatrix} = [T(UDOF, KDOF, \omega_i)]^{(f)} \begin{bmatrix} F_{K6,u}^{m_u}(\omega_i) \\ F_{K6,w}^{m_u}(\omega_i) \\ F_{K11,u}^{fictitious} = 0 \\ F_{K11,w}^{fictitious} = 0 \end{bmatrix} \tag{4.3}$$

The displacement transmissibility matrix $[T_{UK}{}^{A}]^{(d)}$, assuming $A = U$, can be finally determined from $[T_{UK}]^{(f)}$, resorting to *(3.15)*:

$$[T_{UK}]^{(d)} = -\left(\left([T_{UK}]^{(f)}\right)^T\right)^+ = \begin{bmatrix} T_{1,21} & T_{1,22} & T_{1,41} & T_{1,42} \\ T_{2,21} & T_{2,22} & T_{2,41} & T_{2,42} \\ T_{61,21} & T_{61,22} & T_{61,41} & T_{61,42} \\ T_{62,21} & T_{62,22} & T_{62,41} & T_{62,42} \end{bmatrix}^{(d)} \tag{4.4}$$

From this problem, one is interested in plotting the force transmissibilities obtained directly from *(4.1)* and presented in *(4.2)*, with the reconstructed from the displacement transmissibility problem (section 4.3), resorting to *(3.14).* For the purpose, the force transmissibility values obtained from *(3.14)* are compared with the measured ones, resorting to the correlation function *CSF (Cross Signature Scale Factor)* presented in Saldanha [14], applied through the following equation:

$$CSF(\omega_i) = \frac{2\left|[T_{UK}^{measured}]^H [T_{UK}^{reconstructed}]\right|}{[T_{UK}^{measured}]^H [T_{UK}^{measured}] + [T_{UK}^{reconstructed}]^H [T_{UK}^{reconstructed}]} \tag{4.5}$$

where $[\,]^H$, represents the complex conjugate (Hermitian) transpose. Note that, the values of this correlation function range from zero (no correlation) to one (perfect correlation).

The force transmissibility values in *(4.2)*, that relate the forces of sets *U* and *K*, through *(4.3),* can now be plotted as illustrated in *Figure 4.3*. Note that, only some entries of the force transmissibility matrix *(4.2)* are plotted, namely the ones that relate the DOF of nodes 1 ($\in$ set U) and 6 ($\in$ set K).

Force Transmissibility $\left[T_{Unode,Knode}\right]^{(f)} = \left[T_{1,6}\right]^{(f)} = \begin{bmatrix} T_{1,21} & T_{1,22} \\ T_{2,21} & T_{2,22} \end{bmatrix}$

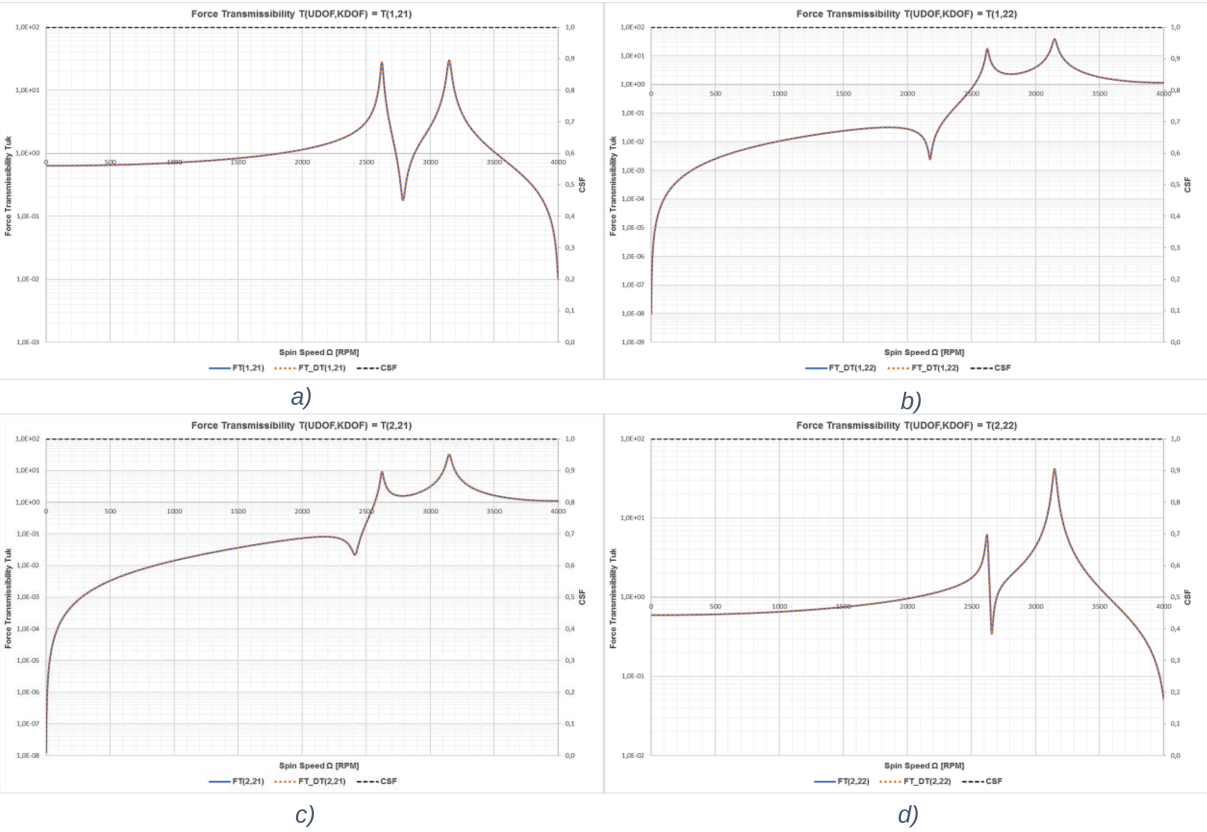


Figure 4.3 – Force Transmissibility $[T(UDOF,KDOF)]^{(f)}$: a) $T_{1,21}$; b) $T_{1,22}$; c) $T_{2,21}$; d) $T_{2,22}$.

As one can observe from *Figures 4.3,* the force transmissibilities obtained from the displacement transmissibility problem of section 4.3, resorting to *(3.15)*, exactly ($CSF = 1$ and note that it is numerical simulation without noise) reproduce those obtained directly from the force reconstruction problem. Also, one verifies that, force transmissibilities involving DOF in transverse directions (u and w), e.g. $\left[T_{1,22}\right]$ (transmissibility that relates DOF u of node 1($\in$ set U) with DOF w of node 6 ($\in$ set K)), become coupled/dependent as the spin speed Ω increases, i.e. the transmissibility starts from exhibiting a value ≈ 0 at $\Omega = 0\ [RPM]$ and then varies as function of the spin speed Ω. This behavior is evidenced in *Figure 4.3 b) and c).*

4.3. Force Transmissibility from Displacement Transmissibility

To find the equivalent Displacement Transmissibility Rotor configuration that allows to obtain $[T_{UK}]^{(f)}$ from $[T_{UK}]^{(d)}$, one deconstructs the force identification problem based on MDOF displacement transmissibility concept from the results obtained with force transmissibility (section 4.2).

Recall the force identification problem based on displacement transmissibility in section 3.1. To analyse the displacement transmissibility, one is interested in the case where the structure is considered free (no supports). Due to this, *one can assume that sets A and U coincide* ($A = U$). Also, to relate both transmissibilities, the forces to identify with both concepts must be the same. For this to occur, and since for the case of displacement transmissibility the only non-zero forces are in set $A = U$ and for the case of force transmissibility the reaction forces are the ones reconstructed (direct problem), the boundary conditions must be different and the only forces in the displacement transmissibility model must be the reaction forces. Therefore, in the equivalent Displacement Transmissibility Rotor configuration, the mass unbalance excitation forces, or any other forces except reactions, are not considered. Instead, are the reaction forces that these excitation forces cause at the nodes where supports were located (for the case of force transmissibility) that are included in the structure matrices (constrained/reduced model).

Since only u, w displacements and F_u, F_w forces along the X and Z axis are considered, the set U is immediately identified: $\boldsymbol{F_U} = \{\boldsymbol{F_1} \quad \boldsymbol{F_{16}}\}$, $\boldsymbol{Y_U} = \{\boldsymbol{Y_1} \quad \boldsymbol{Y_{16}}\}$ and $\boldsymbol{UDOF} = (\mathbf{1} \quad \mathbf{2} \quad \mathbf{61} \quad \mathbf{62})$**.** To identify the set K, substitute the displacement transmissibility matrix $[T_{UK}]^{(d)}$ obtained from $[T_{UK}]^{(f)}$ in *(3.1)*:

$$\{Y_U(\omega_i)\} = [T_{UK}]^{(d)}\{Y_K(\omega_i)\} \leftrightarrow \begin{bmatrix} Y_{U1,u}(\omega_i) \\ Y_{U1,w}(\omega_i) \\ Y_{U16,u}(\omega_i) \\ Y_{U16,w}(\omega_i) \end{bmatrix} = \begin{bmatrix} Y_1^U(\omega_i) \\ Y_2^U(\omega_i) \\ Y_{61}^U(\omega_i) \\ F_{62}^U(\omega_i) \end{bmatrix} = \begin{bmatrix} T_{1,21} & T_{1,22} & T_{1,41} & T_{1,42} \\ T_{2,21} & T_{2,22} & T_{2,41} & T_{2,42} \\ T_{61,21} & T_{61,22} & T_{61,41} & T_{61,42} \\ T_{62,21} & T_{62,22} & T_{62,41} & T_{62,42} \end{bmatrix}^{(d)} \{Y_K(\omega_i)\} \tag{4.5}$$

From *(4.5)* one can extract the measured/calculated dynamic responses $\{Y_K(\omega_i)\}$:

$$\{Y_K(\omega_i)\} = \begin{bmatrix} Y_{21}^K(\omega_i) \\ Y_{22}^K(\omega_i) \\ Y_{41}^K(\omega_i) \\ Y_{42}^K(\omega_i) \end{bmatrix} = \begin{bmatrix} Y_{K6,u}(\omega_i) \\ Y_{K6,w}(\omega_i) \\ Y_{K11,u}(\omega_i) \\ Y_{K11,w}(\omega_i) \end{bmatrix} \tag{4.6}$$

From *(4.6),* one immediately identifies the set K of known displacement responses $\{Y_K\}$: $\boldsymbol{Y_K} = \{\boldsymbol{Y_6} \quad \boldsymbol{Y_{11}}\}$ and $\boldsymbol{KDOF} = (\mathbf{21} \quad \mathbf{22} \quad \mathbf{41} \quad \mathbf{42})$, where $\boldsymbol{Y_K} = \{\boldsymbol{Y_{11}}\}$ and $\boldsymbol{KDOF} = (\mathbf{41} \quad \mathbf{42})$ are fictitious coordinates that result from the added fictitious force in the force transmissibility problem in section 4.1. The displacement transmissibility configuration for the rotor FEM model for which it is possible to obtain $[T_{UK}]^{(f)}$ from $[T_{UK}]^{(d)}$, is illustrated in *Figure 4.3*.

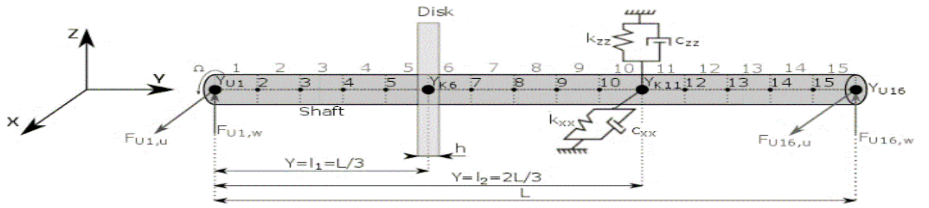
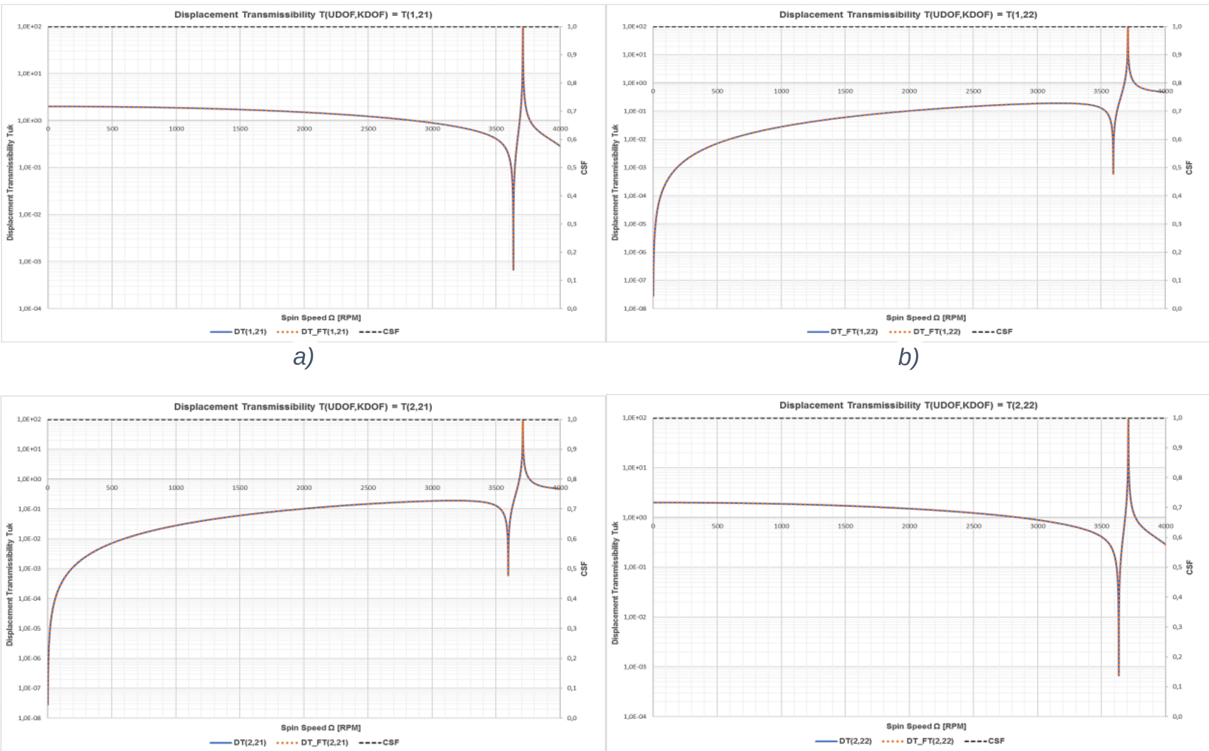


Figure 4.3 – Displacement Transmissibility Rotor configuration – coordinate sets U=A, K.

From this problem, one is interested in plotting the displacement transmissibilities obtained directly from *(3.2),* with the reconstructed from the force transmissibility problem (section 4.2), resorting to *(3.15).*

Displacement Transmissibility $\left[T_{Unode,Knode}\right]^{(d)} = \left[T_{1,6}\right]^{(d)} = \begin{bmatrix} T_{1,21} & T_{1,22} \\ T_{2,21} & T_{2,22} \end{bmatrix}$



a) *b)*

c) *d)*

Figure 4.4 – Displacement Transmissibility $[T(UDOF, KDOF)]^{(d)}$: a) $T_{1,21}$; b) $T_{1,22}$; c) $T_{2,21}$; d) $T_{2,22}$.

5. CONCLUSIONS

The aim and challenge of this thesis was achieved and consisted on to extend existing transmissibility relations as well as transmissibility-based load identification location and reconstruction to rotordynamics. In order to achieve this, the force identification using the MDOF transmissibility concepts, previously applied to beams, was adapted to rotating systems. The proposed methods were successfully applied to several numerical examples including a Propfan bi-rotor engine [15].

To relate both transmissibilities, one verified that, the forces to identify with both concepts (force and displacement transmissibility) must be the same. For this to occur, and since for the case of displacement transmissibility the only non-zero forces are in set $A = U$ and for the case of force transmissibility the reaction forces are the ones reconstructed (direct problem), the boundary conditions must be different and the only forces in the displacement transmissibility model must be the reaction forces. Therefore, in the equivalent Displacement Transmissibility configuration, the excitation forces, or any other forces except reactions, are not considered. Instead, are the reaction forces that these excitation forces cause at the nodes where supports were located (force transmissibility) that are included in the structure matrices (reduced model). Also, one verified that, in rotating structural dynamics, both force and displacement transmissibilities involving DOF in transverse directions (u and w) become coupled as the spin speed Ω increases, i.e. the transmissibility starts from exhibiting a value ≈ 0 at rest (non-rotating structural dynamics) and then varies as function of the spin speed Ω.

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