Investment in Clean Energy: a Real Options Approach

Hugo Miguel Ladeiro Pinto

October 2014

Abstract

In this paper we analyse the problem of investment in clean energy, using a valuation model that takes into account the allowance prices following diffusion processes like Geometric Brownian Motion, Geometric Mean Reversion and as proposed by Carmona and Hinz ([1]). The Hamilton-Jacobi-Bellman (HJB) equation associated to this last case involves a PDE with explicit time dependency, due to the digital nature of the terminal allowance price.

For such a problem we propose a numerical method based on finite differences to find an approximate solution for the optimal investment decision. One of the major problems is related to the free boundary (in 2 dimensions). We use an argument based on a change-of-variable formula, as proposed by Peskir ([2]).

Some numerical results are presented, showing the effect of the parameters in the investment decision.

Keywords: Real options, Energy market, Investment in clean energy, Hamilton-Jacobi-Bellman, Free boundary, Finite difference method

1 Introduction

The energy market is constantly changing over time. Many resources and policies have been devoted to control greenhouse gas (GHG) emissions in different industrial sectors. One of the last, and more notorious policy, was the Kyoto protocol. The Kyoto protocol brought to the table a new market. The carbon emissions market. How this market operates? The quantity of carbon emissions a certain country can emit is bounded by the quantity they buy. Carbon emissions can now be sold and bought. This introduces a new setting in the analysis of power plants, namely, this introduces allowance certificates to the analysis. Those allow the countries to emit carbon. As the production of energy generates a lot of carbon emissions, the price of allowance certificates for carbon emissions plays a key role in the electricity market.

Throughout this article, we will denote the price of an allowance certificate at a given time $t \geq 0$ by $A_t \in \mathbb{R}_+$. This price can depend not only on time but on different market factors, such as demand for electricity, fuel prices, etc... We will not be concerned by the market factors. Instead, we will just assume that the price of allowance certificates follows a given diffusion process.

The problem that we propose to study consists in finding the optimal investment time in a new, more clean source to produce electricity.

Mathematically, the problem can be reduced to

$$\sup_{\tau \in \mathbf{S}} \mathbb{E}\left[\int_0^\tau e^{-rt}(P_t^1 - \overline{e}A_t)dt + \int_\tau^{+\infty} e^{-rt}(P_t^2 - \underline{e}A_t)dt - e^{-r\tau}K\mathbf{1}_{\{\tau<\infty\}}\right] \tag{1}$$

where r is the interest rate, P_t^1 and P_t^2 are, respectively, the prices at which the electricity is sold for the not cleaner and cleaner sources, $\overline{e}$ and $\underline{e}$ are, respectively, the emission rates for the not cleaner and cleaner sources, and K is the investment cost.

We assume that the firm has only one possibility of investing in such cleaner source, and that once the investment is taken, it will hold for ever.

2 Optimal Stopping Problems

2.1 Dynamic Programming and HJB Equations

The problem presented in the previous section is an example of an optimal stopping problem. Therefore, we present the most important results that will be used to solve it.

The main idea is to reduce the optimal stopping problem to a differential equation. This is done recurring to the Hamilton-Jacobi-Bellman (HJB) equations. Still, and despite the use of the HJB equations, an extra condition has to be imposed to the differential equations, as in some cases the boundary is not well defined. It is known as the smooth-fit principle. In this section, we will focus on these topics, giving a brief description of them and presenting the most important results in each of them.

Before anything else, the setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with the filtration $\{\mathcal{F}_t\}$ and a $d-$dimensional $\{\mathcal{F}_t\}$-Wiener process $\{\mathcal{W}_t\}$. We assume $a \in \mathbb{R}^n$ to be the initial state at time $t \in [0, \infty)$ of the process $\{A_t, t \geq 0\}$ with state space $\mathbb{S} \subseteq \mathbb{R}^n$ that evolves according to a diffusion process

$$dA_t = \mu(A_t, t)dt + \sigma(A_t, t)dW_t. \tag{2}$$

A random variable τ taking values in $[0, \infty) \cup \{\infty\}$ is a $\{\mathcal{F}_t\}$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. We denote by $\mathbf{S}$ the set of all $\{\mathcal{F}_t\}$-stopping times.

We introduce the following functional, for the infinite case, defined for $a \in \mathbb{S}$ and $\tau \in \mathbf{S}$:

$$J(a, \tau) = \mathbb{E}[\int_0^\tau e^{-rs} g(A_s)ds + e^{-r\tau} h(A_\tau)\mathbf{1}_{\{\tau<\infty\}}] \tag{3}$$

where $g(\cdot)$ and $h(\cdot)$ are certain functions (assumed to have the necessary properties in order for equation (3) to be well defined) and r is the discount factor.

Moreover, we let $V : \mathbb{R} \to \mathbb{R}$ denote the following functional:

$$V(a) = \inf_{\tau \in \mathbf{S}} J(a, \tau) \tag{4}$$

and, therefore, the goal is to find an optimal stopping time $\tau^* \in \mathbb{S}$ such that $J(a, \tau*) = V(a)$. In the literature, $V(\cdot)$ is frequently denoted by the "value function".

For the finite case, the functional J, defined for $a \in \mathbb{S}$ and $\tau \in \mathbf{S}$, is given by:

$$J(t, a, \tau) = \mathbb{E}[\int_t^\tau e^{-rs} g(A_s)ds + e^{-r\tau} h(A_\tau)\mathbf{1}_{\{\tau<T\}}] \tag{5}$$

and thus, the value function $V : [0, T) \times \mathbb{R} \to \mathbb{R}$ of the optimal stopping problem is given by

$$V(t, a) = \inf_{\tau \in \mathbf{S}} J(t, a, \tau) \tag{6}$$

where T is the maturity time and the other arguments maintain the same properties as for the infinite case.

After applying the dynamic programming principle, we end up with the following HJB equations:

- For the infinite horizon case:

$$\min\{-rV(a) + \mathcal{L}V(a) + g(a), h(a) - V(a)\} = 0, a \in \mathbb{R}; \tag{7}$$

- For the finite horizon case:

$$\min\{-rV(t, a) + \frac{dV(t, a)}{dt} + \mathcal{L}V(t, a) + g(t, a), h(t, a) - V(t, a)\} = 0, \quad 0 \leq t \leq T, a \in \mathbb{R}. \tag{8}$$

These functions, $g(\cdot)$ and $h(\cdot)$, will have a precise meaning in the real options approach that we will use in order to analyse the problem of investment in cleaner energies. In fact, $g(\cdot)$ is the "running cost" (and denotes the cost of producing with the actual technology) and $h(\cdot)$ is the "terminal cost" (and denotes the overall cost of investing and producing with a different technology). Here, we use just g and h, without, for the time being, specifying the actual functions.

Furthermore, $\mathcal{L}f(a) = \mu(a)f'(a) + \frac{\sigma^2(a)}{2}f''(a)$ is the infinitesimal generator of the specific diffusion at hands with drift given by μ and volatility given by σ.

We use the verification theorem and the smooth-fit principle to guarantee uniqueness and smoothness of our solution.

Therefore, we address a problem in which we assume particular dynamics for the stochastic process $\{A_t, t \geq 0\}$ and propose specific drift and volatility functions μ and σ (see equation (2)) to find the solution of equations (7) and (8) with

$$
\begin{aligned}
g(a) &= \mathbb{E}\left[P_s^1 - \overline{e}A_s\right] \\
h(a) &= \mathbb{E}\left[\int_0^\infty e^{-rs}(P_s^2 - \underline{e}A_s)ds - K\right]
\end{aligned}
$$

such that $A_0 = a$, and

$$
\begin{aligned}
g(t,a) &= \mathbb{E}\left[P_s^1 - \overline{e}A_s\right] \\
h(t,a) &= \mathbb{E}\left[\int_t^T e^{-rs}(P_s^2 - \underline{e}A_s)ds - K\right],
\end{aligned}
$$

such that $A_t = a$, and T is the maturity time.

Both cases involve the solution of a differential equation (a partial one, on the finite horizon case). For the most relevant cases of dynamics for $\{A_t, t \geq 0\}$, the numerical approach has to be used, as we cannot hope to find a closed solution.

2.2 Free-boundary

To find the free boundary for the finite horizon case explained in section 4, we recur to a local space-time formula explained in [2].

The problem in hands is to solve the following expression:

$$
\begin{aligned}
\mathbb{E}_{t,b(t)}\left(h(T,A_T)\right) = h(t,b(t)) &+ \int_0^{T-t} \mathbb{E}_{t,b(t)}\left[(h_t + \mathcal{L}(h))(t+u, A_{t+u})I(A_{t+u} > b(t+u))\right] \\
&+ \int_0^{T-t} \mathbb{E}_{t,b(t)}\left[-g(X_{t+u})I(A_{t+u} < b(t+u))\right] du
\end{aligned}
\tag{9}
$$

where $\mathbb{E}_{t,b(t)}$ represents the expected value conditional to time t and value $b(t)$, $b(\cdot)$ is the boundary function, $\mathcal{L}$ represents the infinitesimal generator and I represents the indicator function, and for which we adapt a solution proposed for a similar problem presented in [3, Chapter VII].

3 Infinite Horizon Case

In this section, we assume that $P_t^1 = P^1$, $P_t^2 = P^2$, with P^1 and P^2 constants, and that $\{A_t, t \geq 0\}$ follows a diffusion process. Here, we consider that the process $\{A_t, t \geq 0\}$ follows, initially, a Geometric Brownian Motion (GBM) and, later, a Geometric Mean Reversion modified (GMR).

Our problem is, then, given by

$$\inf_{\tau \in \mathbf{S}} \mathbb{E}\left[\int_0^\tau e^{-rt}(\overline{e}A_t)dt + e^{-r\tau}\left(\int_\tau^{+\infty} e^{-r(t-\tau)}(\underline{e}A_t)dt + \frac{P^1 - P^2 + rK}{r}\right)\right]. \tag{10}$$

Therefore, if we consider $g(A_t) = \overline{e}A_t$ we obtain a familiar optimal stopping problem which has a terminal cost function given by

$$h(a^*) = \mathbb{E}\left[\int_\tau^{+\infty} e^{-r(t-\tau)}(\underline{e}A_t)dt + \frac{P^1 - P^2 + rK}{r}|A_\tau = a^*\right]. \tag{11}$$

Following the dynamic programming principle and the HJB's equations, the final differential equations for both cases are:

$$\begin{cases} -rV(a) + \mathcal{L}V(a) + \overline{e}a = 0, \quad 0 < a < a^* \\ V(a) = h(a), \quad a = a^* \\ V'(a) = h'(a), \quad a = a^* \\ V(0) = 0 \end{cases} \tag{12}$$

where $\mathcal{L}$ is the infinitesimal generator.

Thus, the solution is given by

$$\begin{cases} V(a) = \frac{\overline{e}a}{r-\mu} + a^{\beta_2}C_2 \\ a^* = \frac{(P^1-P^2+Kr)(r-\mu)(-2\mu+\sigma^2+\sqrt{8r\sigma^2+(-2\mu+\sigma^2)^2})}{(\overline{e}-\underline{e})r(-2\mu-\sigma^2+\sqrt{8r\sigma^2+(-2\mu+\sigma^2)^2})} \\ C_2 = -\frac{1}{r(-2\mu-\sigma^2+\sqrt{8r\sigma^2+(-2\mu+\sigma^2)^2})}a^{*-\frac{-2\mu+\sigma^2+\sqrt{8r\sigma^2+(-2\mu+\sigma^2)^2}}{2\sigma^2}}(2(P^1 - P^2 + Kr)\sigma^2) \end{cases} \tag{13}$$

where C_2 is a constant and $\beta_2 = \frac{-2\mu+\sigma^2+\sqrt{8r\sigma^2+(-2\mu+\sigma^2)^2}}{2\sigma^2} > 1$, for the Geometric Brownian Motion and

$$\begin{cases} -rV(a) + \mu(\theta - Log(a))aV'(a) + \frac{\sigma^2}{2}a^2V''(a) + \overline{e}a = 0, \quad 0 < a < a^* \\ V(a) = h(a), \quad a = a^* \\ V'(a) = h'(a), \quad a = a^* \\ V(0) = 0 \end{cases} \tag{14}$$

with

$$h(a^*) = \int_0^{+\infty} \underline{e}(e^{-rv+e^{-\mu v}Log(a^*)+\frac{(1-e^{-2\mu v})\sigma^2}{4\mu}+(1-e^{\mu v})(\theta-\frac{\sigma^2}{2\mu})})dv + \frac{P^1 - P^2 + rK}{r}$$

for the Geometric Mean Reversion (modified). This last problem can only be solved recurring to numerical methods.

3.1 Results

One of the initial elements of the optimal investment policy is the trigger value a^* (see, for instance, equation (13)). As already mentioned, this value is such that investment should take place as soon as the allowance certificate prices hits it. So, a^* is, in fact, the boundary value for the stopping region. That boundary value will be the critical value from now on. The continuation region increases with the increase of the critical value and, therefore, it is of great interest to see the influence of certain parameters of the problem in the behaviour of a^*.

The increase of the continuation region is related with the increase of the critical value in a direct way.

Besides the fact that the two diffusion processes considered are different, we choose to join them in one section, as the conclusions are similar for both cases.

A major difference between the two cases is the fact that one has an explicit solution (the Geometric Brownian motion) for the critical value, while for the other (the Geometric Mean Reversion (modified)), the solution can only be achieve recurring to numerical methods.

Another difference to consider is the fact that the solution of equation (14) has one more parameter of influence than the solution to equation (12). This parameter is the long term Log-price, and represents the logarithm of the price to which the process tends to.

Throughout this analysis, we presented a detailed result for both cases (GBM and GMR) for the most influential parameters. For all the other parameters we just present a brief conclusion.

- *Selling prices.*

  Starting with the selling prices, we can say that the influence of the selling prices for the Geometric Mean Reversion case does not differ from the Geometric Brownian Motion. The selling prices are an important factor of analysis. However, we do not present the explicit expression for the critical value in the GBM case, and do not present the numerical results in a figure for the GMR case. First of all, the influence of the selling prices is linear. When the selling price of the not cleaner source increases, the critical value also increases, leading to a later investment in a cleaner source. For the selling price of the cleaner source the conclusions are reverse. Another conclusion that can be draw from equation 13 is the fact that the influence of the selling prices on the critical value depends on the absolute difference between the two. Mathematically, the critical value a^* depends on $(P^1 - P^2)$ and not on each parameter separately.

- *Investment cost.*

  The investment cost is always a key parameter in this analysis. In both cases the conclusions are the same. The influence is linear in the GBM case and almost linear in the GMR case, and as the investment cost increases, the critical value also increases, i.e., the later a company invests in a new cleaner source.

  As an example, we provide the results for this case. For the Geometric Brownian motion the influence is given by:

  $$a^* = C_1 K + C_2 \tag{15}$$

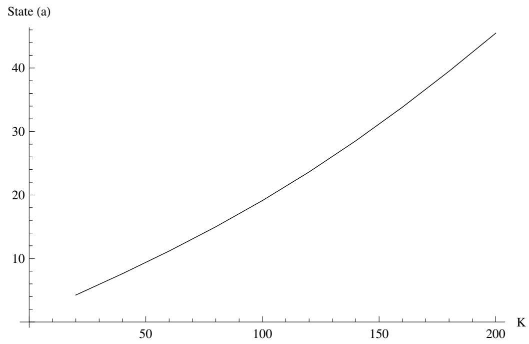
  where both $C_1 > 0$ and C_2 are constants.

  For the Geometric Mean Reversion, the numerical results are described in figure 1(a).

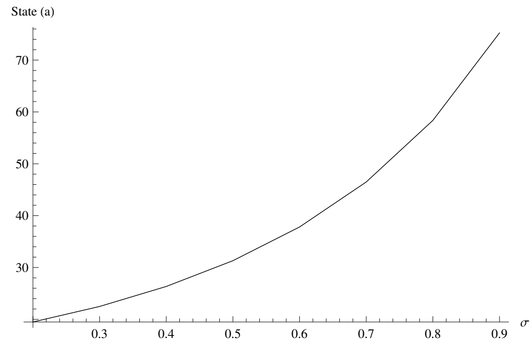
  As we can see the results are the same for both cases. This stands for the conclusions presented before.

- *Interest rate.*

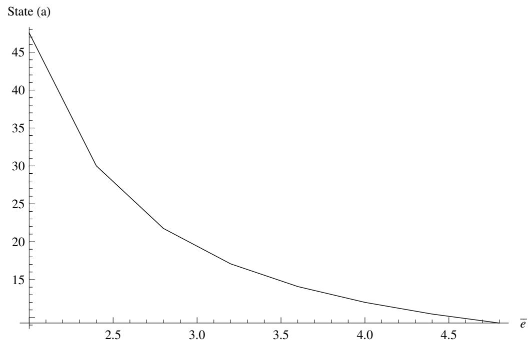
  The interest rate is a parameter that is not greatly analysed in general. Here, we just say that the influence is almost linear, and that as the interest rate increases, the critical value also increases, in both cases. This leads to a later investment in a clean source by the company.



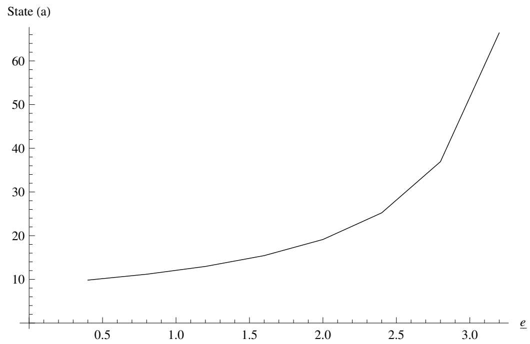
(a) Influence of the investment cost k.



(b) Influence of the volatility σ.



(c) Influence of the emission rate $\overline{e}$



(d) Influence of the emission rate $\underline{e}$

Figure 1: Influences of parameters in the GMR case.

- *Rate μ.*

  The influence of the rate μ is similar in both cases, as well. This parameter has not a linear influence on the critical value. It is not one of the most influential in the final solution, as the shifts provoked by it are small. Nevertheless, as the rate μ increases, the critical value decreases, i. e., the sooner a company invests in a cleaner source.

- *Emission rates.*

  The emission rates also have similar influence in both cases. The influence of these two parameters is linked. Meaning that, it does not make sense to study the cases separately. Therefore, the influence of the emission rates is not linear. It depends on the difference of the values for the cleaner and not cleaner source. If the value of the difference increases, then the critical value decreases.

  This last result can be confirmed from the figures 1(c) and 1(d) for the GMR case and noticing that, in this case, the equation for the critical value in the GBM case is given by

$$a^* = \frac{C_1}{\overline{e} - \underline{e}} \tag{16}$$

  where $C_1 > 0$ is constant, and $\overline{e} \geq \underline{e}$.

  In an economic point of view, this conclusion is expected. If the emission rate for the cleaner source is much smaller than the emission rate for the not cleaner source, then the investment in that cleaner source is a good investment.

- *Volatility.*

  Lastly, in the joint conclusions, the volatility. The influence of the volatility is not linear. As it increases, the critical value also increases. Economically, as the volatility measures the uncertainty of the market, it is natural that if the market is constantly changing, then the investment in a new cleaner source might not be a good investment.

The results can be confirmed from the equation for the critical value in the GBM case

$$a^* = C_1 O(\sigma^2) + C_2 \tag{17}$$

where both $C_1 > 0$ and $C_2 > 0$ are constants and $O(\sigma^2)$ represents a polynomial of order two, and from the figure 1(b) for the GMR case.

- *Long term Log-price.*

  Finally, the influence of the long term Log-price for the Geometric Mean Reversion is the one that has more convergence problems in the numerical approach. Nevertheless, some conclusions can still be drawn. As the long term Log-price increases, the critical value decreases. Meaning that, if the long term Log-price is large, then a company should use a clean source because it has a smaller emission rate, leading to an investment in such source.

As we can see for the detailed results presented, the influence of the parameters is similar in both cases, despite the fact that for the GMR case, we are only able to achieve results through numerical methods.

4 Finite Horizon Case

In this section, we analyse the problem of the optimal investment in a cleaner power source no longer assuming infinite horizon.

Moreover, we assume other models for the certificate allowance prices, as we will explain later in the section.

Returning to the problem we propose to solve, now with finite horizon, we need to solve the optimal stopping problem given by

$$\inf_{\tau \in \mathbf{S}} \mathbb{E}\left[\int_0^\tau e^{-rt}(\overline{e}A_t)dt + e^{-r\tau}\left(\int_\tau^T e^{-r(t-\tau)}(\underline{e}A_t)dt + \frac{P^1 - P^2 + rK}{r}\right)\right] \tag{18}$$

where T is the maturity time, and the other parameters are as described for the infinite horizon case.

In this case, we assume that π is a penalty that must be paid in case of failure of the Kyoto protocol, that limits the carbon emissions up to a pre-established value. Following Carmona and Hinz ([1]) and using the transformation $a_t := \frac{1}{\pi}A_t$ to simplify the notation, and assuming constant volatility ($\sigma_s = \sigma$), we end up with the following process

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{T} + W_t}{\sqrt{T-t}}\right) \tag{19}$$

with dynamics

$$da_t = \Phi'(\Phi^{-1}(a_t))\frac{1}{\sqrt{T-t}}dW_t. \tag{20}$$

Then, the HJB equation and the differential system of equations that must be solved is given by

$$\begin{aligned}
-rV(t,x) + V_t(t,x) + \mathcal{L}(V(t,x)) + \overline{e}x = 0 &\quad 0 \le x \le b(t) && (21)\\
V(t,x) = h(t,x) &\quad \text{for} \quad x = b(t) \quad \text{or} \quad t = T && (22)\\
V_x(t,x) = h_x(t,x) &\quad \text{for} \quad x = b(t) \quad \text{or} \quad t = T && (23)\\
V(t,0) = 0, &\quad 0 \le t \le T && (24)
\end{aligned}$$

$$\mathbb{E}_{t,b(t)}\left[h(T,A_T)\right] = h(t,b(t)) + \int_0^{T-t} \mathbb{E}_{t,b(t)}\left[(h_t + \mathcal{L}(h))(t+u, A_{t+u})I(A_{t+u} > b(t+u))\right] \tag{25}$$
$$+ \int_0^{T-t} \mathbb{E}_{t,b(t)}\left[-g(X_{t+u})I(A_{t+u} < b(t+u))\right] du.$$

where $\mathcal{L}$ is the infinitesimal generator for the process $\{a_s, s \in (0,T)\}$ and which can only be solved by numerical methods.

The terminal cost function $h(t,x)$ is given by

$$h(\tau, a^*) = \pi \int_\tau^T e^{-r(s-\tau)}(\underline{e}(\int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(\frac{1}{\pi}a^*)\sqrt{T}+w}{\sqrt{T-(s-\tau)}}\right) f_{N(0,s-\tau)}(w)dw))ds + \frac{P^1 - P^2 + rK}{r}$$

which can only be computed numerically.

The boundary of the continuation region is going to depend on both time and state. In section 2.2, we provided references for the numerical method that we need in order to approximate this (unknown) boundary function.

4.1 Results

In this section, we present the numerical results for the finite horizon case. In this case, the continuation region depends on the time and therefore extra care needs to be taken in order to interpret the plots. We remark that each plot represents one approximation (i.e., one approximation for a specific value of the parameters) of the boundary function.

For computation purposes, and after trying to put the terminal time T equal to 1, we consider $T = 365$, as it is the value for which the numerical method has better convergence.

It is important to refer that, in this situation, we are presented with two options in the maturity time. Either invest or not.

The option to invest will not be presented here because it is not appealing. The reason is that, if we impose the investment in a new source until the terminal time (including it), then the investment cost does not have any influence in the decision. This fact can be proved analytically. For more details see [4].

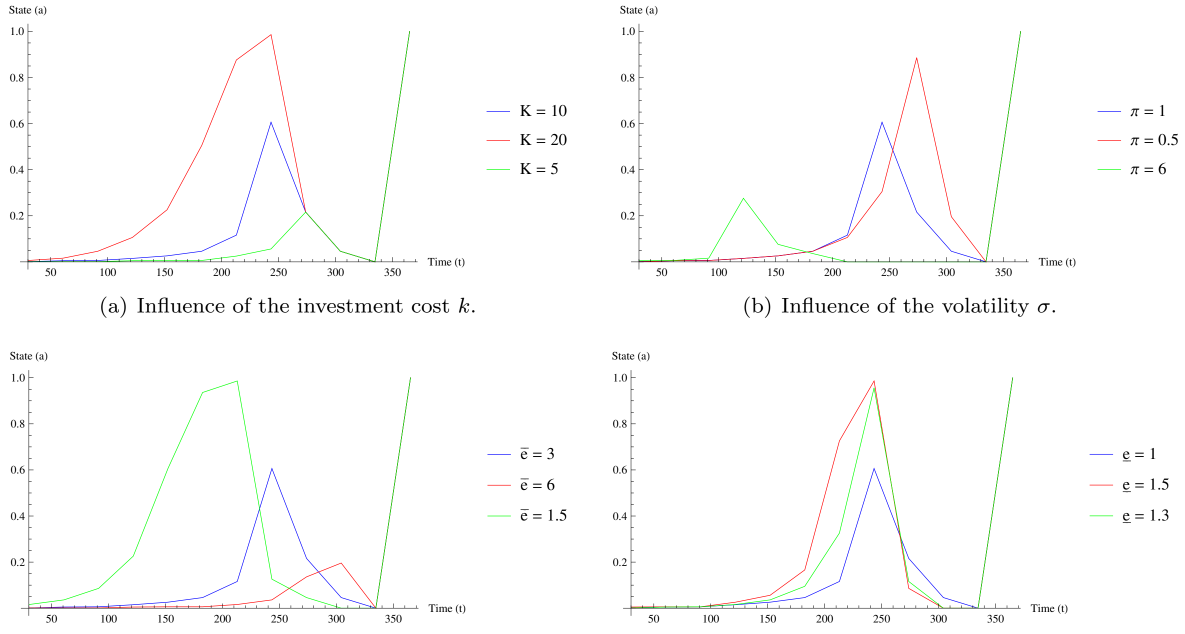
Thus, we assume that, if at time T the investment did not took place, then it will not occur at time T.

In the following analysis we just consider the parameters that we thought more influential, this means that we analyse the influence of the investment cost and the emission rates (both these cases were some of the most influential parameters in the previous analysis). Notice that this time the volatility is not a influential parameter, given that we assumed that it is constant, which in this case makes the boundary independent of the value of the volatility. Finally, we also analyse the influence of the penalty cost.

Before we present results, a quick word about their convergence. In some cases we are not able to get convergence of some of the numerical steps that we need to use in order to get an approximation for the boundary. Still, in most cases, convergence is obtained, and for cases without convergence in all the interval $[0,T]$, conclusions are still possible to be drawn.

- *Investment cost.*

  The influence of the investment cost is the same as it was for the infinite horizon cases. As the investment cost increases, the continuation region also increases, and later the company invests in a new cleaner source. This result is demonstrated in figure 2(a), where three cases are compared, each one of them with a different investment cost. As we can see, the largest investment cost ($K = 20$) results in the largest continuation region. In short and as expected, the company invests if the investment cost is small, and postpone the investment otherwise.



(a) Influence of the investment cost k.

(b) Influence of the volatility σ.

(c) Influence of the emission rate $\overline{e}$

(d) Influence of the emission rate $\underline{e}$

Figure 2: Influences of parameters in the Finite Horizon case case.

- *Emission rates.*

  The influence of the emission rates is different for the finite horizon case. Not in the general conclusion, but in the importance of the parameters to the continuation region. Although, in the infinite horizon case, the continuation region (critical value) depends on the difference of the values, making then both equally important. Here, in the finite horizon case, the emission rate for the not cleaner source has a more significant impact on the continuation region. Therefore, we separate the analysis of the two parameters. The influence of the emission rate for the not cleaner source is, as expected, similar to the one of the infinite horizon cases. As that emission rate increases the continuation region decreases, i.e., the sooner a company invests in a new cleaner source. This result can be proved by figure 2(c), where the influence of the emission rate $\overline{e}$ is analysed. Figure 2(d) shows the influence of the emission rate $\underline{e}$ on the continuation region. It is not as important as it was in the infinite horizon cases, because, the continuation region remains almost unaltered with shifts of this parameter. Nevertheless, as this emission rate increases, the larger the continuation region is. Meaning that, the later a company invests in a cleaner source. A quick word about the influence of these two rates together. The continuation region depends on the absolute value of both parameters, and not on the difference or the rate between them.

- *Penalty.*

  Finally, we study the influence of the penalty, the parameter that did not appear in the infinite horizon cases. This influence is demonstrated in figure 2(b). As we can see by the comparison of figures 2(b) and 2(a), the influence of the penalty is the opposite of the influence of the investment cost. As expected, if the penalty increases, then the continuation region is smaller. Meaning that, the sooner a company invests in a cleaner source. Economically, this results makes sense. If the penalty for surpassing a given limit of emissions is large, then it is worth to invest in a source that has a smaller emission rate, as soon as possible.

For more details on the results we refer to [4].

5 Conclusions

This work presented an approach based on optimal stopping techniques, as a tool to analyse the investment in cleaner sources of energy. Initially, we assumed that the allowance price followed a Geometric Brownian Motion. Despite having several advantages such as simplicity of calculations and analytical solution, it has serious limitations in describing the allowance price.

Then, we analysed the problem using the Geometric Mean Reversion as the process to model the allowance certificates prices.

For such model, the solution of the associated stochastic differential equation is known, but the ordinary differential equation derived by the HJB equation cannot be solved analytically, and for that reason we had to use numerical methods to find/describe the optimal investment policy.

In the infinite horizon assumption, all the purposed models indicate that the investment cost and the marginal emission rates are the most significant parameters of our problem. Additionally, increasing the volatility of the market postpones the investment decision, as it is commonly accepted in real options.

When we consider the finite horizon case, new challenges appear as in this case we have to use numerical approximations in a greater extent than in the infinite case. The qualitative conclusions that we get in the infinite horizon still hold in the finite horizon case. Additionally, the penalty cost plays the (expected) influence: the larger it is, the more incentives to invest in a cleaner source.

Despite numerical problems, the qualitative results are very clear both in the finite and infinite cases.

The obvious extension of this work is to address the finite horizon case without the assumption of constant volatility, as also Carmona ([1]) proposes. Surely this will raise more problems associated with the complexity and convergence issues of the numerical methods that will be needed.

As a further extension of this work, one may assume that the selling prices of electricity are not constant, but instead are defined as a function of various market factors, as mentioned in section 1. On this theme we refer to [5], where Schwarz gives a proposal for the electricity price.

References

[1] R. Carmona and J. Hinz. Risk-neutral models for emission allowance prices and option valuation. *Journal Management Science*, 57(8):1453–1468, August 2011.

[2] G. Peskir. A change-of-variable formula with local time on curves. *Journal of Theoretical Probability*, 18(3):499–535, July 2005.

[3] G. Peskir and A. Shiryaev. *Optimal Stopping and Free Boundary Problems*. Birkhäuser Verlag, 2006.

[4] H. Pinto. *Investment in Clean Energy: a Real Options Approach*. Master thesis, Instituto Superior Técnico, 2014.

[5] D. C. Schwarz. *Price Modelling and Asset Valuation in Carbon Emission and Electricity Markets*. Doctoral thesis, University of Oxford, 2012.